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Varying discrete Laguerre–Sobolev orthogonal polynomials: asymptotic behavior and zeros

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Abstract

We consider a varying discrete Sobolev inner product involving the Laguerre weight. Our aim is to study the asymptotic properties of the corresponding orthogonal polynomials and of their zeros. We are interested in Mehler–Heine type formulas because they describe the asymptotic differences between these Sobolev orthogonal polynomials and the classical Laguerre polynomials. Moreover, they give us an approximation of the zeros of the Sobolev polynomials in terms of the zeros of other special functions. We generalize some results appeared very recently in the literature for both the varying and non–varying cases.

Keywords: Laguerre–Sobolev orthogonal polynomials, Mehler–Heine formulae, Asymptotics, Zeros.
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1. Introduction

In this paper we deal with sequences of polynomials orthogonal with respect to a varying Sobolev inner product involving the Laguerre weight $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, on the real nonnegative semiaxis $[0, +\infty)$. More precisely, we consider the inner product

$$(f,g)_n = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + M_n f^{(j)}(0)g^{(j)}(0), \quad j \geq 0, \tag{1}$$

with $\alpha > -1$ and where $\{M_n\}_n$ is a sequence of nonnegative numbers satisfying

$$\lim_{n\to\infty} M_n n^\beta = M > 0, \quad \text{with} \quad \beta \in \mathbb{R}. \tag{2}$$

This inner product generalizes one considered in [5] and [7], i.e., for $\alpha > -1$,

$$(f,g) = \int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + Nf^{(j)}(0)g^{(j)}(0), \quad j,\, N \geq 0. \tag{3}$$

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Thus, we will recover the results appearing in those papers when $\{M_n\}_n$ is a constant sequence. Note that for $M_n = N/\Gamma(\alpha+1)$, for all n, we have

$$\Gamma(\alpha+1)(f,g)_n = (f,g),$$

and it is necessary to take this into account for the technical details.

Moreover, we want to give a qualitative interpretation of the asymptotic behavior of the orthogonal polynomials with respect to (1) in this general case. In such a sense, we prove that the *size* of the sequence $\{M_n\}_n$ has an influence on the asymptotic behavior of the orthogonal polynomials with respect to (1), but this influence is only local, that is, around the point where we have introduced the perturbation. In our case, this point is located at the origin. Thus, denoting by $L_n^{(\alpha)}(x)$ the classical Laguerre polynomials and by $L_n^{(\alpha,M_n)}(x)$ the orthogonal polynomials with respect to (1), first we will prove that

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M_n)}(x)}{L_n^{(\alpha)}(x)} = 1, \tag{4}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,\infty)$. When $M_n = M$ for all n, (4) was already observed by several authors (see, for example, [1]). Then, we focus our attention on the local asymptotic behavior to find the differences between both sequences of orthogonal polynomials. In fact, we focus our attention on the limit behavior of the ratio

$$\frac{L_n^{(\alpha,M_n)}(x/n)}{n^\alpha}, \quad \text{when} \quad n\to\infty,$$

and we will describe how the size of the sequence $\{M_n\}_n$ influences on the local asymptotics, i.e., essentially we have three possible cases: one of them is when the size of $\{M_n\}_n$ is negligible and therefore the Mehler–Heine type asymptotics for $\{L_n^{(\alpha,M_n)}\}_n$ and $\{L_n^{(\alpha)}\}_n$ are the same; another one is when the size of $\{M_n\}_n$ influences on the asymptotics; and in the third one we will prove that it is a convex combination of the two other cases. Thus, we generalize the results obtained in [3] and [4] for particular cases.

We also analyze the zeros of the polynomials $L_n^{(\alpha,M_n)}(x)$ and their asymptotic behavior as a consequence of the Mehler–Heine type formula.

According to our objectives, the structure of the paper is the following. In Section 2, we introduce the varying Laguerre–Sobolev type orthogonal polynomials and their basic properties. In Section 3, we provide our main results about the asymptotics of the polynomials $L_n^{(\alpha,M_n)}(x)$. Finally, Section 4 is devoted to the zeros of $L_n^{(\alpha,M_n)}(x)$, as well as we show some numerical computations for illustrating the results previously obtained.

