mathematics

Article

Faces and Renormings of ℓ_1

Juan Carlos Navarro-Pascual * and **Antonio Zarauz-Moreno**

Departamento de Matemáticas, Facultad de Ciencias Experimentales, Universidad de Almería, 04120 Almería, Spain
* Correspondence: jcnav@ual.es

Abstract: The faces of the unit ball of a finite-dimensional Banach space are automatically closed. The situation is different in the infinite-dimensional case. In fact, under this last condition, the closure of a face may not be a face. In this paper, we discuss these issues in an expository style. In order to illustrate the described situation we consider an equivalent renorming of the Banach space ℓ_1.

Keywords: convex set; face of a convex set; equivalent renorming of a Banach space

MSC: 46B20; 46B45

check for updates

Citation: Navarro-Pascual, J.C.; Zarauz-Moreno, A. Faces and Renormings of ℓ_1. *Mathematics* **2023**, *11*, 193. https://doi.org/10.3390/math11010193

Academic Editor: Enrique Alfonso Sánchez Pérez

Received: 7 December 2022
Revised: 23 December 2022
Accepted: 28 December 2022
Published: 29 December 2022

1. Introduction

Throughout this note the symbol $\mathbb{K}$ is used to indistinctly denote the field $\mathbb{R}$ of the real numbers or the field $\mathbb{C}$ of the complex numbers. We also incorporate the usual notations $\mathbb{D} = \{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$ and $\mathbb{T} = \{\alpha \in \mathbb{K} : |\alpha| = 1\}$.

Let X be a vector space over $\mathbb{K}$ and A a nonempty subset of X. The real affine subspace generated by A is written as $\mathcal{V}_A$:

$$\mathcal{V}_A = \left\{ \sum_{j=1}^{n} t_j a_j : n \in \mathbb{N},\ t_j \in \mathbb{R},\ a_j \in A \text{ for all } j \in \{1,\dots,n\},\ \sum_{j=1}^{n} t_j = 1 \right\}. \tag{1}$$

Suppose A is convex. A nonempty, convex subset F of A is said to be a face of A if the following condition is satisfied:

$$a, b \in A,\ t \in]0,1[\,,\ (1-t)a + tb \in F \Rightarrow a, b \in F. \tag{2}$$

A itself is a face of A. In addition, faces of A reduced to a single element, if any, are called extreme points of A.

The dimension of (the nonempty, convex set) A is the dimension (over $\mathbb{R}$) of $\mathcal{V}_A$, which coincides with the dimension of the real subspace attached to $\mathcal{V}_A$; i.e., the only real subspace M of X such that $\mathcal{V}_A = a + M$ for any $a \in A$.

Given a nonempty subset A of a topological vector space X, the intrinsic boundary of A is defined as the boundary of A relative to $\mathcal{V}_A$. Following the same argument, one can define the intrinsic interior of A. On the other hand, the symbols $\mathrm{co}(A)$ and $\overline{\mathrm{co}}(A)$ represent the convex hull and the closed convex hull of the set A, respectively.

The notion of face is purely algebraic. However, if the ambient space is endowed with a vector topology, we can adequately express some of its properties. In the following result we cite two of them, incorporating their proof for the sake of completeness. Additional information on the concepts covered in Propositions 1 and 2 can be found in [1].

Proposition 1. *Let X be a Hausdorff topological vector space and A a nonempty convex subset of X.*

(i) Every face F of A, with $F \neq A$, is contained in the intrinsic boundary of A.

(ii) *If A is closed, so is every finite-dimensional face of A.*

Proof. (i) Consider a point $x_0 \in A \backslash F$. If a is an element of the intrinsic interior of A, then there exists $\delta \in \mathbb{R}^+$ such that $a \pm \delta(x_0 - a) \in A$. Defining $a_0 = a - \delta(x_0 - a)$, it is readily seen that $a = \frac{1}{1+\delta} a_0 + \frac{\delta}{1+\delta} x_0$, hence $a \notin F$.

(ii) Let F be a finite-dimensional face of A. Since the statement is clearly true if F consists of a single point we can assume that the dimension n of F is non-zero. Thus, there exist $a_0, a_1, \ldots, a_n \in F$ (affinely independent) such that $\mathcal{V}_F$, the real affine subspace generated by F, is given by

$$\mathcal{V}_F = \{t_0 a_0 + t_1 a_1 + \cdots + t_n a_n \colon t_0, t_1, \ldots, t_n \in \mathbb{R},\ t_0 + t_1 + \cdots + t_n = 1\}. \tag{3}$$

It is clear that $\mathcal{V}_F$ is closed and $F \subset \mathcal{V}_F \cap A$, hence it only lefts to prove that $\mathcal{V}_F \cap A \subset F$. To this end, fix an element x of $V_F \cap A$. Then there exist real numbers $t_0, t_1, \ldots, t_n$ with $t_0 + t_1 + \cdots + t_n = 1$ and

$$x = t_0 a_0 + t_1 a_1 + \cdots + t_n a_n. \tag{4}$$

The point $a = \frac{1}{n+1}(a_0 + a_1 + \cdots + a_n)$ belongs to F and we can choose $s \in]0,1[$ small enough that $\frac{1-s}{n+1} + st_j > 0$ for every $j \in \{0, 1, \ldots, n\}$. Taking into account that

$$(1-s)a + sx = \left(\tfrac{1-s}{n+1} + st_0\right)a_0 + \left(\tfrac{1-s}{n+1} + st_1\right)a_1 + \cdots + \left(\tfrac{1-s}{n+1} + st_n\right)a_n, \tag{5}$$

and $\sum_{j=0}^{n}\left(\frac{1-s}{n+1} + st_j\right) = 1$, we see that $(1-s)a + sx \in F$. To conclude that $x \in F$, it suffices to remember that $a, x \in A$ and that F is a face of A. □

