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DIFFERENTIABLE FUNCTIONS AND NICE OPERATORS

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Abstract. The aim of this paper is to describe the operators between spaces of continuously differentiable functions whose adjoint preserves extreme points. It is important to mention that no condition regarding injectivity or surjectivity of the operators is assumed. Previously known results characterizing surjective isometries can be immediately derived from such descriptions.

1. Introduction and preliminaries

Let K_1 be a compact Hausdorff space, and let $C(K_1)$ be the space of scalar-valued continuous functions defined on K_1 endowed with the uniform norm. A well-known result of Arens and Kelley [3] describes the extreme points of the closed unit ball of the dual of $C(K_1)$ as the functions of the form $\alpha\delta_s$, where α is a scalar with $|\alpha| = 1$ and δ_s is the evaluation functional at an arbitrary point $s \in K_1$.

The scalar field ($\mathbb{R}$ or $\mathbb{C}$) is denoted hereafter by $\mathbb{K}$.

If $\mathbb{K}$ is identified with the space of continuous functions defined on a compact Hausdorff space K_2, reduced to a point, then the mappings $\alpha\delta_s$ match the weighted composition operators $T : C(K_1) \to C(K_2)$ given by

$$(Tf)(t) = e(t)f\big(\varphi(t)\big), \quad \text{for any } f \in C(K_1) \text{ and } t \in K_2, \tag{1.1}$$

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where $e : K_2 \to \mathbb{K}$ and $\varphi : K_2 \to K_1$ are (continuous) functions with $|e(t)| = 1$, for every $t \in K_2$.

If K_1 and K_2 are arbitrary compact Hausdorff spaces, then a linear and continuous operator $T : C(K_1) \to C(K_2)$ is of the form of (1.1) if and only if the adjoint of T maps every extreme point of the closed unit ball of $C(K_2)^*$, the dual of $C(K_2)$, to an extreme point of the closed unit ball of $C(K_1)^*$. This kind of behavior leads to the concept of *nice operators*.

Given a normed space X, the symbols B_X and S_X stand for the closed unit ball and the unit sphere of X, respectively:

$$B_X = \{x \in X : \|x\| \leq 1\}, \qquad S_X = \{x \in X : \|x\| = 1\}.$$

We use the abbreviation E_X to refer to the set of extreme points of B_X. As usual, if $A \subset X$, then the linear span of the set A will be written in the form $\operatorname{span} A$. By means of X^* we will express the dual Banach space of X.

If Y is also a normed space over the same field $\mathbb{K}$, then $L(X, Y)$ will denote the space of linear and continuous mappings from X into Y, provided with the operator norm. According to custom, we write $L(X)$ instead of $L(X, X)$. For each $T \in L(X, Y)$, the adjoint of T is denoted by T^*.

An operator $T \in L(X, Y)$ is said to be *nice* if $T^*(E_{Y^*}) \subset E_{X^*}$. This notion first appeared in [16]. But, without an explicit designation, such operators had been considered previously in [5].

Any surjective linear isometry (and in particular the identity mapping on every normed space) is a nice operator. Moreover, each nice operator $T : X \to Y$ is an extreme point of $B_{L(X,Y)}$. Regarding this last comment, it should be noted that the elements of the set $E_{L(X,Y)}$ are known as extreme operators or extreme contractions.

The two important references already mentioned—[5] and [16]—later gave rise to remarkable contributions to deepen the study of extreme operators and their connection with nice operators. Early works on this subject include results as interesting as those obtained by Sharir in [21] and the references therein. Originally, extreme operators emerged within the context of continuous function spaces, but they have also been studied in $L_1(\mu)$-spaces (see [20]), l_p-spaces (see [4], [12], [13]), and some other structures (see [10], [22], [14]). Some spaces of vector-valued functions have also been considered (see [1], [2]). More recent contributions can be found in [7], [6], [15], [17], and [8]. The survey [19] deserves special mention by virtue of its excellent exposition.

The study of the interplay between nice operators and isometric isomorphisms is the goal of [17]. In particular, it is shown that there exist of infinite-dimensional Banach spaces X such that any nice operator $T \in L(X)$ is an isometric isomorphism. The reference [17] also contains a description of the nice operators between spaces of continuously differentiable functions with respect to two natural norms.

Now let K be a compact interval of $\mathbb{R}$, and let $C^1(K)$ be the vector space of scalar-valued continuously differentiable functions defined on K. Given $x \in C^1(K)$, by putting

$$p(x) = \max\{\|x\|_\infty, \|x'\|_\infty\},$$

$$q(x) = \max\{|x(t)| + |x'(t)| : t \in K\},$$
$$r(x) = \|x\|_\infty + \|x'\|_\infty,$$

three equivalent norms on $C^1(K)$ are obtained which make this space into a Banach algebra. The study of nice operators between continuously differentiable function spaces performed in [17] specifically contemplates the norms p and q.

As we have already suggested, from the topological point of view, there is no distinction between norms p, q, and r. However, their geometrical behavior differs markedly.

