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Topological reflexivity of isometries on algebras of C(Y)-valued Lipschitz maps

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ABSTRACT

Let X be a compact metric space and let $C(Y)$ be the space of all complex-valued continuous functions on a Hausdorff compact space Y. We prove that the isometry group of the algebra $\mathrm{Lip}(X, C(Y))$ of all $C(Y)$-valued Lipschitz maps on X, equipped with the sum norm, is topologically reflexive and 2-topologically reflexive whenever the isometry group of $C(Y)$ is topologically reflexive. The same results are established for the sets of isometric reflections and generalized bi-circular projections of $\mathrm{Lip}(X)$.

1. Introduction

Given a compact metric space (X, d_X) and a complex Banach algebra $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$, we denote by $\mathrm{Lip}(X, \mathcal{A})$ the complex Banach algebra of all Lipschitz maps $F\colon X \to \mathcal{A}$, under the norm

$$\|F\|_{\Sigma} = \|F\|_{\infty} + \mathrm{Lip}(F),$$

where

$$\|F\|_{\infty} = \sup\{\|F(x)\|_{\mathcal{A}} : x \in X\}$$

and

$$\mathrm{Lip}(F) = \sup\left\{\frac{\|F(x) - F(y)\|_{\mathcal{A}}}{d_X(x, y)} : x, y \in X,\ x \neq y\right\}.$$

Moreover, $\mathrm{lip}(X, \mathcal{A})$ stands for the closed subalgebra of $\mathrm{Lip}(X, \mathcal{A})$ formed by all those functions F satisfying the condition:

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$$\forall \varepsilon > 0,\ \exists \delta > 0\colon x, y \in X,\ 0 < d_X(x,y) < \delta \Rightarrow \frac{\|F(x) - F(y)\|_{\mathcal{A}}}{d_X(x,y)} < \varepsilon.$$

The space $\mathrm{Lip}(X, \mathcal{A})$ separates the points of X, but $\mathrm{lip}(X, \mathcal{A})$ may contain only constant functions. In order to avoid the triviality, we will only consider the algebras $\mathrm{lip}(X^\alpha, \mathcal{A})$ with $0 < \alpha < 1$, where X^α denotes the metric space (X, d_X^α). Also $\mathrm{Lip}(X, \mathbb{C})$ and $\mathrm{lip}(X^\alpha, \mathbb{C})$ are denoted by $\mathrm{Lip}(X)$ and $\mathrm{lip}(X^\alpha)$, respectively. These algebras were introduced by Sherbert [22,23] with the name of Lipschitz algebras.

A continuous linear operator T of $\mathrm{Lip}(X, \mathcal{A})$ into itself is said to be a *local isometry* if for every $F \in \mathrm{Lip}(X, \mathcal{A})$, there exists a surjective linear isometry T_F of $\mathrm{Lip}(X, \mathcal{A})$ (depending possibly on F) such that $T_F(F) = T(F)$. The operator T is an *approximate local isometry* if for every $F \in \mathrm{Lip}(X, \mathcal{A})$, there exists a sequence $\{T_{F,n}\}_{n\in\mathbb{N}}$ of surjective linear isometries of $\mathrm{Lip}(X, \mathcal{A})$ such that $\lim_{n\to\infty} T_{F,n}(F) = T(F)$.

The isometry group of $\mathrm{Lip}(X, \mathcal{A})$ is said to be *algebraically reflexive* (respectively, *topologically reflexive*) if every local isometry (respectively, approximate local isometry) of $\mathrm{Lip}(X, \mathcal{A})$ is a surjective isometry. The algebraic and topological reflexivity for automorphism groups are similarly defined.

It is straightforward to verify that the topological reflexivity of an isometry group implies the algebraic reflexivity of the group. The flexibility that approximate local isometries possess allows us to study the reflexivity phenomena in a general context.

For a compact metric space X, the algebraic reflexivity of the isometry group and the automorphism group of $\mathrm{Lip}(X, \mathcal{A})$ has been studied for various Banach algebras: $\mathcal{A} = \mathbb{C}$ [6,15,20]; $\mathcal{A} = M_n(\mathbb{C})$, the algebra of complex matrices of order n with the spectral norm, [5,20]; $\mathcal{A} = B(H)$, the algebra of bounded linear operators on a separable complex Hilbert space H with the operator norm, [2,6]; and $\mathcal{A} = C(Y)$, the algebra of complex-valued continuous functions on a Hausdorff compact space Y with the supremum norm, [20]. Also the algebraic and topological reflexivity of some families of bounded linear operators on $\mathrm{Lip}(X, E)$ has been studied in [3,4] for a smooth Banach space E.

It is proved in [20] that, if the isometry group of $C(Y)$ is algebraically reflexive, then so is the isometry group of $\mathrm{Lip}(X, C(Y))$. Also in [16] the 2-algebraic reflexivity of the isometry group of $\mathrm{Lip}(X, E)$ is established under some assumptions on X and E. The purpose of this note is to prove a topological-reflexivity counterpart of these results: $\mathrm{Lip}(X, C(Y))$ is topologically reflexive and 2-topologically reflexive whenever $C(Y)$ is topologically reflexive. There are compact Hausdorff spaces whose homeomorphism groups are compact groups, or even the trivial groups ([1,12,18]). By the Banach–Stone theorem and a direct compactness argument, we see that, for each such space Y, $C(Y)$ is topologically reflexive and our theorem applies to every such space Y.

The proof is based on the characterization of the isometries on $\mathrm{Lip}(X, C(Y))$ as generalized weighted composition operators. Given an approximate local isometry or an approximate 2-local isometry, we apply a result of Hatori and Oi [10] and Izumi and Takagi [13] to recognize these maps as generalized weighted composition operators, and conclude that they are indeed isometries.