2. Laguerre–Sobolev type orthogonal polynomials: the varying case

We consider the nonstandard and varying inner product

$$(f,g)_n = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + M_n f^{(j)}(0)g^{(j)}(0), \quad j\geq 0, \tag{5}$$

with $\alpha > -1$, and where $\{M_n\}_n$ is a sequence of nonnegative numbers such that

$$\lim_{n\to\infty} M_n n^{\beta} = M > 0, \quad \beta \in \mathbb{R}. \tag{6}$$

This inner product is nonstandard because $(xf, g)_n \neq (f, xg)_n$, and thus the nice properties (three–term recurrence relation, Christoffel–Darboux formula, etc) that we can deduce for standard orthogonal polynomials do not hold for the orthogonal polynomials with respect to (5). We should pay attention to the following fact: denoting by $L_n^{(\alpha,M_n)}(x)$ the orthogonal polynomials with respect to (5), then $(L_n^{(\alpha,M_n)}, x^i)_n = 0$, for $i = 0, \ldots, n-1$, but $(L_n^{(\alpha,M_n)}, x^i)_{n-1}$ may be different from zero. In fact, for a sequence $\{M_n\}_n$ we have a sequence of orthogonal polynomials for each n, so we have a square tableau $\{L_k^{(\alpha,M_n)}\}_k$. Here, we treat with the diagonal of this tableau, i.e. $\{L_n^{(\alpha,M_n)}\}_n = \{L_0^{(\alpha,M_0)}, L_1^{(\alpha,M_1)}, \ldots, L_i^{(\alpha,M_i)}, \ldots\}$.

When $M_n = 0$ for all n, the inner product (5) becomes the Laguerre inner product whose orthogonal polynomials are denoted by $L_n^{(\alpha)}(x)$. We choose the same normalization for both sequences of orthogonal polynomials $\{L_n^{(\alpha,M_n)}\}_n$ and $\{L_n^{(\alpha)}(x)\}_n$. The leading coefficient of the polynomial of degree n in each family is $(-1)^n/n!$.

It is easy to observe that $L_n^{(\alpha,M_n)}(x) = L_n^{(\alpha)}(x)$ for $n = 0, \ldots, j-1$. A first step to get asymptotic properties is to obtain an adequate expression of the polynomials $L_n^{(\alpha,M_n)}(x)$ in terms of the classical Laguerre polynomials, i.e., to solve the connection problem. When $\{M_n\}_n$ is a constant sequence, this problem was solved in [6] where the author introduced Sobolev type orthogonal polynomials involving more derivatives. Very recently, in [7] the authors have given the explicit expression of those coefficients. Now, we rewrite Theorem 1 in [7] for the varying case.

Proposition 1. *We assume $L_{-1}^{(\alpha)}(x) \equiv 0$, and $\alpha > -1$. We have, for every $n \geq j$,*

$$L_n^{(\alpha,M_n)}(x) = L_n^{(\alpha)}(x) + \sum_{k=1}^{j+1} B_{n,k}^{[j]} L_{n-k}^{(\alpha+k)}(x),$$

where $B_{n,k}^{[j]} = \dfrac{A_{n,k}^{[j]}}{A_{n,0}^{[j]}}$, with

$$A_{n,k}^{[j]} = \frac{(-1)^k j!\Gamma(\alpha+1)M_n}{\Gamma(\alpha+j+1)} \binom{n+\alpha}{n-j} \binom{n-k}{j+1-k} \quad k = 1, \ldots, j+1,$$

$$A_{n,0}^{[j]} = 1 + \frac{j!\Gamma(\alpha+1)M_n}{\Gamma(\alpha+j+1)} \sum_{k=1}^{\min\{n-j,j+1\}} \binom{n+\alpha}{n-j-k} \binom{n-k}{j+1-k}.$$

The proof can be followed from Theorem 1 in [7] taking into account the relation between the inner products (1) and (3), as we have commented in the introduction, and the fact that we have the same leading coefficients for both families of orthogonal polynomials. We would like to remark that we have expressed the coefficients in such a way that they can be used directly on the computer directly.

Now, we give the asymptotics of the coefficients $B_{n,k}^{[j]}$ in the above connection formula when $n \to \infty$.