The main purpose of this paper is to describe nice operators between spaces of continuously differentiable functions endowed with norms of type r. The results are formally similar to those obtained previously for the norms p and q, but the proofs must overcome many additional hurdles of considerable difficulty. The operators we are going to examine need not be injective or surjective. Therefore, our results contain, as a particular case, the description of surjective isometries between $C^1(K)$-spaces, which can be found in [9], [11], and [18].

Below we will make some additional comments on the notation to be used. As is customary, we will write

$$\mathbb{T} = \{\alpha \in \mathbb{K} : |\alpha| = 1\}.$$

The length of a compact interval K of $\mathbb{R}$ will be denoted by $l(K)$. On the other hand, u_K and i_K will stand for the elements of $C^1(K)$ defined by

$$u_K(t) = 1, \qquad i_K(t) = t, \quad \text{for every } t \in K.$$

If $t \in K$, we will also consider the functions δ_t and δ'_t, from $C^1(K)$ into $\mathbb{K}$, defined by

$$\delta_t(x) = x(t), \qquad \delta'_t(x) = x'(t), \quad \text{for every } x \in C^1(K).$$

Concerning these functionals, we define the following sets:

$$\nabla_K = \{\delta_t : t \in K\} \qquad \text{and} \qquad \nabla'_K = \{\delta'_t : t \in K\}.$$

The next basic result contains Proposition 1 of [17] as a special case.

Proposition 1.1. *The set $\nabla_K \cup \nabla'_K$ is linearly independent and, in particular,*

$$\operatorname{span} \nabla_K \cap \operatorname{span} \nabla'_K = \{0\}. \tag{1.2}$$

Consequently, given $n \in \mathbb{N}$, $\alpha_i, \beta_i \in \mathbb{T}$, and $t_i, s_i \in K$, for every $i \in \{1, \dots, n\}$, the vectors

$$\alpha_1\delta_{t_1} + \beta_1\delta'_{s_1}, \dots, \alpha_n\delta_{t_n} + \beta_n\delta'_{s_n}$$

are linearly independent if $t_1, \dots, t_n$, or $s_1, \dots, s_n$, are pairwise different.

Proof. Let us see first that $\operatorname{span} \nabla_K \cap \operatorname{span} \nabla'_K = \{0\}$. To this end, consider $n, m \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \mathbb{K}$ and $t_1, \dots, t_n, s_1, \dots, s_m \in K$ such that

$$\alpha_1\delta_{t_1} + \dots + \alpha_n\delta_{t_n} = \beta_1\delta'_{s_1} + \dots + \beta_m\delta'_{s_m}. \tag{1.3}$$

One can assume that $t_i \neq t_j$ and $s_i \neq s_j$, for any $i, j \in \{1, \ldots, m\}$ with $i \neq j$. Fix a natural number $j_0 \leq m$. If $s_{j_0} \notin \{t_1, \ldots, t_n\}$, then define $\eta \in C^1(K)$ as follows:

$$\eta(t) = \prod_{j=1}^{n}(t - t_j) \prod_{j \neq j_0}(t - s_j)^2.$$

On the contrary, if $s_{j_0} \in \{t_1, \ldots, t_n\}$, then η will be the function given by

$$\eta(t) = \prod_{t_j \neq s_{j_0}}(t - t_j) \prod_{j \neq j_0}(t - s_j)^2.$$

In addition, let $x \in C^1(K)$ be defined by $x(t) = (t - s_{j_0})\eta(t)$, for every $t \in K$. In any of the above cases, $x(t_j) = 0$, for every $j \in \{1, \ldots, n\}$, and $x'(s_j) = 0$, for every $j \in \{1, \ldots, m\} \backslash \{j_0\}$. It is also clear that $x'(s_{j_0}) = \eta(s_{j_0}) \neq 0$. By virtue of (1.3), $\beta_{j_0} x'(s_{j_0}) = 0$ and hence $\beta_{j_0} = 0$. This concludes the proof of (1.2) and shows the linear independence of the (pairwise different) functionals $\delta'_{s_1}, \ldots, \delta'_{s_m}$. (Note that, in the above argument, we could have considered $\alpha_1 = \cdots = \alpha_n = 0$.)

Moreover, the linear independence of the functionals $\delta_{t_1}, \ldots, \delta_{t_n}$ (where $t_i \neq t_j$, for any $i, j \in \{1, \ldots, n\}$, with $i \neq j$) is immediate simply by using, for each $j_0 \in \{1, \ldots, n\}$, the function $x \in C^1(K)$ given by

$$x(t) = \prod_{j \neq j_0}(t - t_j).$$

It is thus clear that the set $\nabla_K \cup \nabla'_K$ is linearly independent.