This paper consists of three sections. Section 2 contains the necessary notation and some preliminary results. For the proof of our main result in Section 4, we first need to prove in Section 3 that the set of linear isometries of $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ is topologically reflexive and 2-topologically reflexive. We also show that the sets of isometric reflections and generalized bi-circular projections of $\mathrm{Lip}(X)$ are both topologically reflexive and 2-topologically reflexive. We close both sections establishing analogous results for little Lipschitz algebras $\mathrm{lip}(X^\alpha)$ and $\mathrm{lip}(X^\alpha, C(Y))$ with $\alpha \in (0, 1)$.

2. Notation and preliminaries

We begin introducing some notation. As usual, $\mathbb{T}$ stands for the unit circle of $\mathbb{C}$. For a set X, the symbols Id_X and 1_X represent the identity function and the function with constant value 1 on X, respectively.

Given two Hausdorff compact spaces X and Y, we denote

$$C(X,Y) = \{f \text{ is a continuous map of } X \text{ to } Y\},$$
$$\mathrm{Homeo}(X,Y) = \{f \text{ is a homeomorphism of } X \text{ onto } Y\}.$$

For two compact metric spaces X and Y, we denote

$$\mathrm{Lip}(X,Y) = \{f \text{ is a Lipschitz map of } X \text{ to } Y\},$$
$$\mathrm{Iso}(X,Y) = \{f \text{ is an isometry of } X \text{ onto } Y\},$$
$$\mathrm{Iso}^2(X) = \left\{f \text{ is an isometry of } X \text{ onto } X \text{ such that } f^2 = \mathrm{Id}_X\right\}.$$

In particular, we write $\mathrm{Iso}(X)$ instead of $\mathrm{Iso}(X,X)$. Given a map $\phi \in \mathrm{Lip}(Y,X)$, we write $\widehat{\phi}$ for the composition operator from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ given by $\widehat{\phi}(f) = f \circ \phi$.

Given a Banach space E, let us recall that an *isometric reflection* of E is a linear isometry $T\colon E \to E$ such that $T^2 = \mathrm{Id}_E$, and a *generalized bi-circular projection* of E is a linear projection $P\colon E \to E$ such that $P + \lambda(\mathrm{Id}_E - P)$ is a linear surjective isometry for some $\lambda \in \mathbb{T}$ with $\lambda \neq 1$.

For two Banach spaces E and F, we denote

$$B(E,F) = \{T \text{ is a continuous linear operator of } E \text{ to } F\},$$
$$\mathrm{Iso}(E,F) = \{T \text{ is a linear isometry of } E \text{ onto } F\},$$
$$\mathrm{Iso}^2(E) = \{T \text{ is an isometric reflection of } E\},$$
$$\mathrm{GBP}(E) = \{T \text{ is a generalized bi-circular projection of } E\}.$$

Complete descriptions of surjective linear isometries and algebra homomorphisms between $\mathrm{Lip}(X)$-algebras and also isometric reflections and generalized bi-circular projections of $\mathrm{Lip}(X)$ are known (see [10,14,22,15]).

We now recall the concepts of reflexivity studied in this paper. Let E and F be Banach spaces and $\mathcal{S}$ be a nonempty subset of $B(E,F)$. We define the *algebraic reflexive closure* of $\mathcal{S}$ by

$$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \{T \in B(E,F)\colon \forall e \in E,\ \exists T_e \in \mathcal{S} \,|\, T_e(e) = T(e)\}$$

and the *topological reflexive closure* of $\mathcal{S}$ by

$$\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \left\{T \in B(E,F)\colon \forall e \in E,\ \exists \{T_{e,n}\}_{n\in\mathbb{N}} \subset \mathcal{S} \,|\, \lim_{n\to\infty} T_{e,n}(e) = T(e)\right\}.$$

The set $\mathcal{S}$ is said to be *algebraically reflexive* (respectively, *topologically reflexive*) if $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \mathcal{S}$ (respectively, $\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \mathcal{S}$).

Consider now the *2-algebraic reflexive closure* of $\mathcal{S}$, $2\text{-}\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$, defined by

$$\left\{\Delta \in F^E\colon \forall e,u \in E,\ \exists T_{e,u} \in \mathcal{S} \,|\, T_{e,u}(e) = \Delta(e),\ T_{e,u}(u) = \Delta(u)\right\}$$

and the *2-topological reflexive closure* of $\mathcal{S}$, $2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$, given by

$$\left\{\Delta \in F^E\colon \forall e,u \in E,\ \exists \{T_{e,u,n}\}_{n\in\mathbb{N}} \subset \mathcal{S} \,|\, \lim_{n\to\infty} T_{e,u,n}(e) = \Delta(e),\ \lim_{n\to\infty} T_{e,u,n}(u) = \Delta(u)\right\}.$$

We say that the set $\mathcal{S}$ is *2-algebraically reflexive* (respectively, *2-topologically reflexive*) if $2\text{-}\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \mathcal{S}$ (respectively, $2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \mathcal{S}$).

We will also need some facts on algebras $\mathrm{Lip}(X, C(Y))$. Given $f \in \mathrm{Lip}(X)$ and $g \in C(Y)$, the function $f \otimes g\colon X \to C(Y)$ defined by

$$(f \otimes g)(x) = f(x)g \qquad (x \in X)$$

belongs to $\mathrm{Lip}(X, C(Y))$ with $\|f \otimes g\|_\infty = \|f\|_\infty \|g\|_\infty$ and $\mathrm{Lip}(f \otimes g) = \mathrm{Lip}(f) \|g\|_\infty$. It is known (see [13, Proposition 2.3]) that $\mathrm{Lip}(X, C(Y))$ is a semisimple unital commutative complex Banach algebra whose unit is the map $1_X \otimes 1_Y$.

We will make use of the following representation of type BJ of surjective linear isometries between algebras $\mathrm{Lip}(X, C(Y))$, due to Hatori and Oi [10] (see also [8,9]). A Lipschitz map $F \in \mathrm{Lip}(X, C(Y))$ is identified with a continuous function $F\colon X \times Y \to \mathbb{C}$ and $F(x)(y)$ is written as $F(x, y)$ for $x \in X$, $y \in Y$. Following [11, Definition 2.6], a unital homomorphism $T\colon \mathrm{Lip}(X_1, C(Y_1)) \to \mathrm{Lip}(X_2, C(Y_2))$ is of type BJ if there exist two maps $\varphi_1 \in C(X_2 \times Y_2, X_1)$ and $\varphi_2 \in C(Y_2, X_2)$ such that

$$T(F)(x, y) = F(\varphi_1(x, y), \varphi_2(y)) \qquad ((x, y) \in X_2 \times Y_2),$$

for all $F \in \mathrm{Lip}(X_1, C(Y_1))$. Observe that φ_2 depends only on the second variable.