Proposition 2. *We have,*

$$\lim_{n\to\infty} n^k B_{n,k}^{[j]} =$$
$$\begin{cases} 0, & \text{if } \beta > 2j+\alpha+1, \\ \dfrac{(-1)^k\Gamma(j+1)\Gamma(\alpha+1)(\alpha+2j+1)M}{\Gamma(j-k+2)\left(\Gamma^2(\alpha+j+1)(\alpha+2j+1)+M\Gamma(\alpha+1)\right)}, & \text{if } \beta = 2j+\alpha+1, \\ \dfrac{(-1)^k\Gamma(j+1)}{\Gamma(j-k+2)}, & \text{if } \beta < 2j+\alpha+1. \end{cases}$$

Proof. From Proposition 1 we have

$$B_{n,k}^{[j]} = \frac{(-1)^k j!\Gamma(\alpha+1)M_n\binom{n+\alpha}{n-j}\binom{n-k}{j+1-k}}{\Gamma(\alpha+j+1)+j!\Gamma(\alpha+1)M_n\sum_{k=1}^{\min\{n-j,j+1\}}\binom{n+\alpha}{n-j-k}\binom{n-k}{j+1-k}},$$

On the other hand, using the well–known Stirling's formula (see, for example, [2, f. (5.11.13)]) we have the ratio asymptotics

$$\lim_{n\to\infty}\frac{n^{b-a}\Gamma(n+a)}{\Gamma(n+b)} = 1. \tag{7}$$

Then, using adequately (6) and (7) in the above expression of $B_{n,k}^{[j]}$ and after some technical computations we deduce the result. □

3. Mehler–Heine type asymptotics versus outer strong asymptotics

In the previous section we have introduced the tools to tackle with the asymptotics. Now, first we will prove that the polynomials $L_n^{(\alpha,M_n)}(x)$ and $L_n^{(\alpha)}(x)$ have the same outer asymptotics.

Proposition 3. *We have,*

$$\lim_{n\to\infty}\frac{L_n^{(\alpha,M_n)}(x)}{L_n^{(\alpha)}(x)} = 1,$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,\infty)$.

Proof. From the outer asymptotics for classical Laguerre polynomials $L_n^{(\alpha)}(x)$ (see [9, Th.8.22.3]), we can deduce the following relation

$$\lim_{n\to\infty} n^{(\ell-j)/2}\frac{L_{n+k}^{(\alpha+j)}(x)}{L_{n+h}^{(\alpha+\ell)}(x)} = (-x)^{(\ell-j)/2}, \quad j,\ell\in\mathbb{R}, \quad h,k\in\mathbb{Z}, \tag{8}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,\infty)$. Then, using Proposition 1 we can write for n large enough

$$\frac{L_n^{(\alpha,M_n)}(x)}{L_n^{(\alpha)}(x)} = 1+\sum_{k=1}^{j+1} B_{n,k}^{[j]}\frac{L_{n-k}^{(\alpha+k)}(x)}{L_n^{(\alpha)}(x)} = 1+\sum_{k=1}^{j+1} n^k B_{n,k}^{[j]}\frac{L_{n-k}^{(\alpha+k)}(x)}{n^{k/2}L_n^{(\alpha)}(x)}\frac{1}{n^{k/2}}.$$

Taking limits when $n \to \infty$ in the above expression and using Proposition 2 and (8) we get the result. $\square$

We want to know how the discrete part in the inner product (5) influences on the asymptotic behavior of the corresponding orthogonal polynomials and how the size of the sequence $\{M_n\}_n$ also influences on it, and in this way we generalize the results appearing in [3] and [4]. From the above proposition we have that the perturbation introduced in the classical Laguerre inner product does not affect the asymptotic behavior of the corresponding orthogonal polynomials $L_n^{(\alpha,M_n)}$ at least on compact subsets of $\mathbb{C} \setminus [0,\infty)$. But, what happens around the origin? The answer is given by the local asymptotics known as Mehler–Heine asymptotics.

Theorem 1. *Let $\alpha > -1$ and let $\{M_n\}_n$ a sequence of nonnegative numbers satisfying (6). We have,*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M_n)}(x/n)}{n^\alpha} = \begin{cases} d_\alpha(x), & \text{if } \beta < 2j+\alpha+1, \\ \lambda d_\alpha(x) + (1-\lambda)c_{\alpha,0}(x), & \text{if } \beta = 2j+\alpha+1, \\ c_{\alpha,0}(x), & \text{if } \beta > 2j+\alpha+1, \end{cases}$$

uniformly on compact subsets of $\mathbb{C}$, with

$$\lambda = \frac{M\Gamma(\alpha+1)}{(\alpha+2j+1)\Gamma^2(\alpha+j+1) + M\Gamma(\alpha+1)},$$

where J_α is the Bessel function of the first kind, and

$$\begin{aligned} c_{\alpha,k}(x) &= x^{-(\alpha+k)/2} J_{\alpha+k}(2\sqrt{x}), \\ d_\alpha(x) &= c_{\alpha,0}(x) + (\alpha+2j+1)\Gamma(j+1) \sum_{k=1}^{j+1} \frac{(-1)^k}{\Gamma(j-k+2)} c_{\alpha,k}(x). \end{aligned}$$