Finally, consider $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{T}$ and $t_1, \ldots, t_n, s_1, \ldots, s_n \in K$ such that $t_i \neq t_j$, for any $i, j \in \{1, \ldots, n\}$, with $i \neq j$. Given $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that

$$\sum_{j=1}^{n} \lambda_j (\alpha_j \delta_{t_j} + \beta_j \delta'_{s_j}) = 0,$$

it must be that $\sum_{j=1}^{n} \lambda_j \alpha_j \delta_{t_j} = -\sum_{j=1}^{n} \beta_j \delta'_{s_j}$. Therefore, taking into account that $\operatorname{span} \nabla_K \cap \operatorname{span} \nabla'_K = \{0\}$, we have $\sum_{j=1}^{n} \lambda_j \alpha_j \delta_{t_j} = 0$. The linear independence of the functionals $\delta_{t_1}, \ldots, \delta_{t_n}$ ensures that $\lambda_j \alpha_j = 0$, and thus

$$\lambda_j = 0, \quad \text{for every } j \in \{1, \ldots, n\}.$$

In the remaining situation ($s_i \neq s_j$, for any $i, j \in \{1, \ldots, n\}$, with $i \neq j$), we proceed analogously. □

Later we shall use the following consequence of the previous result.

Corollary 1.2. *Consider $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{K}$ and $t_1, t_2, s_1, s_2 \in K$ such that*

$$\alpha_1 \delta_{t_1} + \beta_1 \delta'_{s_1} = \alpha_2 \delta_{t_2} + \beta_2 \delta'_{s_2}.$$

Then $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. In particular, $\alpha_1 \neq 0$ if and only if $\alpha_2 \neq 0$. Moreover, in the latter case, it holds also that $t_1 = t_2$. Similarly, $\beta_1 \neq 0$ if and only if $\beta_2 \neq 0$ and, in such a case, $s_1 = s_2$.

Proof. According to the hypothesis, $\alpha_1\delta_{t_1} - \alpha_2\delta_{t_2} = \beta_2\delta'_{s_2} - \beta_1\delta'_{s_1}$ and, taking into account Proposition 1.1, $\alpha_1\delta_{t_1} - \alpha_2\delta_{t_2} = 0 = \beta_2\delta'_{s_2} - \beta_1\delta'_{s_1}$. The first equality implies either $\alpha_1 = \alpha_2 = 0$ or $t_1 = t_2$ and $\alpha_1 = \alpha_2$. The second one only holds if either $\beta_1 = \beta_2 = 0$ or $s_1 = s_2$ and $\beta_1 = \beta_2$. □

It is worth mentioning that if $C^1(K)$ is provided with any one of the aforementioned norms, then, with respect to the weak-$*$ topology of its dual space, the following is true: given two sequences $\{t_n\}$ and $\{s_n\}$ in K and a sequence $\{\beta_n\}$ in $\mathbb{T}$ such that, for certain $t, s \in K$ and $\beta \in \mathbb{T}$, the sequence $\{\delta_{t_n} + \beta_n\delta'_{s_n}\}$ converges to $\delta_t + \beta\delta'_s$, it must be that $\{t_n\} \to t$, $\{s_n\} \to s$ and $\{\beta_n\} \to \beta$.

To check the preceding statement, assume for a moment that the sequences $\{t_n\}$, $\{s_n\}$, and $\{\beta_n\}$ are convergent. Thus let $t', s' \in K$ and $\beta' \in \mathbb{T}$ be their respective limits. Obviously, the sequence $\{\delta_{t_n} + \beta_n\delta'_{s_n}\}$ converges in the weak-$*$ topology to $\delta_{t'} + \beta'\delta'_{s'}$ and consequently $\delta_{t'} + \beta'\delta'_{s'} = \delta_t + \beta\delta'_s$. By virtue of Corollary 1.2, $t' = t$, $s' = s$, and $\beta' = \beta$. It is clear in view of the above that all the convergent subsequences of $\{(t_n, s_n, \beta_n)\}$ have the same limit, namely, (t, s, β). Equivalently, the sequence $\{(t_n, s_n, \beta_n)\}$ converges to (t, s, β).

For later use we state a last basic fact, whose (elementary) proof can be consulted, if desired, in [17, Proposition 3].

Proposition 1.3. *Let I and J be two intervals of $\mathbb{R}$ with nonempty interior, and let $\varphi : J \to I$ be a function such that $x' \circ \varphi$ belongs to $C^1(J)$ for every $x \in C^1(I)$. Then φ is a constant function.*

Finally, let K_1 and K_2 be compact intervals of $\mathbb{R}$. It is evident that the existence of isometries from K_2 into K_1 is equivalent to the condition $l(K_2) \leq l(K_1)$. Furthermore, in this case, there is $c \in \mathbb{R}$ such that either

$$\varphi(t) = t + c, \quad \text{for every } t \in K_2, \qquad \text{or} \qquad \varphi(t) = -t + c, \quad \text{for every } t \in K_2.$$

In particular, if $l(K_2) = l(K_1)$, there are exactly two isometries (in this case surjectives) from K_2 onto K_1. It is also clear that a mapping $\varphi : K_2 \to K_1$ is an isometry if and only if φ is differentiable and $|\varphi'(t)| = 1$, for every $t \in K_2$.

2. The results

As we indicated in the Introduction, our aim is to describe the nice operators between continuously differentiable function spaces endowed with norms of type r.