In what follows, given a map φ defined on $X \times Y$, for each $y \in Y$, we denote by φ^y the map given by $\varphi^y(x) = \varphi(x, y)$ for all $x \in X$.

Theorem 1. *[10, Corollary 14] Let X_1, X_2 be compact metric spaces and Y_1, Y_2 be Hausdorff compact spaces. A map $T\colon \mathrm{Lip}(X_1, C(Y_1)) \to \mathrm{Lip}(X_2, C(Y_2))$ is a surjective linear isometry with respect to the norms $\|\cdot\|_\Sigma$ if and only if there exist a function $h \in C(Y_2, \mathbb{T})$, a map $\varphi \in C(X_2 \times Y_2, X_1)$ with $\varphi^y \in \mathrm{Iso}(X_2, X_1)$ for each $y \in Y_2$ and a map $\tau \in \mathrm{Homeo}(Y_2, Y_1)$ such that*

$$T(F)(x, y) = h(y)F(\varphi(x, y), \tau(y)) \qquad ((x, y) \in X_2 \times Y_2),$$

for all $F \in \mathrm{Lip}(X_1, C(Y_1))$. $\square$

The study of homomorphisms between algebras $\mathrm{Lip}(X, C(Y))$ apparently began in the work [11] by Hatori, Oi and Takagi. The first characterization of such maps appears in the paper [19] by Oi. We need the following theorem by Izumi and Takagi [13] which improves [19, Theorem 1].

Theorem 2. *[13, Theorem 1] Let X_1, X_2 be compact metric spaces and Y_1, Y_2 be Hausdorff compact spaces. If T is an algebra homomorphism from $\mathrm{Lip}(X_1, C(Y_1))$ to $\mathrm{Lip}(X_2, C(Y_2))$, then there exist a clopen subset $\mathcal{D} \subseteq X_2 \times Y_2$ and two maps $\varphi \in C(\mathcal{D}, X_1)$ and $\tau \in C(\mathcal{D}, Y_1)$ with the properties (i) and (ii) below such that T has the form:*

$$T(F)(x, y) = \begin{cases} F(\varphi(x, y), \tau(x, y)) & \text{if} \quad (x, y) \in \mathcal{D}, \\ 0 & \text{if} \quad (x, y) \in (X_2 \times Y_2) \setminus \mathcal{D}, \end{cases} \tag{1}$$

for all $F \in \mathrm{Lip}(X_1, C(Y_1))$.

(i) There exists a bound $L \geq 0$ such that if $(x_1, y), (x_2, y) \in \mathcal{D}$ and $x_1 \neq x_2$, then

$$\frac{d_{Y_1}(\varphi(x_1, y), \varphi(x_2, y))}{d_{X_2}(x_1, x_2)} \leq L.$$

(ii) For any $y \in Y_2$, there exists $r > 0$ such that the set $\mathcal{D}^y = \{x \in X_2 \colon (x, y) \in \mathcal{D}\}$ is a union of finitely many disjoint clopen subsets $V_1^y, \ldots, V_{n_y}^y$ of X_2 such that τ^y is constant on V_i^y for $i = 1, \ldots, n_y$ and $d_{X_2}(V_i^y, V_j^y) \geq r$ for $i \neq j$. Here r is a positive constant independent of y.

Conversely, if $\mathcal{D}$, φ and τ are given as above, then T defined by (1) is an algebra homomorphism from $\mathrm{Lip}(X_1, C(Y_1))$ *to* $\mathrm{Lip}(X_2, C(Y_2))$. *Moreover, T is unital if and only if $\mathcal{D} = X_2 \times Y_2$.* $\square$

3. Topological reflexivity in Lip(X)-algebras

We first prove that every approximate local isometry from $\mathrm{Lip}(X)$ to $\mathrm{Lip}(Y)$ is a surjective isometry. This improves Theorem 2.3 of [15] and Theorem 3.1 of [20]. Our proof consists of showing the equality $\mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y))) = \mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y)))$, from which the conclusion is derived by the algebraic reflexivity.

Theorem 3. *Let X and Y be compact metric spaces. Then* $\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ *is topologically reflexive.*

Proof. Let $T \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y)))$. Using [10, Corollary 15], for each $f \in \mathrm{Lip}(X)$, there exist sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ and $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Iso}(Y, X)$ such that $\lim_{n\to\infty} ||\lambda_{f,n}\widehat{\phi_{f,n}}(f) - T(f)||_\Sigma = 0$. By the compactness of X and the Arzelá-Ascoli theorem (see, for example, [7, Chapter XII]), the set $\mathrm{Iso}(Y, X)$, endowed with the topology induced by the metric

$$d^+(f, g) = \sup\{d(f(y), g(y)) \colon y \in Y\}$$

for bounded continuous maps f, g of Y into X, is compact. Together with the compactness of $\mathbb{T}$, we may assume that there are subsequences $\{\lambda_{f,n_k}\}_{k\in\mathbb{N}}$ and $\{\phi_{f,n_k}\}_{k\in\mathbb{N}}$ such that $\{\lambda_{f,n_k}\}_{k\in\mathbb{N}} \to \lambda_f$ and $\{\phi_{f,n_k}\}_{k\in\mathbb{N}} \to \phi_f$ for some $\lambda_f \in \mathbb{T}$ and $\phi_f \in \mathrm{Iso}(Y, X)$. Hence $\lim_{k\to\infty} ||\lambda_{f,n_k}\widehat{\phi_{f,n_k}}(f) - \lambda_f\widehat{\phi_f}(f)||_\infty = 0$. By the definition of the norm $\|\cdot\|_\Sigma$, we have $\lim_{k\to\infty} ||\lambda_{f,n_k}\widehat{\phi_{f,n_k}}(f) - T(f)||_\infty = 0$, hence $T(f) = \lambda_f\widehat{\phi_f}(f)$. Therefore T belongs to $\mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y)))$, and since it is known that $\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ is algebraically reflexive, we conclude that $T \in \mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y))$. $\square$

We now state the topological reflexivity of $\mathrm{Iso}^2(\mathrm{Lip}(X))$ and $\mathrm{GBP}(\mathrm{Lip}(X))$. The next two results improve, respectively, the conclusions obtained in Theorem 2.5 and Corollary 3.3. of [15].