The extreme points of convex sets play an important role in functional analysis and have a significant repercussion in other areas. A good source of information on this matter is [2]. The more general notion of face of a convex set enjoys great presence in the framework of the geometry and structure of Banach spaces, as can be seen in [3]. Mention should also be made of the introduction in [4] of the so-called facial topology on the set of extreme points of a convex, compact set in a Hausdorff locally convex space. Some recent applications of this topology can be seen in [5,6].

It is also worth reviewing some results about the interaction between convexity and topology.

Proposition 2. *Let A be a bounded, closed, convex set in a Hausdorff topological vector space X, and B a compact, convex subset of X. Then,* $\operatorname{co}(A \cup B)$ *is closed. Furthermore, if A is also compact, then so is* $\operatorname{co}(A \cup B)$.

Proof. Since A and B are convex,

$$\operatorname{co}(A \cup B) = \{(1-t)a + tb : t \in [0,1], a \in A, b \in B\}. \tag{6}$$

Let $\{w_\lambda\}_{\lambda \in \Lambda}$ be a convergent net of elements in $\operatorname{co}(A \cup B)$, and w its corresponding limit. Then, for every $\lambda \in \Lambda$, there exist $t_\lambda \in [0,1]$, $a_\lambda \in A$, and $b_\lambda \in B$ such that

$$w_\lambda = (1-t_\lambda)a_\lambda + t_\lambda b_\lambda. \tag{7}$$

The compactness of the sets B and $[0,1]$ allows us to assume, considering subnets if necessary, that $\{b_\lambda\}_{\lambda \in \Lambda}$ and $\{t_\lambda\}_{\lambda \in \Lambda}$ converge to a point $b \in B$ and a scalar $t \in [0,1]$, respectively.

If $t = 1$ the net $\{(1-t_\lambda)a_\lambda\}_{\lambda \in \Lambda}$ converges to zero and, therefore, $\{w_\lambda\}_{\lambda \in \Lambda}$ converges to b. Thus, $w = b$ and it can be concluded that $w \in \operatorname{co}(A \cup B)$ (in fact, $w \in B$).

Suppose now that $t < 1$. Using subnets if necessary, it can be assumed that $t_\lambda < 1$ for every $\lambda \in \Lambda$. That way, $a_\lambda = \frac{w_\lambda - t_\lambda b_\lambda}{1-t_\lambda}$ for each $\lambda \in \Lambda$. Hence, the net $\{a_\lambda\}_{\lambda \in \Lambda}$ converges to $\frac{w-tb}{1-t}$ and, taking into account that A is closed, the point $a = \frac{w-tb}{1-t}$ belongs to A. Finally, $w = (1-t)a + tb$ and consequently, $w \in \mathrm{co}(A \cup B)$.

Regarding the last assertion, if A is also compact, so is the set $\mathrm{co}(A \cup B)$, as from (6) it can be seen as the continuous image of a compact set. □

The convex hull of every compact subset of a finite-dimensional topological vector space is automatically compact. In infinite-dimensional spaces, the same cannot be said. However, if the closure of the convex hull is considered, we find positive results. It is well known, without going any further, that the closed convex hull of every compact subset of a Banach space is also a compact set.

Let X be a normed space. The symbols B_X and S_X stand for the unit ball and unit sphere of X, respectively:

$$B_X = \{x \in X : \|x\| \leq 1\}, \;\; S_X = \{x \in X : \|x\| \leq 1\}. \tag{8}$$

According to Proposition 1, every proper face of B_X is contained in S_X, which applies to the potential extreme points of B_X in particular. Plus, if X is a finite dimensional normed space, every face of B_X is closed.

The last statement is not always true when it comes to infinite dimensional normed spaces; in fact, the closure of a face might not even be a face. In the forthcoming section, an example is provided based on a renorming of the space ℓ_1 of absolutely summable sequences of (real or complex) scalars.

As it is well known, two norms $\|\cdot\|$ and $\|\cdot\|_0$ on the same vector space X are said to be equivalent if they induce the same topology on X. This occurs if, and only if, there exist positive real numbers α and β such that

$$\alpha\|x\|_0 \leq \|x\| \leq \beta\|x\|_0 \text{ for all } x \in X. \tag{9}$$

Therefore, two equivalent norms also share uniform properties and, in particular, one of such norms is complete if and only if the other is. The properties not common to two equivalent norms are of a geometric nature.

Given a normed space X, any other norm in the underlying vector space is called a renorming of X. In Banach space literature, this concept usually includes the requirement that the new norm is equivalent to the original one.

For an enlightening discussion of renormings in Banach spaces the reader is referred to [7,8] and to the new monograph [9].

2. Main Results

The unit ball of certain Banach spaces contain faces which closure is not a face. We illustrate this fact by equivalently renorming ℓ_1.