Let K be a compact interval of $\mathbb{R}$. From this point we will consider the following norm on $C^1(K)$:

$$\|x\| = \|x\|_\infty + \|x'\|_\infty, \quad \text{for every } x \in C^1(K). \tag{2.1}$$

If $X = (C^1(K), \|\cdot\|)$, it is known (see [18]) that the set of extreme points of the closed unit ball of X^* is given by

$$E_{X^*} = \mathbb{T}\nabla_K + \mathbb{T}\nabla'_K.$$

First, we observe that the nice operators between spaces of continuously differentiable functions preserve the constant functions. This is a property previously known for isometric isomorphisms and also for nice operators between $C^1(K)$

spaces provided with norms of type q. The proof is similar to that of Lemma 6 of [17], but it will be included here for the sake of completeness.

Let K_1 and K_2 be compact intervals of $\mathbb{R}$, $X = C^1(K_1)$ and $Y = C^1(K_2)$ endowed with their corresponding type (2.1) norms. Furthermore, consider a nice operator $T : X \to Y$.

Lemma 2.1. *The function Tu_{K_1} is constant. Indeed, there is $\alpha_0 \in \mathbb{T}$ such that*

$$(Tu_{K_1})(t) = \alpha_0, \quad \textit{for every } t \in K_2.$$

Proof. Let $e = Tu_{K_1}$, and pick $t, s \in K_2$ and $\beta \in \mathbb{T}$. Since T is nice, there are $\alpha_1, \beta_1 \in \mathbb{T}$ and $t_1, s_1 \in K_1$ such that $T^*(\delta_t + \beta\delta'_s) = \alpha_1\delta_{t_1} + \beta_1\delta'_{s_1}$. Therefore,

$$\big|e(t) + \beta e'(s)\big| = \big|\alpha_1 u_{K_1}(t_1) + \beta'_1 u'_{K_1}(s_1)\big| = |\alpha_1| = 1. \tag{2.2}$$

In the real case, $|e(t) \pm e'(s)| = 1$, and hence $2e(t) \in \{-2, 0, 2\}$; that is to say, $e(t) \in \{-1, 0, 1\}$, for every $t \in K_2$. It is thus clear that e is a constant function, and in view of (2.2), either $e(t) = 1$ for every $t \in K_2$ or $e(t) = -1$ for every $t \in K_2$.

In the complex case, one can assign to β the values ± 1 and $\pm i$ in order to obtain that

$$\big|e(t)\big|^2 + \big|e'(s)\big|^2 \pm 2\operatorname{Re}\big(e(t)\overline{e'(s)}\big) = 1,$$
$$\big|e(t)\big|^2 + \big|e'(s)\big|^2 \mp 2\operatorname{Im}\big(e(t)\overline{e'(s)}\big) = 1.$$

It follows that $e(t)\overline{e'(s)} = 0$, or, equivalently, $e(t)e'(s) = 0$. Therefore, the images of the functions e and e' are contained in $\{0\} \cup \mathbb{T}$. By virtue of (2.2), $e(K_2) \subset \mathbb{T}$ and, since $ee' = 0$, necessarily $e'(K_2) \subset \{0\}$. □

Assume, for a moment, that $Tu_{K_1} = u_{K_2}$. Then, given $t, s \in K_2$ and $\beta \in \mathbb{T}$, there are $\varphi(t, \beta, s), \psi(t, \beta, s) \in K_1$ and $\eta(t, \beta, s) \in \mathbb{T}$ such that

$$T^*(\delta_t + \beta\delta'_s) = \delta_{\varphi(t,\beta,s)} + \eta(t, \beta, s)\delta'_{\psi(t,\beta,s)}. \tag{2.3}$$

Henceforth, we will use the notation

$$Q = K_2 \times \mathbb{T} \times K_2.$$

Lemma 2.2. *The mappings $\varphi, \psi : Q \to K_1$ and $\eta : Q \to \mathbb{T}$ are continuous.*

Proof. Let $\{(t_n, \beta_n, s_n)\}$ be a sequence in Q, and let (t, β, s) be an element of this last set such that $\{(t_n, \beta_n, s_n)\} \to (t, \beta, s)$. Then, the sequence $\{T^*(\delta_{t_n} + \beta_n\delta'_{s_n})\}$ converges in the weak-$*$ topology to $T^*(\delta_t + \beta\delta'_s)$. That is, the sequence

$$\big\{\delta_{\varphi(t_n,\beta_n,s_n)} + \eta(t_n, \beta_n, s_n)\delta'_{\psi(t_n,\beta_n,s_n)}\big\}$$

converges in such topology to $\delta_{\varphi(t,\beta,s)} + \eta(t, \beta, s)\delta'_{\psi(t,\beta,s)}$. In accordance with the comment subsequent to Corollary 1.2, the sequences $\{\varphi(t_n, \beta_n, s_n)\}$, $\{\eta(t_n, \beta_n, s_n)\}$, and $\{\psi(t_n, \beta_n, s_n)\}$ converge to $\varphi(t, \beta, s)$, $\eta(t, \beta, s)$, and $\psi(t, \beta, s)$, respectively. □

