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East Asian Journal on Applied Mathematics

2023



Galina Filipuk, Juan F. Mañas-Mañas, Juan J. Moreno-Balcázar. *Second-Order Difference Equation for Sobolev-Type Orthogonal Polynomials. Part II: Computational Tools*, East Asian J. Appl. Math. **13** (2023), 960-979.

https://doi.org/10.4208/eajam.2022-235.190223

https://global-sci.org/intro/article_detail/eajam/22070.html

In addition, the program developed in this article can be downloaded freely from the web https://w3.ual.es/GruposInv/Tapo/SODE.nb or used freely on the Wolfram Notebook Archive website: https://notebookarchive.org/second-order-difference-equation-for-sobolev-type-orthogonal-polynomials-part-ii-computational-tools--2024-03-4mfbsuv/

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Galina Filipuk, Juan F. Manas-Manas, Juan J. Moreno-Balcazar, "Second-Order Difference Equation for Sobolev-Type Orthogonal Polynomials. Part II: Computational Tools" from the Notebook Archive (2024), https://notebookarchive.org/2024-03-4mfbsuv

# Second-Order Difference Equation for Sobolev-Type Orthogonal Polynomials. Part II: Computational Tools

Galina Filipuk[1], Juan F. Mañas-Mañas[2,*] and Juan J. Moreno-Balcázar[2,3]

March 18, 2024

[a]Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, Warsaw, 02-097, Poland.
[b]Departamento de Matemáticas, Universidad de Almería, Spain.
[c]Instituto Carlos I de Física Teórica y Computacional, Spain.

*E-mail addresses:* (filipuk@mimuw.edu.pl) G. Filipuk, (jmm939@ual.es) J.F. Mañas–Mañas, (balcazar@ual.es) J.J. Moreno–Balcázar.

**Abstract**

We consider polynomials which are orthogonal with respect to a nonstandard inner product. In fact, we deal with Sobolev-type orthogonal polynomials in the broad sense of the expression. This means that the inner product under consideration involves the Hahn difference operator, thus including the difference operators $\mathscr{D}_q$ and $\Delta$ and, as a limit case, the derivative operator. In a previous work, we studied properties of these polynomials from a theoretical point of view. There, we obtained a second-order differential/difference equation satisfied by these polynomials. The aim of this paper is to present an algorithm and a symbolic computer program that provides us with the coefficients of the second-order differential/difference equation in this general context. To illustrate both, the algorithm and the program, we will show three examples related to different operators.

**Keywords:** Sobolev orthogonal polynomials, second-order difference equation, symbolic computation.
**Mathematics Subject Classification (2020):** Mathematics Subject Classification 2020: 33C47, 42C05, 34A05.

## 1 Introduction

In this paper we tackle the problem of computing symbolically the coefficients of the second-order differential/difference equation satisfied by the monic orthogonal polynomials $Q_n(x)$ with respect to the general discrete Sobolev-type inner product

$$(f,g)_S = \int f(x)g(x)\varrho(x)dx + M\mathscr{D}_{q,\omega}^{(j)}f(c)\mathscr{D}_{q,\omega}^{(j)}g(c), \tag{1}$$

where $\varrho(x)$ is a weight function on the real line, $c$ is located on the real axis, $M>0$, $j$ is a nonnegative integer, and $\mathscr{D}_{q,\omega}$ is the operator introduced by Hahn [5, Eq. (1.3)] defined by

$$\mathscr{D}_{q,\omega}f(x) = \begin{cases} \dfrac{f(qx+\omega)-f(x)}{(q-1)x+\omega}, & \text{if } x \neq \omega_0; \\ f'(\omega_0), & \text{if } x = \omega_0, \end{cases} \tag{2}$$

where $0 < q < 1$, $\omega \geq 0$, and $\omega_0 = \frac{\omega}{1-q}$, cf. [8, Eq. (2.1.1)]. Besides, following [1], we define

$$\mathscr{D}_{q,\omega}^{(0)} f := f, \quad \mathscr{D}_{q,\omega}^{(j)} f := \mathscr{D}_{q,\omega} \mathscr{D}_{q,\omega}^{(j-1)} f, \quad j \geq 1.$$

It is well known that this class of operators includes the $q$-difference operator $\mathscr{D}_q$ by Jackson when $\omega = 0$, the forward difference operator $\Delta$ when $q = 1$ and $\omega = 1$, and the derivative operator $\frac{d}{dx}$ as a limit case when $\omega = 0$ and $q \to 1$.