Henceforth, $\{e_n\}$ will denote the canonical basis of ℓ_1. Then, for $n, k \in \mathbb{N}$,

$$e_n(k) = 1, \text{ if } k = n, \;\; e_n(k) = 0, \text{ if } k \neq n. \tag{10}$$

The sequences $\{u_n\}$ and $\{v_n\}$ of vectors in ℓ_1 given by

$$u_1 = e_1 + e_2, \; u_2 = e_1 - e_2, \; u_n = e_n \text{ for every } n \geq 3. \tag{11}$$

$$v_n = \tfrac{n+1}{n+2}\, e_1 + \tfrac{1}{n+1}\, e_{n+2} \text{ for every } n \in \mathbb{N}. \tag{12}$$

will also be considered. In addition, we define $v_0 = e_1$ and

$$K = \{\alpha v_n : \alpha \in \mathbb{T}, n \in \mathbb{N} \cup \{0\}\}. \tag{13}$$

About the canonical basis $\{e_n\}$ of ℓ_1, only the following elementary fact will be necessary: the unique representation of any vector $x \in \ell_1$ in such a basis is given by $x = \sum_{n=1}^{\infty} x(n)e_n$. We must mention, however, that this basis plays a fundamental role in numerous works related to the space ℓ_1. An outstanding exponent of this is the study of the cone positive of ℓ_1 which can be seen in [10] and references therein.

The most interesting set in this section is the following:

$$B_0 = \overline{\text{co}}\big(B_{\ell_1} \cup K \cup \mathbb{T}u_1 \cup \mathbb{T}u_2\big). \tag{14}$$

As it can be readily seen, $B_{\ell_1} \subset B_0 \subset 2B_{\ell_1}$ and hence the set B_0 is absorbing. Moreover, B_0 is convex and radially compact (for each $x \in \ell_1 \backslash \{0\}$ the set $\{t \in \mathbb{R} : t \geq 0$ and $tx \in B_0\}$ is compact). As a consequence, Minkowski's functional, $\|\cdot\|_0$, of B_0 is a norm in ℓ_1 which unit ball is exactly B_0. Using the previous chain of inclusions we can see that both norms are equivalent:

$$\|x\|_0 \leq \|x\|_1 \leq 2\|x\|_0 \text{ for all } x \in \ell_1. \tag{15}$$

To reach our goal, an appropriate description of B_0 will be required. To that end, the set $\overline{\text{co}}(K)$ (a subset of B_0) will be studied in first place.

Lemma 1. *Let $x \in \ell_1$. The following three statements are equivalent:*

(i) $x \in \overline{\text{co}}(K)$.

(ii) $x(2) = 0$ *and the following series are convergent with*

$$\left| x(1) - \sum_{k=1}^{\infty} \tfrac{(k+1)^2}{k+2}\, x(k+2) \right| + \sum_{k=1}^{\infty} (k+1)|x(k+2)| \leq 1. \tag{16}$$

(iii) *For every $n \in \mathbb{N} \cup \{0\}$, there exists a scalar r_n such that*

$$\sum_{n=0}^{\infty} |r_n| \leq 1 \text{ and } x = \sum_{n=0}^{\infty} r_n v_n. \tag{17}$$

Proof. The set M given by the elements x of ℓ_1 such that the series $\sum(k+1)|x(k+2)|$ converges is a (dense but not total) subspace of ℓ_1. For each $x \in M$, the series

$$\sum \tfrac{(k+1)^2}{k+2}\, x(k+2) \tag{18}$$

is (absolutely) convergent and, as it can be seen, the linear maps

$$x \mapsto \sum \tfrac{(k+1)^2}{k+2}\, x(k+2) \text{ and } x \mapsto \sum (k+1)\, x(k+2)\,, \text{ from } M \text{ to } \mathbb{R}, \tag{19}$$

are not continuous. It is clear that the set A containing the elements x of ℓ_1 for which the statement (ii) holds is contained in M. It will be shown that A is closed in ℓ_1. A relevant property to achieve the equivalence of the first two statements ($A = \overline{\text{co}}(K)$).

In light of the aforementioned observations regarding M and its associated functionals, suppose that $\{x_n\}$ is a sequence in A converging to $x_0 \in \ell_1$ and let n, m be natural numbers. Then, $x_n(2) = 0$ and

$$\begin{aligned}
& \left|x_n(1) - \sum_{k=1}^{m} \tfrac{(k+1)^2}{k+2}\, x_n(k+2)\right| + \sum_{k=1}^{m} (k+1)|x_n(k+2)| \\
= & \left|x_n(1) - \sum_{k=1}^{\infty} \tfrac{(k+1)^2}{k+2}\, x_n(k+2) + \sum_{k=m+1}^{\infty} \tfrac{(k+1)^2}{k+2}\, x_n(k+2)\right| \\
& \qquad + \sum_{k=1}^{m} (k+1)|x_n(k+2)| \\
\leq & \left|x_n(1) - \sum_{k=1}^{\infty} \tfrac{(k+1)^2}{k+2}\, x_n(k+2)\right| \\
& \quad + \sum_{k=m+1}^{\infty} \tfrac{(k+1)^2}{k+2} |x_n(k+2)| + \sum_{k=1}^{m} (k+1)|x_n(k+2)| \\
\leq & \left|x_n(1) - \sum_{k=1}^{\infty} \tfrac{(k+1)^2}{k+2}\, x_n(k+2)\right| + \sum_{k=1}^{\infty} (k+1)|x_n(k+2)| \\
\leq & \ 1.
\end{aligned} \tag{20}$$

Taking limits when $n \to \infty$ in the previous inequality,

$$\left|x_0(1) - \sum_{k=1}^{m} \frac{(k+1)^2}{k+2}\, x_0(k+2)\right| + \sum_{k=1}^{m} (k+1)|x_0(k+2)| \leq 1. \tag{21}$$

Since this last inequality holds for any $m \in \mathbb{N}$, we get that $x_0 \in M$. On the other hand, the sequence $\{x_n(2)\}$ converges to zero (it is, indeed, the null sequence). Thus, one can assume that $x_0(2) = 0$. By taking limits in (21), with $m \to \infty$,

$$\left|x_0(1) - \sum_{k=1}^{\infty} \frac{(k+1)^2}{k+2}\, x_0(k+2)\right| + \sum_{k=1}^{\infty} (k+1)|x_0(k+2)| \leq 1. \tag{22}$$

Therefore, $x_0 \in A$ and the latter is a closed set.