Theorem 2.3. *Let K_1 and K_2 be compact intervals of $\mathbb{R}$, $X = C^1(K_1)$ and $Y = C^1(K_2)$ provided with their respective type (2.1) norms. Let $T \in L(X, Y)$.*

(i) *Suppose that* $l(K_1) < l(K_2)$*. Then* T *is nice if and only if there are* $t_0, s_0 \in K_1$ *and* $\alpha_0, \beta_0 \in \mathbb{T}$ *such that*

$$(Tx)(t) = \alpha_0 x(t_0) + \beta_0 x'(s_0), \quad \textit{for any } x \in X \textit{ and } t \in K_2. \tag{2.4}$$

(ii) *Assume now that* $l(K_1) \geq l(K_2)$*. Then* T *is nice if and only if one of the following two claims holds:*

(a) T *is of the form (2.4);*

(b) *there is a scalar* $\alpha_0 \in \mathbb{T}$ *and an isometry* $\varphi : K_2 \to K_1$ *such that*

$$(Tx)(t) = \alpha_0 x\big(\varphi(t)\big), \quad \textit{for any } x \in X \textit{ and } t \in K_2. \tag{2.5}$$

Proof. Let us first make some preliminary comments.

Obviously the operators described by (2.4) and (2.5) are nice. Therefore, it suffices to show that any nice operator from X into Y can be expressed in one of the two ways mentioned.

From now on, T will be a nice operator. By virtue of Lemma 2.1, there is $\alpha_0 \in \mathbb{T}$ such that $(Tu_{K_1})(t) = \alpha_0$, for every $t \in K_2$. Since $T = \alpha_0(\alpha_0^{-1}T)$ and $\alpha_0^{-1}T$ is a nice operator which maps u_{K_1} to u_{K_2}, also assume from this point that T itself sends u_{K_1} to u_{K_2}. Consequently, T can be expressed as indicated in (2.3) for certain functions $\varphi, \psi : Q \to K_1$ and $\eta : Q \to \mathbb{T}$. To reach the general description of the nice operators it is sufficient to multiply by α_0 the expression of T we are going to obtain under the assumed hypothesis.

To get a clearer proof, we will address separately the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Complex case. We first observe that the functions φ and ψ do not depend on β. To this end, fix t and s in K_2. If some of the functions $\beta \mapsto \varphi(t, \beta, s)$ or $\beta \mapsto \psi(t, \beta, s)$, from $\mathbb{T}$ into K_1, were nonconstant, then the right side of equation (2.3) would contain infinite linearly independent vectors of X^* (see Proposition 1.1). This is not possible since the left side of this equation is included in a two-dimensional subspace of X^*. Therefore, we can write the referred equation in the form

$$T^*(\delta_t) + \beta T^*(\delta'_s) = \delta_{\varphi(t,s)} + \eta(t, \beta, s)\delta'_{\psi(t,s)}, \quad \text{for every } (t, \beta, s) \in Q. \tag{2.6}$$

Define $y = Ti_{K_1}$. According to the above equality,

$$y(t) + \beta y'(s) = \varphi(t, s) + \eta(t, \beta, s).$$

By fixing t and s again we see that there are only two possibilities:

(1) $y'(s) = 0$, for every $s \in K_2$, or

(2) $|y'(s)| = 1$, for every $s \in K_2$.

In the first case, y is constant and obviously

$$y(t) = \varphi(t, s) + \eta(t, \beta, s). \tag{2.7}$$

Consequently, there is a real number c such that

$$c = \operatorname{Im}\big(y(t)\big) = \operatorname{Im}\big(\eta(t, \beta, s)\big).$$

This means that $\eta(t, \beta, s)$ is contained in the intersection of a line and the circumference $\mathbb{T}$. Since $\eta(Q)$ is connected, η is necessarily constant. From (2.7) it also follows that φ is constant. In this way, equation (2.6) is expressed in the form

$$T^*(\delta_t) + \beta T^*(\delta'_s) = \delta_{t_0} + \lambda_0 \delta'_{\psi(t,s)},$$

for appropriate $t_0 \in K_1$ and $\lambda_0 \in \mathbb{T}$. If we now fix t and s, from this latter equality it readily follows that $T^*(\delta'_s) = 0$, for every $s \in K_2$. Therefore,

$$T^*(\delta_t) = \delta_{t_0} + \lambda_0 \delta'_{\psi(t,s)},$$

and consequently ψ does not depend on s (thus, we write $\psi(t)$ instead of $\psi(t,s)$). Nor does it depend on t, as we shall see immediately.