Theoretical backgrounds of this problem are established by [4, Theorems 4.1, 4.2]. In particular, it was shown that the nonstandard orthogonal polynomials $Q_n(x)$ satisfy the second-order difference equation

$$\sigma_{1,c,n}(x)\mathscr{D}_{q,\omega}^{(2)} Q_n(x) + \sigma_{2,c,n}(x)\mathscr{D}_{q,\omega} Q_n(x) + \sigma_{3,c,n}(x)Q_n(x) = 0, \quad n \geq 2, \tag{3}$$

where $\sigma_{1,c,n}(x)$, $\sigma_{2,c,n}(x)$, and $\sigma_{3,c,n}(x)$ are explicitly known functions. Moreover, there exist two operators $\Phi_n$ and $\widehat{\Phi}_n$, known as ladder operators, involving the operator of Hahn defined by (2) such that

$$\Phi_n Q_n(x) = \varphi_{c,n}^{1,2}(x) Q_{n-1}(x), \tag{4}$$

$$\widehat{\Phi}_n Q_{n-1}(x) = \varphi_{c,n}^{3,4}(x) Q_n(x), \tag{5}$$

where $\varphi_{c,n}^{1,2}(x)$ and $\varphi_{c,n}^{3,4}(x)$ can also be computed explicitly.

The study of second-order differential/difference equations and their solutions appear in several theoretical and applied contexts. Thus it is worth studying how to compute explicitly the polynomial coefficients $\sigma_{1,c,n}(x)$, $\sigma_{2,c,n}(x)$ and $\sigma_{3,c,n}(x)$ of (3). In this paper we present an algorithm highlighting its more important steps. Later, the symbolic program will be built using the programming language *Mathematica® 13.1.0*.[1] The corresponding code is freely available at
`https://w3.ual.es/GruposInv/Tapo/SODE.nb`

The literature related to symbolic programs in the framework of Sobolev orthogonality is very recent. As far as we know, the first paper is [10], where Mehler-Heine formulas are computed symbolically — cf.
`https://notebookarchive.org/2022-06-amlp3fh`, Notebook Archive (2022).

This paper is structured as follows. Section 2 is devoted to introducing theoretical results obtained in [4] as well as an algorithm to obtain symbolically the coefficients of the second-order differential/difference equation (3). In Section 3, we show how the program works for three examples related to different operators.

## 2 Theoretical Results and Algorithm

As was indicated in the previous section, $\{Q_n\}_{n\geq 0}$ is the sequence of monic orthogonal polynomials with respect to the Sobolev-type inner product (1). We also denote by $\{P_n\}_{n\geq 0}$ the sequence of monic orthogonal polynomials with respect to the standard inner product

$$(f, g)_\varrho = \int f(x)g(x)\varrho(x)dx.$$

Then,

$$(P_n, P_k)_\varrho = \int P_n(x)P_k(x)\varrho(x)dx = h_n\delta_{n,k}, \quad n, k \in \mathbb{N} \cup \{0\},$$

[1]The program in previous versions of *Mathematica®* may not work properly — e.g. we found malfunctions in the version 13.0.0.

where $\delta_{n,k}$ denotes the Kronecker delta and $h_n$ is the square of the norm of these polynomials. It is well known that the polynomials $P_n$ satisfy a three-term recurrence relation of the form:

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \quad n \geq 0, \tag{6}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$. In addition, we know $\beta_n = h_n/h_{n-1}$ for $n \geq 1$. We have $\alpha_n = 0$ when the weight function $\varrho$ is symmetric.

We assume that the polynomials $P_n(x)$ satisfy the following relation:

$$A(x)\mathscr{D}_{q,\omega}P_n(x) = B_n(x)P_n(x) + C_n(x)P_{n-1}(x), \quad n \geq 1, \tag{7}$$

where $A(x)$ is a polynomial and $B_n(x)$ and $C_n(x)$ are certain functions. Notice that the relation (7) is very general. In fact, it holds for general standard orthogonal polynomials with respect to inner products involving any of the three operators $\mathscr{D}_q$, $\Delta$, and $\frac{d}{dx}$ [6].

Then, we can define the ladder operators — viz. the lowering operator

$$\Psi_n := A(x)\mathscr{D}_{q,\omega} - B_n(x), \quad n \geq 1,$$

so that

$$\Psi_n P_n(x) = C_n(x)P_{n-1}(x), \quad n \geq 1,$$

and the raising operator

$$\widehat{\Psi}_n := B_{n-1}(x) + \frac{C_{n-1}(x)(x-\alpha_{n-1})}{\beta_{n-1}} - A(x)\mathscr{D}_{q,\omega}, \quad n \geq 2.$$

Note that the relation (6) yields

$$\widehat{\Psi}_n P_{n-1}(x) = \frac{C_{n-1}(x)}{\beta_{n-1}} P_n(x), \quad n \geq 2.$$

As soon as the properties of the ladder operators are described [4, Theorem 4.1]), the second-order differential/difference equation for the polynomials $Q_n(x)$ can be obtained.