(i) $\Rightarrow$ (ii). It is all about testing the inclusion $\overline{\mathrm{co}}(K) \subset A$ and, since A is closed, it can be reduced to $\mathrm{co}(K) \subset A$. To that purpose, given $x \in \mathrm{co}(K)$, we can find a natural number m and scalars $r_0, r_1, \ldots, r_m$, satisfying $\sum_{k=0}^{m} |r_k| = 1$ and $x = \sum_{k=0}^{m} r_k v_k$. Then,

$$\begin{aligned}
x &= r_0 e_1 + \sum_{k=1}^{m} r_k \left(\tfrac{k+1}{k+2}\, e_1 + \tfrac{1}{k+1}\, e_{k+2} \right) \\
&= \left(r_0 + \sum_{k=1}^{m} r_k \tfrac{k+1}{k+2} \right) e_1 + \sum_{k=1}^{m} \tfrac{r_k}{k+1}\, e_{k+2}.
\end{aligned} \tag{23}$$

From this equality, we get

$$x(1) = r_0 + \sum_{k=1}^{m} r_k \tfrac{k+1}{k+2}, \;\; x(2) = 0, \;\; x(k+2) = \tfrac{r_k}{k+1} \text{ for all } k \in \{1, \ldots, m\}, \tag{24}$$

$$x(k) = 0 \text{ for every } k > m+2. \tag{25}$$

The convergence of the series $\sum (k+1)|x(k+2)|$ (and hence the convergence of $\sum \frac{(k+1)^2}{k+2}\, x(k+2)$) is clear. Furthermore,

$$\begin{aligned}
& \left|x(1) - \sum_{k=1}^{\infty} \tfrac{(k+1)^2}{k+2}\, x(k+2)\right| + \sum_{k=1}^{\infty} (k+1)|x(k+2)| \\
= & \left|r_0 + \sum_{k=1}^{m} r_k \tfrac{k+1}{k+2} - \sum_{k=1}^{m} \tfrac{(k+1)^2}{k+2} \tfrac{r_k}{k+1}\right| + \sum_{k=1}^{m} (k+1)\left|\tfrac{r_k}{k+1}\right| = \sum_{k=0}^{m} |r_k| = 1.
\end{aligned} \tag{26}$$

Therefore, $x \in A$.

(ii) $\Rightarrow$ (iii). Define

$$r_0 = x(1) - \sum_{k=1}^{\infty} \frac{(k+1)^2}{k+2}\, x(k+2), \;\; r_k = (k+1)\, x(k+2) \;\text{ for all } k \in \mathbb{N}. \tag{27}$$

According to the hypothesis, $\sum_{k=0}^{\infty}|r_k| \leq 1$. On the other hand,

$$\begin{aligned} \sum_{k=0}^{\infty} r_k v_k &= \left(r_0 + \sum_{k=1}^{\infty} r_k \frac{k+1}{k+2}\right) e_1 + \sum_{k=1}^{\infty} \frac{r_k}{k+1}\, e_{k+2} \\ &= x(1)\, e_1 + x(2)\, e_2 + \sum_{k=1}^{\infty} x(k+2)\, e_{k+2} = \sum_{k=1}^{\infty} x(k)\, e_k = x, \end{aligned} \tag{28}$$

where it has been used that $x(2) = 0$.

(iii) $\Rightarrow$ (i). For each natural number n, let $y_n = \sum_{k=0}^{n} r_k v_k$. Given $k \in \mathbb{N} \cup \{0\}$, put $r_k = |r_k|\xi_k$, with $\xi_k \in \mathbb{T}$. Taking into account that

$$\begin{aligned} y_n &= \sum_{k=0}^{n} |r_k|\xi_k v_k \\ &= \frac{1-\sum_{k=0}^{n}|r_k|}{2}(-\xi_0\, v_0) + \left(|r_0| + \frac{1-\sum_{k=0}^{n}|r_k|}{2}\right)\xi_0\, v_0 + \sum_{k=1}^{n} |r_k|\xi_k v_k \end{aligned} \tag{29}$$

it can be ensured that $y_n \in \mathrm{co}(K)$. Since $\|y_n - x\|_1 \to x$, we conclude that $x \in \overline{\mathrm{co}}(K)$. □

The previously announced and still pending description of the set

$$B_0 = \overline{\mathrm{co}}\big(B_{\ell_1} \cup K \cup \mathbb{T}u_1 \cup \mathbb{T}u_2\big) \tag{30}$$

begins with the following considerations:

The set K is compact, and so is its closed convex hull. On the other hand,

$$\mathrm{co}(\mathbb{T}u_1 \cup \mathbb{T}u_2) = \mathrm{co}(\mathbb{D}u_1 \cup \mathbb{D}u_2) \tag{31}$$

and, by the last part of Proposition 2, the set $C = \mathrm{co}(\mathbb{T}u_1 \cup \mathbb{T}u_2)$ is also compact. For the same reason, the set

$$B = \mathrm{co}(\overline{\mathrm{co}}(K) \cup C) \tag{32}$$

is compact (and convex). Furthermore, from the first part of the already mentioned proposition, the set $\mathrm{co}\big(B_{\ell_1} \cup B\big)$ is closed. Since $\overline{\mathrm{co}}(K)$, C and B_{ℓ_1} are contained in B_0, so is $\mathrm{co}\big(B_{\ell_1} \cup B\big)$, which provides the inclusion $\mathrm{co}\big(B_{\ell_1} \cup B\big) \subset B_0$.