Given $x \in C^1(K_1)$, it is clear that

$$(Tx)(t) = x(t_0) + \lambda_0 x'\big(\psi(t)\big),$$

and consequently $x' \circ \psi$ is continuously differentiable. By Lemma 1.3 the mapping ψ is constant. Thus, for some $s_0 \in K_1$,

$$(Tx)(t) = x(t_0) + \lambda_0 x'(s_0), \quad \text{for any } t \in K_2 \text{ and } x \in X.$$

We turn now to the second alternative: $|y'(s)| = 1$, for every $s \in K_2$. In this case, $y(t) = \varphi(t,s)$ and $\beta y'(s) = \eta(t,\beta,s)$, for every $(t,\beta,s) \in Q$. Therefore, φ depends only on t (which allows us to write $\varphi(t)$ instead of $\varphi(t,s)$) and, since it coincides with y, φ is continuously differentiable on K_2 with $|\varphi'(t)| = |y'(t)| = 1$, for every $t \in K_2$. In consequence, φ is an isometry from K_2 into K_1 and, as a result, $l(K_2) \le l(K_1)$. Furthermore, the equality $y = \varphi$ tells us that y is actually a real function and, hence, there exists $\xi_0 \in \{-1, 1\}$ such that $y'(s) = \xi_0$, for every $s \in K_2$. Thus, equation (2.6) becomes

$$T^*(\delta_t) + \beta T^*(\delta'_s) = \delta_{\varphi(t)} + \xi_0 \beta \delta'_{\psi(t,s)}.$$

By evaluating such functionals at the point $x = (i_{K_1})^2$, we find that

$$(Tx)(t) + \beta (Tx)'(s) = \varphi(t)^2 + 2\xi_0 \beta \psi(t,s)$$

and, leaving fixed β and s, we note that the function $t \mapsto \psi(t,s)$ (henceforth denoted by ψ_s) is continuously differentiable on K_2 and

$$(Tx)'(t) = 2\xi_0 \varphi(t) + 2\xi_0 \beta \psi'_s(t).$$

As the above is valid for all $\beta \in \mathbb{T}$, necessarily $\psi'_s = 0$ and consequently ψ depends only on the variable s. Then, for every $x \in X$,

$$(Tx)(t) + \beta (Tx)'(s) = x\big(\varphi(t)\big) + \xi_0 \beta x'\big(\psi(s)\big),$$

and it follows easily that $(Tx)(t) = x(\varphi(t))$, for every $t \in K_2$.

This completes the proof if $\mathbb{K} = \mathbb{C}$.

As we have just seen in the complex case, each of the functions φ, η, and ψ appearing in (2.3) depends on at most one of the variables involved—to be more precise, on t, β, and s, respectively. The same is true in the real case, and the verification of this fact will serve as a guideline for the rest of the proof.

Given $t, s \in K_2$ it is easy to check that $\|\delta_t - \delta_s\| \le |t - s|$ and, if $t \ne s$, $\|\delta'_t - \delta'_s\| = 2$.

Real case. Obviously, given $\beta \in \mathbb{T}$, the mapping $(t,s) \mapsto \eta(t,\beta,s)$, from $K_2 \times K_2$ into $\mathbb{T}$, is constant and consequently η depends only on β. Thus, we will write $\eta(\beta)$ instead of $\eta(t,\beta,s)$.

Fix $\beta_0 \in \mathbb{T}$, $\omega_0 \in K_2$, and let $\varphi_0, \psi_0 : K_2 \to K_1$ be the mappings given by

$$\varphi_0(t) = \varphi(t, \beta_0, \omega_0), \qquad \psi_0(t) = \psi(t, \beta_0, \omega_0), \quad \text{for every } t \in K_2.$$

First, assume that φ_0 is constant on some open interval contained in K_2. Then, there are $\theta_0 \in K_2$, $t_0 \in K_1$, and an open interval V such that $\theta_0 \in V \subset K_2$ and $\varphi_0(t) = t_0$, for every $t \in V$. By (2.3), for any $x \in X$,

$$(Tx)(t) + \beta_0 (Tx)'(\omega_0) = x(t_0) + \eta(\beta_0) x'\big(\psi_0(t)\big), \quad \text{for every } t \in V.$$

In particular, the function $t \mapsto x'(\psi_0(t))$, from K_2 into $\mathbb{R}$, is continuously differentiable on V. This implies, by virtue of Lemma 1.3, the existence of a point $s_0 \in K_1$ such that $\psi_0(t) = s_0$, for every $t \in V$. Therefore, given $x \in X$,

$$(Tx)(t) + \beta_0 (Tx)'(\omega_0) = x(t_0) + \eta(\beta_0) x'(s_0), \quad \text{for every } t \in V.$$

Hence, the function Tx is constant on V and $T^*(\delta'_s) = 0$, for each $s \in V$. In this way, equation (2.3) ensures that

$$T^*(\delta_t) = \delta_{\varphi(t,\beta,s)} + \eta(\beta)\delta'_{\psi(t,\beta,s)}, \quad \text{for every } (t,\beta,s) \in K_2 \times \mathbb{T} \times V,$$

and thus $T^*(\delta_t) = \delta_{\varphi(t,\beta_0,\theta_0)} + \eta(\beta_0)\delta'_{\psi(t,\beta_0,\theta_0)}$, for every $t \in K_2$. Accordingly, given $(t,\beta,s) \in Q$,

$$\delta_{\varphi(t,\beta_0,\theta_0)} + \eta(\beta_0)\delta'_{\psi(t,\beta_0,\theta_0)} + \beta T^*(\delta'_s) = T^*(\delta_t + \beta\delta'_s).$$