Proof. Scaling the variable $x \mapsto x/n$ in Proposition 1, we get for $n \geq j$,

$$\begin{aligned} \frac{L_n^{(\alpha,M_n)}(x/n)}{n^\alpha} &= \frac{L_n^{(\alpha)}(x/n)}{n^\alpha} + \sum_{k=1}^{j+1} B_{n,k}^{[j]} \frac{L_{n-k}^{(\alpha+k)}(x/n)}{n^\alpha} \\ &= \frac{L_n^{(\alpha)}(x/n)}{n^\alpha} + \sum_{k=1}^{j+1} n^k B_{n,k}^{[j]} \frac{L_{n-k}^{(\alpha+k)}(x/n)}{(n-k)^{\alpha+k}} \left(\frac{n-k}{n}\right)^{\alpha+k}. \end{aligned} \tag{9}$$

On the other hand, we can extend slightly the Mehler–Heine formula for Laguerre polynomials appearing in [9, p.193] and we obtain

$$\lim_{n\to\infty} \frac{L_n^{(\alpha)}(x/(n+j))}{n^\alpha} = x^{-\alpha/2} J_\alpha(2\sqrt{x}), \tag{10}$$

uniformly on compact subsets of $\mathbb{C}$, and uniformly in $j \in \mathbb{Z}$.

It only remains to apply Proposition 2 and (10) in (9), and after some computations and simplifications we deduce the result. $\square$

Remark. These results obviously recover the particular ones obtained in [3] and [4] for $j = 0$ and $j = 1$, respectively. The key to solve the general case is Proposition

2 given very recently in [7] for the constant case. In this sense, we highlight the importance of solving the adequate connection problem to get the asymptotic results. Thus, the connection problems considered in those papers are not adequate for the general case. To get the results given in [3] and [4] from Theorem 1, it is useful the well–known relation for Bessel functions of the first kind

$$J_\alpha(2\sqrt{x}) - \frac{\alpha+1}{\sqrt{x}} J_{\alpha+1}(2\sqrt{x}) = -J_{\alpha+2}(2\sqrt{x}).$$

On the other hand, Proposition 3 and Theorem 1 recover all the results obtained for the constant case in [5]. We obtain more general results using easier techniques than in [5].

Remark. According to Theorem 1 the *transition case* (convex linear combination of the two other cases) appears when $M_n \asymp n^{2j+\alpha+1}$. This was conjectured in [4].

4. Zeros and numerical simulation

In this Section we provide some results about the zeros of $L_n^{(\alpha,M_n)}$ and using the *Mathematica* software we compute them up to degree 600 in an efficient and stable way.

We can rewrite literally Theorem 4.1 in [8] for the varying case obtaining the following result.

Proposition 4. *The polynomial* $L_n^{(\alpha,M_n)}$ *has* n *real and simple zeros and at most one of them is in* $(-\infty, 0]$.

Notice that M_n could be 0 for some n, then the corresponding orthogonal polynomial is the classical Laguerre polynomials and therefore all the zeros are real, simple, and positive.

Now, we are looking for a negative zero of the limit functions $c_{\alpha,0}$, d_α, and $\lambda d_\alpha + (1-\lambda)c_{\alpha,0}$ in Theorem 1.

