Acta Mathematica Sinica, English Series
Mar., 2013, Vol. 29, No. 3, pp. 471–476
Published online: February 15, 2013
DOI: 10.1007/s10114-013-1555-z
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

A Note on Extreme Points in Dual Spaces

Juan Francisco MENA-JURADO
Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada,
18071-*Granada, Spain*
E-mail: *jfmena@ugr.es*

Juan Carlos NAVARRO-PASCUAL1)
Departamento de Álgebra y Análisis Matemático, Universidad de Almería, 04120 *Almería, Spain*
E-mail: *jcnav@ual.es*

Abstract Given a normed space X it can be easily proven that every extreme point in B_{X^*}, the unit ball of X^*, is the restriction of an extreme point in $B_{X^{***}}$. Our purpose is to study when the restrictions of extreme points in $B_{X^{***}}$ are extreme points in B_{X^*}. Namely, we characterize L_1-preduals satisfying the aforementioned property.

Keywords Extreme point, L_1-predual, M-ideal

MR(2010) Subject Classification 46B20, 46E15

1 Introduction

In the sequel, X and Y will be real or complex normed spaces. As usual, B_X and E_X will stand for the unit ball of X and the set of extreme points of B_X, respectively. We will denote by $L(X,Y)$ the space of all the linear continuous operators from X into Y and, as usual, $L(X,\mathbb{K})$ will be written in X^*. Elements in $E_{L(X,Y)}$ will be called extreme operators. For any T in $L(X,Y)$, T^* will denote its adjoint. If $x^* \in E_{X^*}$, there is $x^{***} \in E_{X^{***}}$ whose restriction to X coincides with x^*, that is, $x^{***} \circ J_X = x^*$, where J_X is the canonical imbedding from X into X^{**}. Nevertheless, the restriction of an element in $E_{X^{***}}$ does not always belong to E_{X^*} (see Theorem 2.5 and Corollary 2.6).

The aim of this paper is to study conditions on a normed space X ensuring that

$$x^{***} \circ J_X \in E_{X^*}, \qquad \text{for all } x^{***} \in E_{X^{***}},$$

that is,

$$(J_X)^*(E_{X^{***}}) \subseteq E_{X^*}.$$

This last property has been introduced by Blumenthal, Lindenstrauss and Phelps in [1] for an operator between normed spaces. We recall it in the following definition.

Received September 29, 2011, accepted March 9, 2012
Supported by Ministerio de Ciencia e Innovación and Junta de Andalucía Projects (Grant Nos. MTM2009-07498, MTM2010-17687 and FQM-3737)
1) Corresponding author

Definition 1.1 (Nice operators) *Let X and Y be normed spaces. An operator T in $L(X,Y)$ is said to be nice if $T^*(E_{Y^*}) \subseteq E_{X^*}$.*

It is easy to prove that every nice operator is extreme. Nice operators have been widely studied because they are very useful for studying extreme operators. It is well known that J_X, the canonical imbedding of a normed space X in its bidual, is always an extreme operator in $L(X, X^{**})$ (see [2]). Up to date, only L-spaces ($L_1(\mu)$ for some measure μ) and $\mathcal{C}$-spaces ($\mathcal{C}(K)$ for some compact Hausdorff space K) have been considered in order to prove that its canonical injection is nice (see [2, Remark following Proposition 3.4]). The main goal of this paper is to characterize Banach spaces with nice canonical imbedding in a well-known class which we introduce below.

Definition 1.2 (L_1-preduals) *An L_1-predual is a Banach space X such that X^* is isometrically isomorphic to $L_1(\mu)$ for some measure μ.*

Our results generalize those obtained in [2] for L-spaces and $\mathcal{C}$-spaces. As a consequence, we will get examples of classical Banach spaces whose canonical imbedding is not nice.

2 Results

The fact that L-spaces have nice canonical injection will be obtained as a consequence that these spaces belong to the class which we introduce below.

Definition 2.1 (L-summands) *Let X be a Banach space. A linear projection $P : X \to X$ is called an L-projection if it satisfies*

$$\|x\| = \|Px\| + \|x - Px\| \quad \textit{for all } x \in X.$$

The range of an L-projection is called an L-summand. When $J_X(X)$ is an L-summand in X^{**}, we say that X is an L-summand in its bidual.