**Theorem 2.1 (cf. G. Filipuk *et al.* [4])** *The discrete Sobolev-type orthogonal polynomials, $Q_n(x)$, with respect to (1) satisfy the second-order linear differential/difference equation (3) where the coefficients $\sigma_{1,c,n}(x)$, $\sigma_{2,c,n}(x)$ and $\sigma_{3,c,n}(x)$ are explicitly known.*

To built an algorithm providing the explicit functions involved in the relations (3)-(5), we follow the notation in [4]. First of all, we must provide some data related to the discrete Sobolev-type inner product (1), i.e. $c$, $M$, $j$, and the values $q$ and $\omega$ according to the operator. Besides, we need certain properties of the orthogonal polynomials $P_n(x)$. More specifically, we assume that we know an explicit expression for the monic orthogonal polynomials $P_n(x)$, the square of their norms — i.e. $h_n$, the coefficients of the three-term recurrence relation (6), and the functions $A(x)$, $B_n(x)$ and $C_n(x)$ in the relation (7). We remark that these explicit expressions are known for all the orthogonal polynomials in the Askey scheme using their hypergeometric representations [8]. Thus the steps of the algorithm are:

Step 1. Enter all the information described previously for the polynomials $P_n(x)$ and the data of the inner product (1).

Step 2. Compute the functions

$$\rho_{n,j,c} = \frac{\mathscr{D}_{q,\omega}^{(j)} P_n(c)}{1 + M K_{n-1}^{(j,j)}(c,c)}, \tag{8}$$

$$r_c(x) = \prod_{k=0}^{j} (x - q^k c - \omega [k]_q), \tag{9}$$

where $[k]_q$ is the basic $q$-number for $q \neq 0$ and $q \neq 1$ given by

$$[\alpha]_q := \frac{1 - q^\alpha}{1 - q}, \tag{10}$$

cf. [8, Eq. (1.8.1)], and $K_n(x,y)$ are the kernel polynomials of the form

$$K_n(x,y) := \sum_{i=0}^{n} \frac{P_i(x) P_i(y)}{h_i}. \tag{11}$$

Therefore,

$$K_n^{(k,\ell)}(x,y) := \sum_{i=0}^{n} \frac{\mathscr{D}_{q,\omega}^{(k)} P_i(x)\ \mathscr{D}_{q,\omega}^{(\ell)} P_i(y)}{h_i}, \quad k, \ell \in \mathbb{N} \cup \{0\}. \tag{12}$$

Notice (8-10) must be computed adequately according to the operator chosen ($\mathscr{D}_{q,\omega}$, $\mathscr{D}_q$, $\Delta$, or $\frac{d}{dx}$).

Step 3. Compute all the necessary auxiliary functions to obtain the coefficients of the second-order differential/difference equation (3). The theoretical development to obtain these functions was made in [4]. They are:

$$f_{1,c,n}(x) = r_c(x) - \frac{M\rho_{n,j,c}}{h_{n-1}}\left(\sum_{i=0}^{j}\frac{(q;q)_j \mathscr{D}_{q,\omega}^{(i)} P_{n-1}(c)\prod_{k=0}^{i-1}(x-q^k c-\omega[k]_q)}{(q;q)_i(1-q)^{j-i}}\right), \tag{13}$$

$$g_{1,c,n}(x) = \frac{M\rho_{n,j,c}}{h_{n-1}}\left(\sum_{i=0}^{j}\frac{(q;q)_j \mathscr{D}_{q,\omega}^{(i)} P_{n}(c)\prod_{k=0}^{i-1}(x-q^k c-\omega[k]_q)}{(q;q)_i(1-q)^{j-i}}\right), \tag{14}$$

$$\begin{aligned} f_{2,c,n}(x) = {} & \frac{x-q^j c-\omega[j]_q}{q^{j+1}\left(x-q^{-1}c-\omega[-1]_q\right)}\Bigg(\mathscr{D}_{q,\omega} f_{1,c,n}(x) + f_{1,c,n}(qx+\omega)\frac{B_n(x)}{A(x)} \\ & - g_{1,c,n}(qx+\omega)\frac{C_{n-1}(x)}{\beta_{n-1}A(x)} - \frac{[j+1]_q}{\left(x-q^j c-\omega[j]_q\right)} f_{1,c,n}(x)\Bigg), \end{aligned} \tag{15}$$