In consequence, $\delta_{\varphi(t,\beta_0,\theta_0)} + \eta(\beta_0)\delta'_{\psi(t,\beta_0,\theta_0)} \pm T^*(\delta'_s) \in B_{X^*}$ and, since

$$\delta_{\varphi(t,\beta_0,\theta_0)} + \eta(\beta_0)\delta'_{\psi(t,\beta_0,\theta_0)} \in E_{X^*},$$

it holds that $T^*(\delta'_s) = 0$, for each $s \in K_2$. Therefore,

$$T^*(\delta_t) = \delta_{\varphi(t,\beta,s)} + \eta(\beta)\delta'_{\psi(t,\beta,s)}, \quad \text{for every } (t,\beta,s) \in Q.$$

By Corollary 1.2, the mappings φ, η, and ψ are independent of the variables β and s. In particular, η is constant. On the other hand, the condition $T^*(\delta'_s) = 0$, for every $s \in K_2$, is equivalent to saying that Tx is a constant function for every $x \in X$. Consider $x_1 = i_{K_1}$ and $x_2 = x_1^2$. Given $(t,\beta,s) \in Q$,

$$(Tx_1)(t) = \varphi(t,\beta,s) + \eta(\beta), \qquad (Tx_2)(t) = \varphi(t,\beta,s)^2 + 2\eta(\beta)\psi(t,\beta,s).$$

The first equality implies that φ is constant and, according to the second one, the same is true for ψ. Therefore, there is $\lambda_0 \in \mathbb{T}$ such that

$$T^*\delta_t = \delta_{t_0} + \lambda_0\delta'_{s_0}.$$

We conclude that $(Tx)(t) = x(t_0) + \lambda_0 x'(s_0)$, for any $x \in X$ and $t \in K_2$.

Now suppose that φ_0 is not constant on any open interval contained in K_2, and let θ_0 be an arbitrary point of K_2.

Evidently, there is a sequence $\{\theta_n\}$ in K_2 such that $\{\theta_n\} \to \theta_0$ and $\{\varphi_0(\theta_n)\}$ is strictly monotonic. Pick $(\beta,s) \in \mathbb{T} \times K_2$ and define, for every integer $n \geq 0$,

$$\begin{aligned} u_n^* &= \delta_{\theta_n} + \beta_0\delta'_{\omega_0}, & v_n^* &= \delta_{\theta_n} + \beta\delta'_s, \\ \xi_n &= \varphi(\theta_n,\beta_0,\omega_0), & \xi'_n &= \varphi(\theta_n,\beta,s), \\ \tau_n &= \psi(\theta_n,\beta_0,\omega_0), & \tau'_n &= \psi(\theta_n,\beta,s). \end{aligned}$$

Obviously, $\{\xi_n\} \to \xi_0$ and $\{\xi'_n\} \to \xi'_0$. Moreover,

$$\begin{aligned} T^*u_n^* &= \delta_{\xi_n} + \eta(\beta_0)\delta'_{\tau_n} \qquad \text{and} \\ T^*v_n^* &= \delta_{\xi'_n} + \eta(\beta)\delta'_{\tau'_n}. \end{aligned}$$

Since $\{\|u_n^* - u_0^*\|\} \to 0$ and $\{\|v_n^* - v_0^*\|\} \to 0$, the sequences $\{T^* u_n^*\}$ and $\{T^* v_n^*\}$ converge in norm to $T^* u_0^*$ and $T^* v_0^*$, respectively. Thus, for every natural number n large enough, each of the following real numbers

$$|\xi_n - \xi_0|, \qquad |\xi_n' - \xi_0'|, \qquad \|T^* u_n^* - T^* u_0^*\|, \qquad \text{and} \qquad \|T^* v_n^* - T^* v_0^*\|$$

is less than $\frac{1}{2}$. It follows that

$$\begin{aligned} \|\delta'_{\tau_n} - \delta'_{\tau_0}\| &= \big\|\big(\delta_{\xi_n} + \eta(\beta_0)\delta'_{\tau_n}\big) - \big(\delta_{\xi_0} + \eta(\beta_0)\delta'_{\tau_0}\big) + \delta_{\xi_0} - \delta_{\xi_n}\big\| \\ &\le \|T^* u_n^* - T^* u_0^*\| + \|\delta_{\xi_n} - \delta_{\xi_0}\| \\ &< 1. \end{aligned}$$

Analogously, $\|\delta'_{\tau_n'} - \delta'_{\tau_0'}\| < 1$ and, in consequence, $\tau_n = \tau_0$ and $\tau_n' = \tau_0'$. Taking into account that

$$T^*(u_0^* - v_0^*) = T^*(u_n^* - v_n^*) = \delta_{\xi_n} + \eta(\beta_0)\delta'_{\tau_n} - \delta_{\xi_n'} - \eta(\beta)\delta'_{\tau_n'}, \quad \text{for every } n \in \mathbb{N},$$

the sequence $\{\delta_{\xi_n} - \delta_{\xi_n'}\}$ is constant for n large enough. Assume, without loss of generality, that $\{\delta_{\xi_n} - \delta_{\xi_n'}\}$ is constant. In this manner, $\xi_n - \xi_n' = a$ and $(\xi_n)^2 - (\xi_n')^2 = b$, for each $n \in \mathbb{N}$ and suitable real numbers a and b. If $a \neq 0$, it would follow that

$$\xi_n = \frac{a + \xi_n + \xi_n'}{2} = \frac{a^2 + b}{2a}, \quad \text{for every } n \in \mathbb{N},$$

contrary to the strict monotony of $\{\xi_n\}$. Therefore, $a = 0$ and $\xi_n = \xi_n'$, for each natural number n. Thus,

$$\varphi(\theta_0, \beta_0, \omega_0) = \lim \xi_n = \lim \xi_n' = \varphi(\theta_0, \beta, s).$$