To get the other inclusion, first we notice that $B_{\ell_1} \cup K \cup \mathbb{T}u_1 \cup \mathbb{T}u_2 \subset B_{\ell_1} \cup B$, and hence $B_0 \subset \mathrm{co}\big(B_{\ell_1} \cup B\big)$, since the set $\mathrm{co}\big(B_{\ell_1} \cup B\big)$ is closed (and convex). It has just been proved that

$$B_0 = \mathrm{co}\big(B_{\ell_1} \cup B\big). \tag{33}$$

Lemma 2. *Given $x_0 \in B_0$, there exist sequences of scalars $\{\alpha_n\}$ and $\{\beta_n\}$ such that*

$$x_0 = \sum_{n=1}^{\infty} \alpha_n u_n + \sum_{n=1}^{\infty} \beta_n v_n \;\; \textit{and} \;\; \sum_{n=1}^{\infty} |\alpha_n| + \sum_{n=1}^{\infty} |\beta_n| \leq 1, \tag{34}$$

where $\{u_n\}$ and $\{v_n\}$ are the sequences defined in (11) and (12).

Proof. Equality (33) and definitions it requires will be taken into account to complete the proof. By virtue of the convexity of the sets B_{ℓ_1} and B, one can find $y \in B_{\ell_1}$, $z_0 \in B$ and $t \in [0,1]$ satisfying $x_0 = (1-t)y + tz_0$. In a similar way, there exist $x \in \overline{\mathrm{co}}(K)$, $z \in C$ and $s \in [0,1]$ such that $z_0 = (1-s)x + sz$. Furthermore, $z = (1-\rho)\alpha u_1 + \rho\beta u_2$ for some $\rho \in [0,1]$ and $\alpha, \beta \in \mathbb{D}$. Consequently,

$$x_0 = (1-t)y + t(1-s)x + ts(1-\rho)\alpha u_1 + ts\rho\beta u_2\,. \tag{35}$$

According to Lemma 1, $x = \sum_{n=0}^{\infty} r_n v_n$, where $v_0 = e_1$, $r_n \in \mathbb{K}$ for every $n \in \mathbb{N} \cup \{0\}$, and $\sum_{n=0}^{\infty} |r_n| \leq 1$. On the other hand, $y = \sum_{n=1}^{\infty} y(n)\, e_n$ and $\sum_{n=1}^{\infty} |y(n)| \leq 1$. It is now clear that

$$x_0 = \sum_{n=1}^{\infty} \alpha_n u_n + \sum_{n=1}^{\infty} \beta_n v_n\,, \tag{36}$$

$\{\alpha_n\}$ and $\{\beta_n\}$ being the sequences of scalars defined by

$$\alpha_1 = \frac{(1-t)(y(1)+y(2)) + t(1-s)r_0}{2} + ts(1-\rho)\alpha \tag{37}$$

$$\alpha_2 = \frac{(1-t)(y(1)-y(2)) + t(1-s)r_0}{2} + ts\rho\beta \tag{38}$$

$$\alpha_n = (1-t)\, y(n) \;\text{ for all } n \geq 3 \tag{39}$$

$$\beta_n = t(1-s)\, r_n\, y(n) \;\text{ for all } n \in \mathbb{N}. \tag{40}$$

It is not hard to check that $\sum_{n=1}^{\infty} |\alpha_n| + \sum_{n=1}^{\infty} |\beta_n| \leq 1$. □

There is now enough coverage to prove that B_0 contains faces which closure is not another face of B_0.

Theorem 1. *Let $\{v_n\}$ be the sequence defined in (12). Then, the set*

$$F = \text{co}(\{v_n : n \in \mathbb{N}\}) \tag{41}$$

is a face of B_0, although $\overline{F}$ is not.

Proof. Let $x_0 \in F$. Then, there exists a natural number m and scalars $t_1, \ldots, t_m \in [0,1]$ such that $t_1 + \cdots + t_m = 1$ and

$$x_0 = t_1 v_1 + \cdots + t_m v_m \tag{42}$$

It is convenient to show that (42) is the only representation of x_0 that satisfies the constraints described in the previous lemma. Indeed, if

$$x_0 = \sum_{n=1}^{\infty} \alpha_n u_n + \sum_{n=1}^{\infty} \beta_n v_n\,, \;\text{ s.t. } \sum_{n=1}^{\infty} |\alpha_n| + \sum_{n=1}^{\infty} |\beta_n| \leq 1, \tag{43}$$

it is clear that

$$\alpha_1 + \alpha_2 + \sum_{n=1}^{\infty} \beta_n \tfrac{n+1}{n+2} = \sum_{k=1}^{m} t_k \tfrac{k+1}{k+2} \tag{44}$$

$$\alpha_1 - \alpha_2 = 0 \tag{45}$$

$$\alpha_{k+2} + \tfrac{\beta_k}{k+1} = \tfrac{t_k}{k+1} \;\text{ for all } k \in \{1, \ldots, m\} \tag{46}$$

$$\alpha_{k+2} + \tfrac{\beta_k}{k+1} = 0 \;\text{ for each natural number } k > m. \tag{47}$$