Lemma 1. *We have*

(a) The functions $c_{\alpha,k}$*, for all* $k \in \mathbb{N} \cup \{0\}$*, do not have negative real zeros.*

(b) The function d_α *has exactly one negative real zero for* $j \geq 1$*. When* $j = 0$*,* d_α *has a zero at the origin.*

(c) $\lambda d_\alpha + (1-\lambda)c_{\alpha,0}$ *has one zero in* $(-\infty, 0]$ *if and only if*

$$M \geq \frac{(1+\alpha+2j)(1+\alpha+j)\Gamma^2(1+\alpha+j)}{\Gamma(\alpha+1)j}, \quad j \geq 1.$$

For $j = 0$*,* $\lambda d_\alpha + (1-\lambda)c_{\alpha,0}$ *has only positive real zeros.*

Proof. (a) Since $c_{\alpha,k} = x^{-(\alpha+k)/2} J_{\alpha+k}(2\sqrt{x})$, our statement follows from a very well–known result in the theory of Bessel functions (see, for example, [9]).

(b) When $j = 0$, $d_\alpha(x) = -x\, c_{\alpha,2}(x)$ and the result follows. Now, let us consider $j \geq 1$. First, we prove that d_α has a zero in $(-\infty, 0)$. Using the explicit expression of Bessel function of the first kind

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha},$$

we get

$$\begin{aligned} d_\alpha(0) &= \frac{1}{\Gamma(\alpha+1)} + \sum_{k=1}^{j+1} \frac{(-1)^k(\alpha+2j+1)\Gamma(j+1)}{\Gamma(j-k+2)\Gamma(\alpha+k+1)} \\ &= -\frac{j}{(\alpha+j+1)\Gamma(\alpha+2)} < 0. \end{aligned}$$

On the other hand, after some computations we have

$$\begin{aligned} d_\alpha(x) &= \sum_{i=0}^{\infty} \frac{(-x)^i}{i!\Gamma(i+\alpha+1)} \\ &+ (\alpha+2j+1)\Gamma(j+1) \sum_{k=1}^{j+1} \frac{(-1)^k}{\Gamma(j-k+2)} \sum_{i=0}^{\infty} \frac{(-x)^i}{i!\Gamma(i+\alpha+k+1)} \\ &= \sum_{i=0}^{\infty} \frac{(-x)^i}{i!\Gamma(i+\alpha+1)} \Bigg(1+ \\ &\quad (\alpha+2j+1)\Gamma(j+1) \sum_{k=1}^{j+1} \frac{(-1)^k}{\Gamma(j-k+2)\prod_{c=1}^{k}(i+\alpha+c)}\Bigg) \\ &= \sum_{i=0}^{\infty} \frac{(-x)^i}{i!\Gamma(i+\alpha+1)} \left(1 - \frac{\alpha+2j+1}{i+\alpha+j+1}\right) \\ &= \sum_{i=0}^{j+1} \frac{(-x)^i}{i!\Gamma(i+\alpha+1)} \left(1 - \frac{\alpha+2j+1}{i+\alpha+j+1}\right) \\ &+ \sum_{i=j+2}^{\infty} \frac{(-x)^i}{i!\Gamma(i+\alpha+1)} \left(1 - \frac{\alpha+2j+1}{i+\alpha+j+1}\right). \end{aligned}$$

Taking $x < 0$, we can observe that the first term in the above expression is a polynomial of degree $j+1$ whose leading coefficient is positive, and the second term is a series where all the terms are positive. Then, $\lim_{x\to-\infty} d_\alpha(x) = +\infty$. Since $d_\alpha(x)$ is a continuous function, if we gather this limit with the fact that $d_\alpha(0) < 0$, we deduce that $d_\alpha(x)$ has at least one zero in $(-\infty, 0)$. Finally, applying the well–known Hurwitz's Theorem (see [9, p.22]), Proposition 4, and Theorem 1 we have that $d_\alpha(x)$ has at most one zero in $(-\infty, 0)$. Then, the result follows.

(c) For $j = 0$, (5) is a varying standard inner product and so the zeros of the polynomial $L_n^{(\alpha,M_n)}$ are real, simple, and positive. Using this together with the fact that $\lambda d_\alpha(0) + (1-\lambda)c_{\alpha,0}(0) = \left(1 - \frac{M}{M+\Gamma(\alpha+2)}\right)\frac{1}{\Gamma(\alpha+1)} > 0$, it is enough to apply Hurwitz's Theorem in Theorem 1 to get the result.