L-spaces and preduals of a von Neumann algebra are suggestive general examples of Banach spaces which are L-summands in its bidual (see [3]).

Theorem 2.2 *Let X be a Banach space which is an L-summand in its bidual. Then J_X is a nice operator.*

Proof If X is an L-summand in X^{**}, then there exists a closed subspace Z of X^{**} such that $X^{**} = J_X(X) \bigoplus_1 Z$. Hence, we have $X^{***} = J_X(X)^\circ \bigoplus_\infty Z^\circ$, and from here we deduce $E_{X^{***}} = E_{J_X(X)^\circ} + E_{Z^\circ}$, so $J_X^*(E_{X^{***}}) = J_X^*(E_{Z^\circ})$. Now, it is easy to prove that the restriction of J_X^* to Z° is a surjective linear isometry from Z° onto X^*, so $J_X^*(E_{Z^\circ}) = E_{X^*}$ and we conclude that J_X is a nice operator. □

Theorem 2.2 generalizes the result mentioned above about L-spaces, which says that if X is an L-space, then J_X is nice. Moreover, we can use this theorem to obtain the next corollary.

Corollary 2.3 *The canonical imbedding of the predual of a von Neumann algebra is a nice operator.*

The predualization of the concept of L-summand gives us the following definition.

Definition 2.4 (M-ideals) *Let X be a Banach space. A closed subspace Y of X is called an*

M-ideal if Y° is an L-summand in X^*. When $J_X(X)$ is an M-ideal in X^{**}, we say that X is an M-ideal in its bidual.

Again, the main reference is [3]. There are several spaces which are M-ideals in its bidual, the compact operators on a Hilbert space being a representative example.

Theorem 2.5 *Let X be a Banach space which is a proper M-ideal in its bidual. Then the canonical imbedding is not a nice operator.*

Proof Provided X is an M-ideal in X^{**}, we have $X^{***} = J_{X^*}(X^*) \bigoplus_1 J_X(X)^\circ$ (see [3, Proposition III.1.2]). Then, if we consider $x^{***} \in E_{J_X(X)^\circ} \subseteq E_{X^{***}}$, we obtain $J_X^*(x^{***}) = 0 \notin E_{X^*}$, thus J_X is a non-nice operator. □

Taking into account that $c_0(\Gamma)$ is an M-ideal in its bidual [3, Examples III.1.4], we obtain the following result.

Corollary 2.6 *Let Γ be an infinite set and $X = c_0(\Gamma)$. Then the canonical injection J_X is a non-nice operator.*

We introduce now a class of Banach spaces which can be defined by certain properties of intersection of balls (see [4, Theorem 2.2] for details) and which includes L-spaces and $\mathcal{C}$-spaces [5, Example 6].

Definition 2.7 (Property (E)) *Let X be a normed space such that E_X is nonempty. We will say that X satisfies Property (E) if $|x^*(x)| = 1$ whenever $x^* \in E_{X^*}$ and $x \in E_X$.*

For Banach spaces with Property (E), we give a sufficient condition for getting that the canonical injection is nice.

Proposition 2.8 *Let X be a Banach space which satisfies Property (E) and such that $B_X = \overline{\mathrm{co}}(E_X)$. Then J_X is nice.*

Proof By [6, Proposition 2.1], we have $|x^{***}(J_X x)| = |(J_X^* x^{***})(x)| = 1$ whenever $x \in E_X$ and $x^{***} \in E_{X^{***}}$, and so, $|x^{**}(J_X^* x^{***})| = 1$ whenever x^{**} is in $\overline{J_X(E_X)}^{w^*}$ and x^{***} is in $E_{X^{***}}$. Now [2, Lemma 5.1] allows us to get that $|x^{**}(J_X^* x^{***})| = 1$ whenever $x^{**} \in E_{X^{**}}$ and $x^{***} \in E_{X^{***}}$ and we conclude that $J_X^* x^{***} \in E_{X^*}$ for all $x^{***} \in E_{X^{***}}$. □

It suffices to take into account that the unit ball of $L_1(\mu)$ has extreme points if and only if, μ has atoms to see that the known fact that L-spaces have nice canonical injection cannot be deduced from the above proposition and that the condition appearing in it is not necessary for getting that J_X is nice. One can ask if Property (E) is enough to get nice canonical imbedding. We will see below that this is not the case.