From (46) and (47) is readily seen that

$$\tfrac{k+1}{k+2}\alpha_{k+2} + \tfrac{\beta_k}{k+2} = \tfrac{t_k}{k+2} \;\text{ for all } k \in \{1, \ldots, m\} \tag{48}$$

$$\tfrac{k+1}{k+2}\alpha_{k+2} + \tfrac{\beta_k}{k+2} = 0 \;\text{ for each natural number } k > m. \tag{49}$$

By adding (44), (45), (48) and (49), one reaches the inequality

$$\begin{aligned} 1 &= \sum_{k=1}^{m} t_k = \alpha_1 + \alpha_2 + \sum_{n=1}^{\infty} \tfrac{n+1}{n+2}\alpha_{n+2} + \sum_{n=1}^{\infty} \beta_n \\ &\leq |\alpha_1| + |\alpha_2| + \sum_{n=1}^{\infty} \tfrac{n+1}{n+2}|\alpha_{n+2}| + \sum_{n=1}^{\infty} |\beta_n|. \end{aligned} \tag{50}$$

The condition $\sum_{n=1}^{\infty}|\alpha_n| + \sum_{n=1}^{\infty}|\beta_n| \leq 1$ implies that $\alpha_{n+2} = 0$ for all $n \in \mathbb{N}$. By virtue of (46), $\beta_k = t_k$ for every $k \in \{1, \ldots, m\}$ and hence $\alpha_1 = \alpha_2 = 0 = \beta_k$ for each $k > m$.

To see that F is a face of B_0, take $t \in]0,1[$ and $x, x' \in B_0$ such that $(1-t)x + tx' \in F$. Put $x_0 = (1-t)x + tx'$. Using the representations

$$x = \sum_{n=1}^{\infty} \alpha_n u_n + \sum_{n=1}^{\infty} \beta_n v_n \text{ and } x' = \sum_{n=1}^{\infty} \alpha'_n u_n + \sum_{n=1}^{\infty} \beta'_n v_n, \tag{51}$$

with $\sum_{n=1}^{\infty}|\alpha_n| + \sum_{n=1}^{\infty}|\beta_n| \leq 1$ and $\sum_{n=1}^{\infty}|\alpha'_n| + \sum_{n=1}^{\infty}|\beta'_n| \leq 1$, we have that

$$x_0 = \sum_{n=1}^{\infty}(((1-t)\alpha_n + t\alpha'_n))u_n + \sum_{n=1}^{\infty}((1-t)\beta_n + t\beta'_n)v_n. \tag{52}$$

On the other hand, x_0 can be expressed as in (42) and, from what has been proven,

$$(1-t)\alpha_n + t\alpha'_n = 0 \text{ for every natural number } n, \tag{53}$$
$$(1-t)\beta_n + t\beta'_n = 0 \text{ for all } n > m, \tag{54}$$
$$(1-t)\beta_k + t\beta'_k = t_k \text{ for each } k \in \{1, \ldots, m\}. \tag{55}$$

Thus,

$$1 = (1-t)\sum_{k=1}^{m}\beta_k + t\sum_{k=1}^{m}\beta'_k \leq (1-t)\sum_{k=1}^{m}|\beta_k| + t\sum_{k=1}^{m}|\beta'_k| \leq 1 \tag{56}$$

and, necessarily,

$$\beta_k = |\beta_k| \text{ and } \beta'_k = |\beta'_k| \text{ for every } k \in \{1, \ldots, m\}. \tag{57}$$

In the same way, $\sum_{k=1}^{m}|\beta_k| = \sum_{k=1}^{m}|\beta'_k| = 1$. Accordingly, $\alpha_n = \alpha'_n = 0$ for all $n \in \mathbb{N}$, and $\beta_n = \beta'_n = 0$ for each $n > m$. Therefore, we conclude that $x, x' \in F$.

Last, to prove that $\overline{F}$ is not a face of B_0, it is enough to bear in mind that $e_1 \in \overline{F}$ (since $\|v_n - e_1\|_1 \to 0$), $e_1 = \frac{u_1+u_2}{2}$, and, however, $u_1, u_2 \notin \overline{F}$. Indeed, $\|x - u_j\|_1 \geq 1$ for each $x \in F$ and $j \in \{1,2\}$. □

In contrast with the previous results, the natural norm of ℓ_1 has greater synergies with its underlying topological properties. Indeed, it will be proven that the closure of a face of B_{ℓ_1} is also a face of this set.

For every $x \in \ell_1$, the support of x is defined as $\operatorname{supp}(x) = \{n \in \mathbb{N} : x(n) \neq 0\}$. This concept can be easily extended to an arbitrary nonempty subset of ℓ_1 as follows: $\operatorname{supp} F = \bigcup_{x \in F} \operatorname{supp}(x)$.

From now on, F stands for any proper face of B_{ℓ_1}.

Lemma 3. *Let $x, y \in F$ and $m \in supp(x) \cap supp(y)$. Then* $\frac{x(m)}{|x(m)|} = \frac{y(m)}{|y(m)|}$.