For $j \geq 1$, we proceed like in (b). Thus, it will be enough to prove that the continuous function $\lambda d_\alpha + (1-\lambda)c_{\alpha,0}$ has a non–positive zero under the restriction considered. On the one hand, since $\lambda \in (0,1)$ we get

$$\lim_{x\to-\infty} \lambda d_\alpha + (1-\lambda)c_{\alpha,0} = +\infty. \tag{11}$$

On the other hand, after some computations we obtain

$$\begin{aligned}\lambda d_\alpha(0) + (1-\lambda)c_{\alpha,0}(0) &= \frac{1}{\Gamma(\alpha+1)} \\ &- \frac{(\alpha+2j+1)M}{(\alpha+j+1)(M\Gamma(\alpha+1)+(\alpha+2j+1)\Gamma^2(\alpha+j+1))}.\end{aligned}$$

This expression is less than or equal to 0 if and only if

$$M \geq \frac{(\alpha+2j+1)(\alpha+j+1)\Gamma^2(\alpha+j+1)}{\Gamma(\alpha+1)j},$$

and the result follows. □

The perturbation introduced in the classical Laguerre inner product to obtain the varying Laguerre–Sobolev inner product (5) does not influence on the outer strong asymptotics as we can deduce from Proposition 3. However, that perturbation does influence on the local asymptotics as we have proved in Theorem 1 and it depends on the size of the sequence of $\{M_n\}_n$. Therefore, via Hurwitz's Theorem, the Mehler–Heine type formulae given in Theorem 1 provide us with a detailed information about the asymptotic behavior of the zeros of $L_n^{(\alpha,M_n)}$. In fact, we have

Proposition 5. *Let $s_{n,1} < s_{n,2} < \ldots < s_{n,n}$ the zeros of $L_n^{(\alpha,M_n)}$. Then,*

(a) If $\beta < 2j+\alpha+1$,

$$\lim_{n\to\infty} n s_{n,i} = d_{\alpha,i}$$

where $d_{\alpha,i}$ denotes the i–th real zero of the function d_α.

(b) If $\beta = 2j+\alpha+1$,

$$\lim_{n\to\infty} n s_{n,i} = t_{\alpha,i}$$

where $t_{\alpha,i}$ denotes the i–th real zero of the function $\lambda d_\alpha + (1-\lambda)c_{\alpha,0}$.

(c) If $\beta > 2j+\alpha+1$,

$$\lim_{n\to\infty} n s_{n,i} = \frac{j_{\alpha,i}^2}{4}$$

where $j_{\alpha,i}$ are the positive zeros of J_α.

Proof. We deduce the result from Theorem 1 applying Hurwitz's Theorem and Lemma 1. □

The Mehler–Heine type formulae given in Theorem 1 are specially adequate to describe the smallest zeros of $L_n^{(\alpha,M_n)}$. Thus, using the powerful software *Mathematica* we have computed the first four scaled zeros of $L_n^{(\alpha,M_n)}$ up to degree 600. In the following tables, we show some of these numerical experiments paying attention to different cases given in Proposition 5 and Lemma 1. In all the tables, we have taken

$$M_n = \frac{M}{n^\beta}.$$

Tables 1 and 6 correspond to the case (a) and (c) in Proposition 5, respectively. Tables 2, 3, 4, and 5 correspond to the case (b) in Proposition 5, but taking into account the different cases given in Lemma 1 for the zeros of the limit function $\lambda d_\alpha + (1-\lambda)c_{\alpha,0}$. Table 2 (Table 5) shows the case when the first zero of $\lambda d_\alpha + (1-\lambda)c_{\alpha,0}$ is negative (positive). Tables 3 and 4 illustrate the case when this first zero of the limit function is 0. In Table 3, the zeros of $L_n^{(\alpha,M_n)}$ are on the right side of 0, and in Table 4 are on the left side. The results are showed with six decimal digits, but more enough precision is obtained in the numerical experiments.

Table 1: Case $\beta < 2j+\alpha+1,\ \beta = 2/3,\ j=3,\ \alpha=1,\ M=10$

	$ns_{n,1}$	$ns_{n,2}$	$ns_{n,3}$	$ns_{n,4}$
$n=50$	-16.499895	5.649929	17.916242	35.758173
$n=150$	-15.941311	5.787543	18.270102	36.342549
$n=300$	-15.808299	5.823032	18.362221	36.498456
$n=600$	-15.742730	5.840946	18.408856	36.578796
Limit	$\mathbf{d_{1,1}} = -15.677791$	$\mathbf{d_{1,2}} = 5.858974$	$\mathbf{d_{1,3}} = 18.455882$	$\mathbf{d_{1,4}} = 36.658511$