Corollary 2.6 allows us to give an example of a Banach space X having Property (E) such that the canonical imbedding J_X is a non-nice operator.

Example 2.9 Let Y be a Banach space with Property (E), and let us define $X := Y \bigoplus_1 c_0$. Then X has Property (E) and J_X is a non-nice operator.

We start by showing that X has Property (E). As an immediate consequence of the definition of X, we have $X^* = Y^* \bigoplus_\infty \ell_1$. Let x be in E_X and x^* be in E_{X^*}. Then we deduce that $x \in E_Y$, and there exist y^* in E_{Y^*} and z^* in E_{ℓ_1} such that $x^* = y^* + z^*$, thus we have

$$|x^*(x)| = |(y^* + z^*)(x)| = |y^*(x)| = 1,$$

because Y has Property (E).

Now, we see that J_X is a non-nice operator. It is immediate to prove that $J_X = J_Y + J_{c_0}$ and hence $J_X^* = J_Y^* + J_{c_0}^*$. Moreover, we have $X^{***} = Y^{***} \bigoplus_\infty \ell_\infty^*$ and

$$E_{X^{***}} = \left\{ y^{***} + z^{***} : y^{***} \in E_{Y^{***}}, z^{***} \in E_{\ell_\infty^*} \right\}.$$

Finally, let y^{***} be in $E_{Y^{***}}$ and z^{***} be in $E_{\ell_\infty^*}$ such that $J_{c_0}^* z^{***} \notin E_{\ell_1}$ (J_{c_0} is a non-nice operator). Then, we have $x^{***} = y^{***} + z^{***} \in E_{X^{***}}$, but

$$J_X^* x^{***} = J_Y^* y^{***} + J_{c_0}^* z^{***} \notin E_{X^*}$$

and we conclude that J_X is a non-nice operator.

In Corollary 2.6, we obtain L_1-preduals whose canonical imbedding is a non-nice operator. We are going to characterize those L_1-preduals with nice canonical imbedding. In order to prove our result, we will need the following proposition.

Proposition 2.10 *Let X and Y be Banach spaces and $T : X \to Y$ a linear isometry. Then $E_{X^*} \subseteq T^*(E_{Y^*})$. In particular $E_{X^*} \subseteq J_X^*(E_{X^{***}})$.*

Proof For $x^* \in E_{X^*}$, let us define $K = \{y^* \in B_{Y^*} : T^*(y^*) = x^*\}$. It is easy to see that K is a nonempty w^*-closed face of B_{Y^*}. So there is some extreme point in K which is in E_{Y^*} and this finishes the proof. □

From the above proposition we get a result from which it can be easily deduced [6, Proposition 2.1].

Corollary 2.11 ([2, Proposition 3.5]) *Let X be a Banach space. Then*

$$J_X^*(E_{X^{***}}) \subseteq \overline{E_{X^*}}^{w^*}.$$

Proof Let us denote $K = \overline{E_{X^*}}^{w^*}$ and $Y = \mathcal{C}(K)$. Let us define $T : X \to Y$ by

$$T(x)(x^*) = x^*(x).$$

Then T is a linear isometry and so T^{**} is a linear isometry from X^{**} into Y^{**}. Proposition 2.10 gives us $E_{X^{***}} \subseteq T^{***}(E_{Y^{***}})$ and from here $J_X^*(E_{X^{***}}) \subseteq (J_X^* \circ T^{***})(E_{Y^{***}})$. Now

$$J_X^* \circ T^{***} = (T^{**} \circ J_X)^* = (J_Y \circ T)^* = T^* \circ J_Y^*$$

and therefore,

$$J_X^*(E_{X^{***}}) \subseteq T^*(J_Y^*(E_{Y^{***}})) \subseteq T^*(E_{Y^*}).$$

Where in the last inclusion we have used that J_Y is nice which is a consequence of [7, Theorem 1.3] together with the fact that Y^{**} is $\mathcal{C}(H)$ for convenient extremally disconnected compact Hausdorff space H (see [8]). Finally, it can be easily checked that $T^*(E_{Y^*}) \subseteq K$. □

Taking into account Proposition 2.10 and Corollary 2.11, we get the following corollary.

Corollary 2.12 *Let X be a Banach space such that $E_{X^{***}}$ is w^*-closed. Then*

$$J_X^*\left(E_{X^{***}}\right) = \overline{E_{X^*}}^{w^*}.$$

As a result, J_X is a nice operator if and only if, E_{X^*} is w^*-closed.