Theorem 4. *Let X be a compact metric space. Then* $\mathrm{Iso}^2(\mathrm{Lip}(X))$ *is topologically reflexive.*

Proof. We follow the idea of the proof of Theorem 3. Let $T \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}^2(\mathrm{Lip}(X)))$. By [15, Corollary 2.4], for every $f \in \mathrm{Lip}(X)$, there are two sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ in $\{-1, 1\}$ and $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Iso}^2(X)$ satisfying

$$\left\|\lambda_{f,n}\widehat{\phi_{f,n}}(f) - T(f)\right\|_\Sigma \to 0.$$

We claim that $\mathrm{Iso}^2(X)$ is compact in $(C(X, X), d^+)$. Since $(\mathrm{Iso}(X), d^+)$ is compact, it suffices to prove that $\mathrm{Iso}^2(X)$ is closed in $(\mathrm{Iso}(X), d^+)$. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Iso}^2(X)$ and assume that $d^+(\phi_n, \phi) \to 0$ for some $\phi \in \mathrm{Iso}(X)$. Given $x \in X$ and $\epsilon > 0$, since ϕ is continuous and $d(\phi_n(x), \phi(x)) \to 0$, we can take $n_0 \in \mathbb{N}$ such that $d^+(\phi_n, \phi) < \epsilon/2$ and $d(\phi(\phi_n(x)), \phi^2(x)) < \epsilon/2$ for all $n \geq n_0$. Then, for any $n \geq n_0$, we have

$$d(\phi_n^2(x), \phi^2(x)) \leq d(\phi_n(\phi_n(x)), \phi(\phi_n(x))) + d(\phi(\phi_n(x)), \phi^2(x)) < \epsilon,$$

which yields $d(\phi_n^2(x), \phi^2(x)) \to 0$. Since $\phi_n^2(x) = x$ for all $n \in \mathbb{N}$, we deduce that $\phi^2(x) = x$. Hence $\phi^2 = \mathrm{Id}_X$ and this proves our claim.

Taking subsequences, we can suppose that $|\lambda_{f,n_k} - \lambda_f| \to 0$ and $d^+(\phi_{f,n_k}, \phi_f) \to 0$ for some $\lambda_f \in \{-1, 1\}$ and $\phi_f \in \mathrm{Iso}^2(X)$. Hence

$$||\lambda_{f,n_k}\widehat{\phi_{f,n_k}}(f) - \lambda_f\widehat{\phi_f}(f)||_\infty \to 0.$$

Since the convergence in the norm $||\cdot||_\Sigma$ implies convergence in the norm $||\cdot||_\infty$, we have

$$||\lambda_{f,n_k}\widehat{\phi_{f,n_k}}(f) - T(f)||_\infty \to 0.$$

Hence $T(f) = \lambda_f\widehat{\phi_f}(f)$. This shows that $T \in \mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}^2(\mathrm{Lip}(X)))$, and since $\mathrm{Iso}^2(\mathrm{Lip}(X))$ is algebraically reflexive by [15, Theorem 2.5], we conclude that $T \in \mathrm{Iso}^2(\mathrm{Lip}(X))$. $\square$

Corollary 1. *Let X be a compact metric space. Then* $\mathrm{GBP}(\mathrm{Lip}(X))$ *is topologically reflexive.*

Proof. Let $P \in \mathrm{ref}_{\mathrm{top}}(\mathrm{GBP}(\mathrm{Lip}(X)))$. By [15, Theorem 3.1], for every $f \in \mathrm{Lip}(X)$, there are sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ in $\{-1, 1\}$ and $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Iso}^2(X)$ such that

$$\lim_{n\to\infty} \frac{1}{2}\left[f + \lambda_{f,n}(f \circ \phi_{f,n})\right] = P(f).$$

Hence, for every $f \in \mathrm{Lip}(X)$, we have

$$\lim_{n\to\infty} \lambda_{f,n}(f \circ \phi_{f,n}) = 2P(f) - f,$$

and so $2P - \mathrm{Id}_{\mathrm{Lip}(X)} \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}^2(\mathrm{Lip}(X)))$. Hence $2P - \mathrm{Id}_{\mathrm{Lip}(X)} \in \mathrm{Iso}^2(\mathrm{Lip}(X))$ by Theorem 4, and therefore $P \in \mathrm{GBP}(\mathrm{Lip}(X))$. $\square$

We also may apply Theorem 3 to study the 2-local reflexivity of the following three types of maps.

Corollary 2. *Let X and Y be compact metric spaces. Then* $\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y))$, $\mathrm{Iso}^2(\mathrm{Lip}(X))$ *and* $\mathrm{GBP}(\mathrm{Lip}(X))$ *are 2-topologically reflexive.*

Proof. Let $\Delta \in 2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y)))$. We will first prove that for each $y \in Y$, the functional $\Delta_y \colon \mathrm{Lip}(X) \to \mathbb{C}$ defined by

$$\Delta_y(f) = \Delta(f)(y) \qquad (f \in \mathrm{Lip}(X)),$$

is linear. According to a spherical variant of the Kowalski–Słodkowski theorem [17, Proposition 3.2], it suffices to show that Δ_y is 1-homogeneous and satisfies that $\Delta_y(f) - \Delta_y(g) \in \mathbb{T}\sigma(f - g)$ for all $f, g \in \mathrm{Lip}(X)$.