Table 2: Case $\beta = 2j+\alpha+1,\ \beta = 4.5,\ j=2, \alpha=-0.5,\ M=6$

	$ns_{n,1}$	$ns_{n,2}$	$ns_{n,3}$	$ns_{n,4}$
$n=5$	0.361472	4.013769	13.273140	29.911532
$n=25$	0.045234	3.322179	12.692342	27.484004
$n=50$	-0.004229	3.265756	12.708104	27.510178
$n=150$	-0.037898	3.232452	12.729267	27.562443
$n=300$	-0.046386	3.224667	12.735899	27.579910
$n=600$	-0.050639	3.220855	12.739418	27.589310
Limit	$\mathbf{t_{-0.5,1}} = -0.054898$	$\mathbf{t_{-0.5,2}} = 3.217098$	$\mathbf{t_{-0.5,3}} = 12.743072$	$\mathbf{t_{-0.5,4}} = 27.599156$

Table 3: Case $\beta = 2j+\alpha+1,\ \beta = 4.5,\ j=2,\ \alpha=-0.5,\ M=\frac{405\sqrt{\pi}}{128}$

	$ns_{n,1}$	$ns_{n,2}$	$ns_{n,3}$	$ns_{n,4}$
$n=50$	0.046939	3.366935	12.798708	27.592451
$n=150$	0.015775	3.332807	12.818347	27.643538
$n=300$	0.007902	3.324785	12.824596	27.660707
$n=600$	0.004400	3.320850	12.827923	27.669958
Limit	$\mathbf{t_{-0.5,1}} = 0$	$\mathbf{t_{-0.5,2}} = 3.316967$	$\mathbf{t_{-0.5,3}} = 12.831384$	$\mathbf{t_{-0.5,4}} = 27.679654$

Table 4: Case $\beta = 2j + \alpha + 1,\ \beta = 10.5,\ j = 3,\ \alpha = 3.5,\ M = \frac{18261468225\sqrt{\pi}}{4096}$

	$ns_{n,1}$	$ns_{n,2}$	$ns_{n,3}$	$ns_{n,4}$
$n = 50$	-0.768161	18.368255	37.336214	61.058768
$n = 150$	-0.279965	19.080219	38.602105	62.958454
$n = 300$	-0.142968	19.270878	38.942083	63.474855
$n = 600$	-0.072230	19.368235	39.115863	63.739980
Limit	$\mathbf{t_{7/2,1}} = 0$	$\mathbf{t_{7/2,2}} = 19.466977$	$\mathbf{t_{7/2,3}} = 39.292242$	$\mathbf{t_{7/2,4}} = 64.009468$

Table 5: Case $\beta = 2j + \alpha + 1,\ \beta = 4.5,\ j = 2,\ \alpha = -0.5,\ M = 5$

	$ns_{n,1}$	$ns_{n,2}$	$ns_{n,3}$	$ns_{n,4}$
$n = 50$	0.123833	3.536847	12.953595	27.733338
$n = 150$	0.096668	3.502138	12.971011	27.782683
$n = 300$	0.089784	3.493918	12.976693	27.799408
$n = 600$	0.086323	3.489870	12.979735	27.808435
Limit	$\mathbf{t_{-0.5,1}} = 0.082864$	$\mathbf{t_{-0.5,2}} = 3.485872$	$\mathbf{t_{-0.5,3}} = 12.982911$	$\mathbf{t_{-0.5,4}} = 27.817907$

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Table 6: Case $\beta > 2j + \alpha + 1,\ \beta = 20,\ j = 5,\ \alpha = 3,\ M = 32$

	$ns_{n,1}$	$ns_{n,2}$	$ns_{n,3}$	$ns_{n,4}$
$n = 50$	9.789487	22.922933	40.778399	63.406163
$n = 150$	10.043227	23.508339	41.798630	64.950778
$n = 300$	10.109353	23.662250	42.070188	65.368620
$n = 600$	10.142840	23.740411	42.208618	65.582661
Limit	$\frac{\mathbf{j_{3,1}^2}}{\mathbf{4}} = 10.176616$	$\frac{\mathbf{j_{3,2}^2}}{\mathbf{4}} = 23.819393$	$\frac{\mathbf{j_{3,3}^2}}{\mathbf{4}} = 42.348862$	$\frac{\mathbf{j_{3,4}^2}}{\mathbf{4}} = 65.800214$