To state our main result, we will need a previous definition.

Definition 2.13 *Let K be a compact Hausdorff space and $\Sigma : K \to K$ a continuous map such that $\Sigma^2 = \mathrm{Id}$, where Id is the identity map, and such that Σ has no fixed points. We will denote $\mathcal{C}_\Sigma(K, \mathbb{R})$ the (real) Banach space of all the continuous maps f in K such that $f\Sigma = -f$ with the uniform norm.*

Theorem 2.14 *Let X be a real or complex L_1-predual. The following assertions are equivalent*:

i) *The canonical imbedding J_X is a nice operator.*

ii) *E_{X^*} is w^*-closed.*

If X is a real space, the above assertions are equivalent to the following:

iii) *There is a compact Hausdorff space K such that $X = \mathcal{C}_\Sigma(K, \mathbb{R})$.*

Proof To prove the equivalence between i) and ii), we are going to see that X satisfies the hypothesis of Corollary 2.12. If X^* is an L-space, then its dual X^{**} is a $\mathcal{C}(K_0)$ -space (see, for example, [8]), for appropriate compact Hausdorff space K_0, and hence $X^{***} = \mathcal{C}(K_0)^*$, thus $E_{X^{***}} = E_{\mathcal{C}(K_0)^*}$, which is w^*-closed. The third equivalence in the real case is deduced from the classification of L_1-preduals made by Lindenstrauss and Wulbert in [9] (see also [10]). □

Finally, taking into account that the space of real-valued, continuous affine functions defined on a simplex space is always an L_1-predual, we deduce the following result.

Corollary 2.15 *Let K be a simplex, and $X = A(K, \mathbb{R})$ the space of real-valued continuous affine functions defined on K with the supremum norm. The following are equivalent*:

i) *J_X is a nice operator.*

ii) *The extreme points of K are closed.*

iii) *X is a $\mathcal{C}$-space.*

Proof i) ⇒ ii) By Theorem 2.14, E_{X^*} is w^*-closed, and [11, Theorem 3.2] gives this implication.

ii) ⇒ iii) It is an easy consequence of [12, Theorem II.7.5].

iii) ⇒ i) This is known. □

References

[1] Blumenthal, R. M., Lindenstrauss, J., Phelps, R. R.: Extreme operators into $\mathcal{C}(K)$. *Pacific J. Math.*, **15**, 747–756 (1965)

[2] Sharir, M.: Extremal structure in operator spaces. *Trans. Amer. Math. Soc.*, **186**, 91–111 (1973)

[3] Harmand, P., Werner, D., Werner, W.: M-ideals in Banach Spaces and Banach Algebras. Lecture Notes in Math., **1547**, Springer-Verlag, Berlin, 1993

[4] Lima, Å.: Intersection properties of balls in spaces of compact operators. *Ann. Inst. Fourier*, **28**, 35–65 (1978)

[5] Mena-Jurado, J. F., Montiel-Aguilera, F.: A note on nice operators. *J. Math. Anal. Appl.*, **289**, 30–34 (2004)

[6] Dutta, S., Rao, T. S. S. R. K.: On weak*-extreme points in Banach spaces. *J. Convex Anal.*, **10**, 531–539 (2003)

[7] Gendler, A.: Extreme operators in the unit ball of $L(\mathcal{C}(X), \mathcal{C}(Y))$ over the complex field. *Proc. Amer. Math. Soc.*, **57**, 85–88 (1976)

[8] Lacey, H. E.: The Isometric Theory of Classical Banach Spaces, Springer-Verlag, Berlin-New York, 1974

[9] Lindenstrauss, J., Wulbert, D. E.: On the classification of the Banach spaces whose duals are L_1-spaces. *J. Funct. Anal.*, **4**, 332–349 (1969)

[10] Lazar, A. J., Lindenstrauss, J.: Banach spaces whose duals are L_1-spaces and their representing matrices. *Acta Math.*, **126**, 165–193 (1971)

[11] Lazar, A. J.: Affine functions on simplexes and extreme operators. *Israel J. Math.*, **5**, 31–43 (1967)

[12] Asimow, L., Ellis, A. J.: Convexity Theory and Its Applications in Functional Analysis, Academic Press, London, 1980