To prove the 1-homogeneity, let $f \in \mathrm{Lip}(X)$ and $\beta \in \mathbb{C}$. Hence there exists a sequence $\{T_{f,\beta f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ such that $\lim_{n\to\infty} T_{f,\beta f,n}(f) = \Delta(f)$ and $\lim_{n\to\infty} T_{f,\beta f,n}(\beta f) = \Delta(\beta f)$. Hence

$$\Delta(\beta f) = \lim_{n\to\infty} T_{f,\beta f,n}(\beta f) = \beta \lim_{n\to\infty} T_{f,\beta f,n}(f) = \beta\Delta(f).$$

To check the spectral condition, let $f, g \in \mathrm{Lip}(X)$. We can take sequences $\{\lambda_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ and $\{\phi_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Iso}(Y, X)$ such that

$$\lim_{n\to\infty} \lambda_{f,g,n} f(\phi_{f,g,n}(y)) = \Delta(f)(y),$$
$$\lim_{n\to\infty} \lambda_{f,g,n} g(\phi_{f,g,n}(y)) = \Delta(g)(y),$$

and thus

$$\Delta_y(f) - \Delta_y(g) = \lim_{n\to\infty} \lambda_{f,g,n}(f-g)(\phi_{f,g,n}(y)) \in \mathbb{T}\sigma(f-g).$$

Since y was arbitrary in Y, we deduce that Δ is linear. Clearly, $\Delta \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y)))$, and therefore $\Delta \in \mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ by Theorem 3. This proves that $\mathrm{Iso}(\mathrm{Lip}(X), \mathrm{Lip}(Y))$ is 2-topologically reflexive.

A similar reasoning yields the 2-topological reflexivity of $\mathrm{Iso}^2(\mathrm{Lip}(X))$. Indeed, let

$$\Delta \in 2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}^2(\mathrm{Lip}(X))).$$

Since every $T \in \mathrm{Iso}^2(\mathrm{Lip}(X))$ has the form $T = \lambda\widehat{\phi}$ for some $\lambda \in \{\pm 1\}$ and $\phi \in \mathrm{Iso}^2(X)$ (see [15, Corollary 2.4]), we may prove as above that Δ is linear. Hence $\Delta \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}^2(\mathrm{Lip}(X)))$ and, by Theorem 4, we conclude that $\Delta \in \mathrm{Iso}^2(\mathrm{Lip}(X))$.

Using the 2-topological reflexivity of $\mathrm{Iso}^2(\mathrm{Lip}(X))$ and that

$$\mathrm{Iso}^2(\mathrm{Lip}(X)) = 2\mathrm{GBP}(\mathrm{Lip}(X)) - \mathrm{Id}_{\mathrm{Lip}(X)},$$

the 2-topological reflexivity of $\mathrm{GBP}(\mathrm{Lip}(X))$ is deduced easily. □

It should be mentioned that Oi [21] improved the above spherical variant of the Kowalski–Słodkowski theorem by obtaining the same conclusion under the weaker assumption "$\Delta_y(0) = 0$" than the 1-homogeneity.

In order to state the preceding results in the setting of little Lipschitz spaces, it is convenient to make the following comment.

Remark 1. Theorem 1 (Corollary 14 of [10]) characterizes surjective linear isometries between $\mathrm{Lip}(X, C(Y))$ spaces and between $\mathrm{lip}(X^\alpha, C(Y))$ spaces with $0 < \alpha < 1$.

[10, Corollary 15] and [15, Corollary 2.4 and Theorem 3.1] characterize surjective linear isometries, isometric reflections and generalized bi-circular projections on the spaces $\mathrm{Lip}(X)$ and $\mathrm{lip}(X^\alpha)$.

By Remark 1, the same proofs of Theorems 3 and 4 and Corollaries 1 and 2, replacing the Lip-spaces by the lip-spaces, yield the following result.

Theorem 5. *Let X and Y be compact metric spaces and $\alpha \in (0, 1)$. The sets* $\mathrm{Iso}(\mathrm{lip}(X^\alpha), \mathrm{lip}(Y^\alpha))$, $\mathrm{Iso}^2(\mathrm{lip}(X^\alpha))$ *and* $\mathrm{GBP}(\mathrm{lip}(X^\alpha))$ *are topologically reflexive and 2-topologically reflexive.* □

4. Topological reflexivity in Lip(X,C(Y))-algebras

We establish the topological reflexivity of the isometry group between $\mathrm{Lip}(X, C(Y))$-algebras when $\mathrm{Iso}(C(Y))$ is topologically reflexive.

Theorem 6. *Let X_1, X_2 be compact metric spaces and Y_1, Y_2 be Hausdorff compact spaces. Suppose that* $\mathrm{Iso}(C(Y_1), C(Y_2))$ *is topologically reflexive. Then* $\mathrm{Iso}(\mathrm{Lip}(X_1, C(Y_1)), \mathrm{Lip}(X_2, C(Y_2)))$ *is topologically reflexive.*

Proof. Let T be an approximate local isometry of $\mathrm{Lip}(X_1, C(Y_1))$ to $\mathrm{Lip}(X_2, C(Y_2))$. We will prove the next claims to conclude that T has a representation as in Theorem 1, and so $T \in \mathrm{Iso}(\mathrm{Lip}(X_1, C(Y_1)), \mathrm{Lip}(X_2, C(Y_2)))$.