Proof. Given that $\frac{x+y}{2} \in F$ it is readily seen that $\|x+y\|_1 = 2$. Therefore,

$$\sum_{n=1}^{\infty}|x(n) + y(n)| = \sum_{n=1}^{\infty}|x(n)| + \sum_{n=1}^{\infty}|y(n)| \tag{58}$$

and consequently $|x(n) + y(n)| = |x(n)| + |y(n)|$ for every $n \in \mathbb{N}$. Taking into account also that $x(m) \neq 0 \neq y(m)$, there is a real number $t > 0$ satisfying $x(m) = t\,y(m)$. Thus, $\frac{x(m)}{|x(m)|} = \frac{t\,y(m)}{|t\,y(m)|} = \frac{y(m)}{|y(m)|}$. □

Given $n \in \operatorname{supp} F$, there exists $x \in F$ such that $x(n) \neq 0$. The preceding lemma allows us to define the scalar $\phi(n) = \frac{\overline{x(n)}}{|x(n)|}$, as the quotient is not dependent on the choice of x (for $x \in F$, $n \in \operatorname{supp}(x)$). If we define $\phi(n) = 0$ for each $n \in \mathbb{N} \setminus \operatorname{supp} F$, the any $x \in F$ satisfies:

$$\sum_{n=1}^{\infty} x(n)\phi(n) = \sum_{n=1}^{\infty} |x(n)| = 1. \tag{59}$$

Clearly, $\phi \in \ell_\infty$ and $\|\phi\|_\infty = 1$. Supposing $\hat{\phi}$ is the element of ℓ_1^* induced by ϕ (i.e., $\hat{\phi}(x) = \sum_{n=1}^{\infty} \phi(n)x(n)$ for all $x \in \ell_1$), it is readily seen that the set

$$F_\phi = \{x \in B_{\ell_1} : \hat{\phi}(x) = 1\} \tag{60}$$

is a closed face of B_{ℓ_1} and, as it has already been remarked, $F \subset F_\phi$.

Theorem 2. *Under the previous notations, $\overline{F} = F_\phi$. In particular, $\overline{F}$ is a face of B_{ℓ_1}.*

Proof. Fix $m \in \operatorname{supp} F$. Hence there is $x \in F$ such that $x(m) \neq 0$. If $|x(m)| = 1$, then $x(n) = 0$ for any $n \in \mathbb{N}\setminus\{m\}$ and it implies that $x = x(m)e_m = \frac{x(m)}{|x(m)|}e_m$, leading to the conclusion $\frac{x(m)}{|x(m)|}e_m \in F$. Similarly, if $|x(m)| < 1$, define $y_m = \sum_{n\in\mathbb{N}\setminus\{m\}} \frac{x(n)}{1-|x(m)|} e_n$. By definition, $\frac{x(m)}{|x(m)|}e_m$, $y_m \in B_{\ell_1}$ (in fact, $\|\frac{x(m)}{|x(m)|}e_m\|_1 = \|y_m\|_1 = 1$). Consequently, taking into account that F is a face of B_{ℓ_1} and

$$\begin{aligned} x &= \sum_{n=1}^{\infty} x(n)e_n = x(m)e_m + \sum_{n\in\mathbb{N}\setminus\{m\}} x(n)e_n \\ &= |x(m)| \frac{x(m)}{|x(m)|}e_m + (1-|x(m)|)\, y_m, \end{aligned} \tag{61}$$

one can conclude that $\frac{x(m)}{|x(m)|}e_m \in F$.

As the inclusion $F \subset F_\phi$ has already been proved, it only remains to check that $F_\phi \subset \overline{F}$. Take x an arbitrary element of F_ϕ. We have that

$$\sum_{n=1}^{\infty} x(n)\phi(n) = 1 = \sum_{n=1}^{\infty} |x(n)|. \tag{62}$$

Select any $m \in \operatorname{supp}(x)$. If $|x(m)| = 1$, it follows from the previous identities that $x(m)\phi(m) = |x(m)|$. On the other hand, if $|x(m)| < 1$, a scalar α_n satisfying $|\alpha_n| = 1$ and $x(n) = |x(n)|\alpha_n$ can be considered for each natural number n. In light of (62),

$$x(m)\phi(m) + \sum_{n\in\mathbb{N}\setminus\{m\}} x(n)\phi(n) = 1 \tag{63}$$

which can also be written as,

$$|x(m)|\,\alpha_m\phi(m) \;+\; (1-|x(m)|) \sum_{n\in\mathbb{N}\setminus\{m\}} \frac{|x(n)|}{1-|x(m)|}\alpha_n\phi(n) \;=\; 1. \tag{64}$$

From the strict convexity of $\mathbb{K}$, it is easily obtained that $\alpha_m\phi(m) = 1$. As a consequence, $|x(m)|\alpha_m\phi(m) = |x(m)|$ and again $x(m)\phi(m) = |x(m)|$. In particular, $\phi(m) \neq 0$ (equivalently $m \in \operatorname{supp} F$) and $\phi(m) = \frac{\overline{x(m)}}{|x(m)|}$. Note that the inclusion $\operatorname{supp}(x) \subset \operatorname{supp} F$ is also given by the previous argument.