We just proved that φ does not depend on β or s. From this moment, we will write $\varphi(t)$ instead of $\varphi(t, \beta, s)$. In this way, equation (2.3) can be expressed in the form

$$T^*\delta_t + \beta T^*\delta_s' = \delta_{\varphi(t)} + \eta(\beta)\delta'_{\psi(t,\beta,s)}, \quad \text{for every } (t, \beta, s) \in Q.$$

Consequently,

$$(Tx)(t) + \beta(Tx)'(s) = x\big(\varphi(t)\big) + \eta(\beta)x'\big(\psi(t, \beta, s)\big), \tag{2.8}$$

for any $(t, \beta, s) \in Q$ and $x \in X$. From the above equality, it can be deduced, by fixing the variables β and s, that φ is continuously differentiable. It is thus clear that, for any $x \in X$, the function $t \mapsto x'(\psi(t, \beta, s))$, from K_2 into $\mathbb{R}$, is also continuously differentiable. According to Lemma 1.3, the mapping $t \mapsto \psi(t, \beta, s)$ is constant. Therefore, ψ does not depend on t, and we can write $\psi(\beta, s)$ instead of $\psi(t, \beta, s)$:

$$(Tx)(t) + \beta(Tx)'(s) = x\big(\varphi(t)\big) + \eta(\beta)x'\big(\psi(\beta, s)\big).$$

From this equation we deduce, again by fixing β and s, that

$$(Tx)'(t) = \varphi'(t)x'\big(\varphi(t)\big), \quad \forall t \in K_2, \forall x \in X.$$

Equation (2.8) can thus be expressed as

$$(Tx)(t) + \beta\varphi'(s)x'\big(\varphi(s)\big) = x\big(\varphi(t)\big) + \eta(\beta)x'\big(\psi(\beta, s)\big). \tag{2.9}$$

As a result, $(Ti_{K_1})(t) + \beta\varphi'(s) = \varphi(t) + \eta(\beta)$ and, by subtracting the equations corresponding to $\beta = 1$ and $\beta = -1$,

$$\varphi'(s) = \frac{\eta(1) - \eta(-1)}{2}, \quad \text{for every } s \in K_2.$$

Under the hypothesis previously assumed, φ is not constant and therefore $\eta(1) \neq \eta(-1)$. This condition is equivalent to saying that $\eta(-1) = -\eta(1)$. Consequently, $\varphi'(s) = \eta(1)$, for every $s \in K_2$, and φ is an isometry from K_2 into K_1. It is also clear that $\eta(\beta) = \beta\eta(1)$, for every $\beta \in \mathbb{T}$. These facts and equation (2.9) ensure that

$$T^*\delta_t + \beta\eta(1)\delta'_{\varphi(s)} = \delta_{\varphi(t)} + \beta\eta(1)\delta'_{\psi(\beta,s)}, \quad \text{for every } (t, \beta, s) \in Q. \tag{2.10}$$

Fix t and s in K_2 and give to β the values 1 and -1. The subtraction of the resulting equations yields

$$2\delta'_{\varphi(s)} = \delta'_{\psi(1,s)} + \delta'_{\psi(-1,s)}$$

and, according to Proposition 1.1, the real numbers $\varphi(s)$, $\psi(1, s)$, and $\psi(-1, s)$ cannot be pairwise different. It follows that, indeed, $\varphi(s) = \psi(1, s) = \psi(-1, s)$ (and, in particular, ψ does not depend on β). Equality (2.10) then allows us to conclude that $T^*\delta_t = \delta_{\varphi(t)}$, for every $t \in K_2$. In other words,

$$(Tx)(t) = x\big(\varphi(t)\big), \quad \text{for any } t \in K_2 \text{ and } x \in X. \qquad \square$$

In particular, the previous result describes the linear isometries from X onto Y.

Corollary 2.4. *Let X and Y be as in the above theorem. Then X and Y are isometrically isomorphic if and only if $l(K_1) = l(K_2)$. Furthermore, in this case, an operator $T : X \to Y$ is an isometric isomorphism if and only if there is $\alpha_0 \in \mathbb{T}$ and an isometric bijection $\varphi : K_2 \to K_1$ such that*

$$(Tx)(t) = \alpha_0 x\big(\varphi(t)\big), \quad \textit{for any } t \in K_2 \textit{ and } x \in X.$$

This last result was known before, as it can be seen in [11, Example 4].

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