Claim 1. *For every $F \in \mathrm{Lip}(X_1, C(Y_1))$, there exist three sequences $\{h_{F,n}\}_{n\in\mathbb{N}}$ in $C(Y_2, \mathbb{T})$, $\{\varphi_{F,n}\}_{n\in\mathbb{N}}$ in $C(X_2 \times Y_2, X_1)$ such that, for each $y \in Y_2$, $\varphi^y_{F,n} \in \mathrm{Iso}(X_2, X_1)$ for all $n \in \mathbb{N}$, and $\{\tau_{F,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Homeo}(Y_2, Y_1)$ satisfying that*

$$\lim_{n\to\infty} h_{F,n} F(\varphi_{F,n}, \tau_{F,n}) = T(F).$$

Let $F \in \mathrm{Lip}(X_1, C(Y_1))$. Since $T \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(\mathrm{Lip}(X_1, C(Y_1)), \mathrm{Lip}(X_2, C(Y_2))))$, there is a sequence $\{T_{F,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Iso}(\mathrm{Lip}(X_1, C(Y_1)), \mathrm{Lip}(X_2, C(Y_2)))$ such that

$$\lim_{n\to\infty} T_{F,n}(F) = T(F).$$

By Theorem 1, for each $n \in \mathbb{N}$ there exist $h_{F,n} \in C(Y_2, \mathbb{T})$, $\varphi_{F,n} \in C(X_2 \times Y_2, X_1)$ with $\varphi^y_{F,n} \in \mathrm{Iso}(X_2, X_1)$ for each $y \in Y_2$, and $\tau_{F,n} \in \mathrm{Homeo}(Y_2, Y_1)$ such that

$$T_{F,n}(G)(x, y) = h_{F,n}(y)G(\varphi_{F,n}(x, y), \tau_{F,n}(y)) \qquad (x, y) \in X_2 \times Y_2,$$

for all $G \in \mathrm{Lip}(X_1, C(Y_1))$, and the claim holds.

Claim 2. *There exists a function $h \in C(Y_2, \mathbb{T})$ such that $T(1_{X_1} \otimes 1_{Y_1}) = 1_{X_2} \otimes h$.*

Applying Claim 1 to $F = 1_{X_1} \otimes 1_{Y_1}$, we deduce that

$$T(F)(x, y) = \lim_{n\to\infty} h_{F,n}(y)F(\varphi_{F,n}(x, y), \tau_{F,n}(y)) = \lim_{n\to\infty} h_{F,n}(y)$$

for all $(x, y) \in X_2 \times Y_2$. Since $\{h_{F,n}\}_{n\in\mathbb{N}} \subseteq C(Y_2, \mathbb{T})$, it follows that $|T(F)(x, y)| = 1$ for all $(x, y) \in X_2 \times Y_2$. Hence $\|T(F)(x)\|_\infty = 1$ for all $x \in X_2$ and thus $\|T(F)\|_\infty = 1$. On the other hand, since

$$\|T(F)\|_\Sigma = \lim_{n\to\infty} \|h_{F,n}F(\varphi_{F,n}, \tau_{F,n})\|_\Sigma = \lim_{n\to\infty} \|F\|_\Sigma = 1,$$

it follows that $\mathrm{Lip}(T(F)) = 0$. Hence $T(F)$ is a constant function from X_2 to $C(Y_2)$ and we conclude that there exists a function $h \in C(Y_2, \mathbb{T})$ such that $T(F) = 1_{X_2} \otimes h$.

Claim 3. *For each $(x, y) \in X_2 \times Y_2$, the functional $S_{(x,y)} : \mathrm{Lip}(X_1, C(Y_1)) \to \mathbb{C}$ defined by*

$$S_{(x,y)}(F) = \overline{h(y)}T(F)(x, y) \qquad (F \in \mathrm{Lip}(X_1, C(Y_1))),$$

is linear, unital and multiplicative.

Fix $(x, y) \in X_2 \times Y_2$. Since T is linear, so is $S_{(x,y)}$. Clearly, $S_{(x,y)}(1_{X_1} \otimes 1_{Y_1}) = 1$ by Claim 2. To prove the multiplicativity of $S_{(x,y)}$, consider the functional $T_{(x,y)} : \mathrm{Lip}(X_1, C(Y_1)) \to \mathbb{C}$ given by

$$T_{(x,y)}(F) = T(F)(x, y) \qquad (F \in \mathrm{Lip}(X_1, C(Y_1))).$$

As $T_{(x,y)}$ is linear and, for all $F \in \mathrm{Lip}(X_1, C(Y_1))$,

$$|T_{(x,y)}(F)| = |T(F)(x, y)| \le \|T(F)\|_\infty \le \|T(F)\|_\Sigma = \|F\|_\Sigma ,$$

we deduce that $T_{(x,y)}$ is continuous. Take now any $F \in \mathrm{Lip}(X_1, C(Y_1))$. We have

$$\lim_{n\to\infty} h_{F,n} F(\varphi_{F,n}, \tau_{F,n}) = T(F),$$

with $\{h_{F,n}\}_{n\in\mathbb{N}}$, $\{\varphi_{F,n}\}_{n\in\mathbb{N}}$ and $\{\tau_{F,n}\}_{n\in\mathbb{N}}$ being sequences as in Claim 1. Since the convergence in the Σ-norm implies pointwise convergence, we infer that

$$T_{(x,y)}(F) = T(F)(x,y) = \lim_{n\to\infty} h_{F,n}(y)F(\varphi_{F,n}(x,y), \tau_{F,n}(y)) \in \mathbb{T}\sigma(F).$$

Finally, an application of [17, Proposition 2.2] yields that $S_{(x,y)} = \overline{T_{(x,y)}(1_{X_1} \otimes 1_{Y_1})}T_{(x,y)}$ is multiplicative.

Claim 4. *There exist two maps $\varphi \in C(X_2 \times Y_2, X_1)$, with $\varphi^y \in \operatorname{Lip}(X_2, X_1)$ for each $y \in Y_2$, and $\tau \in C(X_2 \times Y_2, Y_1)$ such that*

$$T(F)(x,y) = h(y)F(\varphi(x,y), \tau(x,y)) \qquad ((x,y) \in X_2 \times Y_2),$$

for all $F \in \operatorname{Lip}(X_1, C(Y_1))$.