Back to the initial part of the proof, $\frac{x(n)}{|x(n)|}e_n \in F$ for every $n \in \operatorname{supp}(x)$. Finally, decomposing x as

$$x = \sum_{n=1}^{\infty} x(n)e_n = \sum_{n\in\operatorname{supp}(x)} |x(n)| \frac{x(n)}{|x(n)|}e_n \tag{65}$$

one gets that $x \in \overline{\text{co}}\left\{\frac{x(n)}{|x(n)|}e_n : n \in \operatorname{supp}(x)\right\}$ and hence $x \in \overline{F}$. □

Last, an example of a nonclosed face of B_{ℓ_1} is introduced. Let us consider the set

$$C = \left\{x \in B_{\ell_1} : \sum_{n=1}^{\infty} x(n) = 1\right\}, \tag{66}$$

which is a closed face of B_{ℓ_1}. Given that $e_n \in C$ for each natural number n, $\operatorname{supp} C = \mathbb{N}$. If $x \in C$ and $n \in \operatorname{supp}(x)$ we have that $\frac{x(n)}{|x(n)|} = \frac{e_n(n)}{|e_n(n)|} = 1$, and then $x(n) = |x(n)|$. This equality also holds whenever $n \in \mathbb{N}\backslash\operatorname{supp}(x)$.

In order to finalise this example, we will show that the set

$$F = \left\{x \in B_{\ell_1} : \sum_{n=1}^{\infty} x(n) = 1 \text{ and } \operatorname{supp}(x) \text{ is finite}\right\} \tag{67}$$

is a face of B_{ℓ_1} satisfying $\overline{F} = C$. Observe, as a consequence, that F is not closed. First of all, F is a convex set contained in C. In addition, given $x, y \in B_{\ell_1}$ and $t \in \mathbb{R}$, with $0 < t < 1$, such that $(1-t)x + ty \in F$, there is $m \in \mathbb{N}$ satisfying the following condition:

$$n \in \mathbb{N},\ n > m \Rightarrow (1-t)x(n) + ty(n) = 0. \tag{68}$$

Taking into account that C is a (closed) face of B_{ℓ_1}, $x, y \in C$ and $x(n) = |x(n)|$, $y(n) = |y(n)|$ for any $n \in \mathbb{N}$. Thanks to (68) we get that $x(n) = y(n) = 0$ for every natural number $n > m$. That way, $x, y \in F$ and we have shown that F is a face of B_{ℓ_1}. As $e_n \in F$ for each $n \in \mathbb{N}$, $\operatorname{supp} F = \mathbb{N}$ and its corresponding sequence ϕ is given by $\phi(n) = \frac{e_n(n)}{|e_n(n)|} = 1$ for all n. Consequently,

$$\overline{F} = F_\phi = \left\{x \in B_{\ell_1} : \hat{\phi}(x) = 1\right\} = C. \tag{69}$$

Author Contributions: Conceptualization, J.C.N.-P. and A.Z.-M.; methodology, J.C.N.-P. and A.Z.-M.; formal analysis, J.C.N.-P. and A.Z.-M.; investigation, J.C.N.-P. and A.Z.-M.; writing—original draft preparation, J.C.N.-P. and A.Z.-M.; writing—review and editing, J.C.N.-P. and A.Z.-M.; visualization, J.C.N.-P. and A.Z.-M.; supervision, J.C.N.-P. and A.Z.-M.; project administration, J.C.N.-P. and A.Z.-M.; funding acquisition, J.C.N.-P. and A.Z.-M. All authors have read and agreed to the published version of the manuscript.

Funding: This research has been partially supported by the FQM-194 research group of the University of Almería.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their gratitude to the reviewers for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Lukeš, J.; Malý, J.; Netuka, I.; Spurný, J. *Integral Representation Theory. Applications to Convexity, Banach Spaces and Potential Theory*; Walter de Gruyter GmbH and Co.: Berlin, Germany, 2010.
2. Roy, N.M. Extreme Points of Convex Sets in Infinite Dimensional Spaces. *Am. Math. Mon.* **1987**, *94*, 409–422. [CrossRef]
3. Harmand, P.; Werner, D.; Werner, W. *M-Ideals in Banach Spaces and Banach algebras*; Lecture Notes in Mathematics 1547; Springer: Berlin, Germany, 1993.
4. Alfsen, E.M. *Compact Convex Sets and Boundary Integrals*; Springer: Berlin/Heidelberg, Germany, 1971.
5. Cabrera-Serrano, A.M.; Mena-Jurado, J.F. Facial topology and extreme operators. *J. Math. Anal. Appl.* **2015**, *427*, 899–904. [CrossRef]
6. Mena-Jurado, J.F.; Navarro-Pascual, J.C. Preservation of Extreme Points. *Mathematics* **2022**, *10*, 2268. [CrossRef]
7. Deville, R.; Godefroy, G.; Zizler, V. *Smoothness and Renormings in Banach Spaces*; Pitman Monogr Surveys in Pure and Applied Mathematics 64; Longman: London, UK, 1993.
8. Godefroy, G. Renormings of Banach spaces. In *Handbook of the Geometry of Banach Spaces*; North-Holland: Amsterdam, The Netherlands, 2001; Volume I, pp. 781–835.
9. Guirao, A.J.; Montesinos, V.; Zizler, V. *Renormings in Banach Spaces. A Toolbox*; Monografie Matematyczne 75; Birkhäuser/Springer Nature Switzerland AG: Cham, Switzerland, 2022.
10. Polyrakis, I.A. Cones locally isomorphic to the positive cone of $\ell_1(\Gamma)$. *Linear Algebra Its Appl.* **1986**, *84*, 323–334. [CrossRef]