Using Claim 3, it is easily deduced that $S\colon \operatorname{Lip}(X_1, C(Y_1)) \to \operatorname{Lip}(X_2, C(Y_2))$, defined by

$$S(F)(x,y) = \overline{h(y)}T(F)(x,y) \qquad ((x,y) \in X_2 \times Y_2,\ F \in \operatorname{Lip}(X_1, C(Y_1))),$$

is a unital algebra homomorphism. By Theorem 2, there exist two maps $\varphi \in C(X_2 \times Y_2, X_1)$, with $\varphi^y \in \operatorname{Lip}(X_2, X_1)$ for each $y \in Y_2$, and $\tau \in C(X_2 \times Y_2, Y_1)$ such that

$$S(F)(x,y) = F(\varphi(x,y), \tau(x,y)) \qquad ((x,y) \in X_2 \times Y_2),$$

for all $F \in \operatorname{Lip}(X_1, C(Y_1))$.

Claim 5. *For each $y \in Y_2$, $\varphi^y \in \operatorname{Iso}(X_2, X_1)$.*

Fix $y \in Y_2$ and define $T_y\colon \operatorname{Lip}(X_1) \to \operatorname{Lip}(X_2)$ by

$$T_y(f)(x) = T(f \otimes 1_{Y_1})(x,y) \qquad (x \in X_2,\ f \in \operatorname{Lip}(X_1)).$$

By Claim 4, we have

$$T_y(f)(x) = h(y)f(\varphi(x,y)) = h(y)f(\varphi^y(x)) \qquad (x \in X_2,\ f \in \operatorname{Lip}(X_1)).$$

Clearly, $T_y \in B(\operatorname{Lip}(X_1), \operatorname{Lip}(X_2))$ with $\|T_y(f)\|_\infty \le \|f\|_\infty$ and $\operatorname{Lip}(T_y(f)) \le \operatorname{Lip}(f)\operatorname{Lip}(\varphi^y)$ for all $f \in \operatorname{Lip}(X_1)$. By Claim 1, for every $f \in \operatorname{Lip}(X_1)$, there exist three sequences $\{h_{f\otimes 1_{Y_1},n}\}_{n\in\mathbb{N}}$ in $C(Y_2, \mathbb{T})$, $\{\varphi_{f\otimes 1_{Y_1},n}\}_{n\in\mathbb{N}}$ in $C(X_2 \times Y_2, X_1)$ with $\varphi^y_{f\otimes 1_{Y_1},n} \in \operatorname{Iso}(X_2, X_1)$ for all $n \in \mathbb{N}$, and $\{\tau_{f\otimes 1_{Y_1},n}\}_{n\in\mathbb{N}}$ in $\operatorname{Homeo}(Y_2, Y_1)$ such that

$$\lim_{n\to\infty} h_{f\otimes 1_{Y_1},n}(f \otimes 1_{Y_1})(\varphi_{f\otimes 1_{Y_1},n}, \tau_{f\otimes 1_{Y_1},n}) = T(f \otimes 1_{Y_1}).$$

Consequently, we obtain

$$\lim_{n\to\infty} h_{f\otimes 1_{Y_1},n}(y)(f \circ \varphi^y_{f\otimes 1_{Y_1},n}) = T_y(f).$$

For each $n \in \mathbb{N}$, define $T_{y,f,n}\colon \operatorname{Lip}(X_1) \to \operatorname{Lip}(X_2)$ by

$$T_{y,f,n}(g) = h_{f\otimes 1_{Y_1},n}(y)(g \circ \varphi^y_{f\otimes 1_{Y_1},n}) \qquad (g \in \operatorname{Lip}(X_1)).$$

Since $h_{f\otimes 1_{Y_1},n}(y) \in \mathbb{T}$ and $\varphi^y_{f\otimes 1_{Y_1},n} \in \mathrm{Iso}(X_2, X_1)$, notice that $T_{y,f,n} \in \mathrm{Iso}(\mathrm{Lip}(X_1), \mathrm{Lip}(X_2))$ by [10, Corollary 15]. Therefore T_y is an approximate local isometry of $\mathrm{Lip}(X_1)$ to $\mathrm{Lip}(X_2)$. Hence $T_y \in \mathrm{Iso}(\mathrm{Lip}(X_1), \mathrm{Lip}(X_2))$ by Theorem 3. By [10, Corollary 15] again, we can find a number $\alpha_y \in \mathbb{T}$ and a map $\phi_y \in \mathrm{Iso}(X_2, X_1)$ such that

$$T_y(f)(x) = \alpha_y f(\phi_y(x)) \qquad (x \in X_2,\ f \in \mathrm{Lip}(X_1)).$$

In addition, $\alpha_y = T_y(1_{X_1})(x) = h(y)$ where x is any point in X_2, and thus

$$T_y(f)(x) = h(y)f(\phi_y(x)) \qquad (x \in X_2,\ f \in \mathrm{Lip}(X_1)).$$

Therefore we can write

$$h(y)f(\varphi^y(x)) = T_y(f)(x) = h(y)f(\phi_y(x)) \qquad (x \in X_2,\ f \in \mathrm{Lip}(X_1)).$$

Since $\mathrm{Lip}(X_1)$ separates the points of X_1, we conclude that

$$\varphi^y(x) = \phi_y(x) \qquad (x \in X_2)$$

and so $\varphi^y = \phi_y \in \mathrm{Iso}(X_2, X_1)$, as required.

Claim 6. *There exists a map* $\beta \in \mathrm{Homeo}(Y_2, Y_1)$ *such that*

$$\beta(y) = \tau(x, y) \qquad (y \in Y_2),$$

where x *is any point of* X_2.

Let $x \in X_2$ be fixed and define $T_x \colon C(Y_1) \to C(Y_2)$ by

$$T_x(g)(y) = T(1_{X_1} \otimes g)(x, y) \qquad (y \in Y_2,\ g \in C(Y_1)).$$

Claim 4 yields

$$T_x(g)(y) = h(y)g(\tau(x, y)) \qquad (y \in Y_2,\ g \in C(Y_1)).$$

Clearly, $T_x \in B(C(Y_1), C(Y_2))$. For each $g \in C(Y_1)$, Claim 1 asserts the existence of three sequences $\{h_{1_{X_1}\otimes g,n}\}_{n\in\mathbb{N}}$ in $C(Y_2, \mathbb{T})$, $\{\varphi_{1_{X_1}\otimes g,n}\}_{n\in\mathbb{N}}$ in $C(X_2 \times Y_2, X_1)$ and $\{\tau_{1_{X_1}\otimes g,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Homeo}(Y_2, Y_1)$ for which

$$\lim_{n\to\infty} h_{1_{X_1}\otimes g,n}(1_{X_1} \otimes g)(\varphi_{1_{X_1}\otimes g,n}, \tau_{1_{X_1}\otimes g,n}) = T(1_{X_1} \otimes g).$$

Therefore we have

$$\lim_{n\to\infty} h_{1_{X_1}\otimes g,n}(g \circ \tau_{1_{X_1}\otimes g,n}) = T_x(g)$$

with respect to the norm $||\cdot||_\infty$ on $C(Y_2)$. In particular T_x does not depend on x. For each $n \in \mathbb{N}$, define $T_{g,n} \colon C(Y_1) \to C(Y_2)$ by

$$T_{g,n}(f) = h_{1_{X_1}\otimes g,n}(f \circ \tau_{1_{X_1}\otimes g,n}) \qquad (f \in C(Y_1)).$$

Since $h_{1_{X_1}\otimes g,n} \in C(Y_2, \mathbb{T})$ and $\tau_{1_{X_1}\otimes g,n} \in \mathrm{Homeo}(Y_2, Y_1)$, $T_{g,n} \in \mathrm{Iso}(C(Y_1), C(Y_2))$. Hence T_x is an approximate local isometry of $C(Y_1)$ to $C(Y_2)$. By hypothesis, it follows that $T_x \in \mathrm{Iso}(C(Y_1), C(Y_2))$. Now, the Banach–Stone theorem provides a function $\alpha \in C(Y_2, \mathbb{T})$ and a map $\beta \in \mathrm{Homeo}(Y_2, Y_1)$ such that

$$T_x(g)(y) = \alpha(y)g(\beta(y)) \qquad (y \in Y_2,\ g \in C(Y_1)).$$

In fact, $\alpha(y) = T_x(1_{Y_1})(y) = h(y)$ for all $y \in Y_2$, and therefore

$$T_x(g)(y) = h(y)g(\beta(y)) \qquad (y \in Y_2,\ g \in C(Y_1)).$$

Consequently, we have

$$h(y)g(\beta(y)) = T_x(g)(y) = h(y)g(\tau(x, y)) \qquad (g \in C(Y_1),\ y \in Y_2).$$

Since $C(Y_1)$ separates the points of Y_1, we infer that

$$\beta(y) = \tau(x, y) \qquad (y \in Y_2).$$

This proves Claim 6. Now, the proof of Theorem 6 is complete. □

A proof similar to that of Corollary 2 but applying now Theorem 6 instead of Theorem 3 gives the following.

Corollary 3. *Let X_1, X_2 be compact metric spaces and Y_1, Y_2 be Hausdorff compact spaces. Suppose that* $\mathrm{Iso}(C(Y_1), C(Y_2))$ *is topologically reflexive. Then* $\mathrm{Iso}(\mathrm{Lip}(X_1, C(Y_1)), \mathrm{Lip}(X_2, C(Y_2)))$ *is 2-topologically reflexive.* □

Proof. Let $\Delta \in 2\text{-ref}_{\mathrm{top}}(\mathrm{Iso}(\mathrm{Lip}(X_1, C(Y_1)), \mathrm{Lip}(X_2, C(Y_2))))$. For each $(x, y) \in X_2 \times Y_2$, we claim that the functional $\Delta_{(x,y)} \colon \mathrm{Lip}(X_1, C(Y_1)) \to \mathbb{C}$ defined by

$$\Delta_{(x,y)}(F) = \Delta(F)(x, y) \qquad (F \in \mathrm{Lip}(X_1, C(Y_1))),$$

is linear. By [17, Proposition 3.2], it is sufficient to check that $\Delta_{(x,y)}$ is 1-homogeneous and

$$\Delta_{(x,y)}(F) - \Delta_{(x,y)}(G) \in \mathbb{T}\sigma(F - G), \qquad \forall F, G \in \mathrm{Lip}(X_1, C(Y_1)).$$

Both properties of $\Delta_{(x,y)}$ can be proved as in the proof of Corollary 2 for Δ_y. This proves our claim. By the arbitrariness of (x, y), we infer that Δ is linear. Consequently,

$$\Delta \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(\mathrm{Lip}(X_1, C(Y_1)), \mathrm{Lip}(X_2, C(Y_2)))).$$

Hence $\Delta \in \mathrm{Iso}(\mathrm{Lip}(X_1, C(Y_1)), \mathrm{Lip}(X_2, C(Y_2)))$ by Theorem 6, and this finishes the proof. □

In the proof of Theorem 6, we have used Theorems 1, 2 and 3. Note that Theorem 1 is also valid for surjective linear isometries between $\mathrm{lip}(X^\alpha, C(Y))$ spaces (see Remark 1), Theorem 2 can be established with an analogous proof for algebra homomorphisms between such spaces $\mathrm{lip}(X^\alpha, C(Y))$, and, finally, Theorem 5 shows that the same conclusion as that of Theorem 3 holds for $\mathrm{lip}(X^\alpha)$-spaces. Hence a proof similar to that of Theorem 6 yields the topological reflexivity of the isometry group between the spaces $\mathrm{lip}((X_1)^\alpha, C(Y_1))$ and $\mathrm{lip}((X_2)^\alpha, C(Y_2))$. Using this fact, it is deduced the 2-topological reflexivity of this isometry group in the same way as that of the proof of Corollary 3. Thus we have the following result.

Theorem 7. *Let X_1, X_2 be compact metric spaces, $\alpha \in (0,1)$ and Y_1, Y_2 be Hausdorff compact spaces. Suppose that $\mathrm{Iso}(C(Y_1), C(Y_2))$ is topologically reflexive. Then the set*

$$\mathrm{Iso}(\mathrm{lip}((X_1)^\alpha, C(Y_1)), \mathrm{lip}((X_2)^\alpha, C(Y_2)))$$

is topologically reflexive and 2-topologically reflexive. □

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