$$\begin{aligned} g_{2,c,n}(x) = {} & \frac{x-q^j c-\omega[j]_q}{q^{j+1}\left(x-q^{-1}c-\omega[-1]_q\right)}\Bigg(\mathscr{D}_{q,\omega} g_{1,c,n}(x) + f_{1,c,n}(qx+\omega)\frac{C_n(x)}{A(x)} \\ & + g_{1,c,n}(qx+\omega)\left(\frac{B_{n-1}(x)}{A(x)} + \frac{(x-\alpha_{n-1})C_{n-1}(x)}{A(x)\beta_{n-1}}\right) - \frac{[j+1]_q}{\left(x-q^j c-\omega[j]_q\right)} g_{1,c,n}(x)\Bigg), \end{aligned} \tag{16}$$

$$f_{3,c,n}(x) = -\frac{g_{1,c,n-1}(x)}{\beta_{n-1}}, \tag{17}$$

$$g_{3,c,n}(x) = f_{1,c,n-1}(x) + \frac{(x-\alpha_{n-1})g_{1,c,n-1}(x)}{\beta_{n-1}}, \tag{18}$$

$$f_{4,c,n}(x) = -\frac{g_{2,c,n-1}(x)}{\beta_{n-1}}, \tag{19}$$

$$g_{4,c,n}(x) = f_{2,c,n-1}(x) + \frac{(x-\alpha_{n-1})g_{2,c,n-1}(x)}{\beta_{n-1}}. \tag{20}$$

Using (13-20), we construct the functions $\varphi_{c,n}^{i,\ell}(x)$ given by

$$\varphi_{c,n}^{i,\ell}(x) := \begin{vmatrix} f_{i,c,n}(x) & f_{\ell,c,n}(x) \\ g_{i,c,n}(x) & g_{\ell,c,n}(x) \end{vmatrix}, \quad i,\ell \in \{1,2,3,4\}. \tag{21}$$

They are the key to define the ladder operators

$$\Phi_n := \varphi_{c,n}^{3,2}(x) + \varphi_{c,n}^{1,3}(x)\mathscr{D}_{q,\omega}, \tag{22}$$

$$\widehat{\Phi}_n := \varphi_{c,n}^{1,4}(x) - \varphi_{c,n}^{1,3}(x)\mathscr{D}_{q,\omega}. \tag{23}$$

So, we get (4)-(5).

Step 4. Obtain the coefficients of the second-order linear difference equation (3) through these formulae:

$$\sigma_{1,c,n}(x) = \varphi^{1,3}_{c,n}(x)\varphi^{1,3}_{c,n}(qx+\omega)\varphi^{1,2}_{c,n}(x), \tag{24}$$

$$\begin{aligned}\sigma_{2,c,n}(x) = \varphi^{1,3}_{c,n}(x)\Big(&\varphi^{1,2}_{c,n}(x)\left(\varphi^{3,2}_{c,n}(qx+\omega) + \mathscr{D}_{q,\omega}\varphi^{1,3}_{c,n}(x)\right)\\ &-\varphi^{1,2}_{c,n}(qx+\omega)\varphi^{1,4}_{c,n}(x) - \varphi^{1,3}_{c,n}(x)\mathscr{D}_{q,\omega}\varphi^{1,2}_{c,n}(x)\Big),\end{aligned} \tag{25}$$

$$\begin{aligned}\sigma_{3,c,n}(x) = \varphi^{1,2}_{c,n}(qx+\omega)&\left(\varphi^{1,2}_{c,n}(x)\varphi^{3,4}_{c,n}(x) - \varphi^{1,4}_{c,n}(x)\varphi^{3,2}_{c,n}(x)\right)\\ &+\varphi^{1,3}_{c,n}(x)\left(\varphi^{1,2}_{c,n}(x)\mathscr{D}_{q,\omega}\varphi^{3,2}_{c,n}(x) - \varphi^{3,2}_{c,n}(x)\mathscr{D}_{q,\omega}\varphi^{1,2}_{c,n}(x)\right).\end{aligned} \tag{26}$$

We remark that the algorithm, and therefore the symbolic computer program, returns the desired coefficients of (3) by entering only the information provided in Step 1. In this way, we can say that this information constitutes the **inputs** of the symbolic program.

As we have previously commented, the Hahn difference operator includes the $q$-difference operator $\mathscr{D}_q$ by Jackson when $\omega = 0$, the forward difference operator $\Delta$ when $q = 1$ and $\omega = 1$, and the derivative operator $\frac{d}{dx}$ as a limit case when $\omega = 0$ and $q \to 1$. In these cases, the functions (8)-(26) are substantially simplified. For this reason and taking into account that these three operators are the most widely used ones in the framework of the Sobolev orthogonality [11], three appendices were included in [4] where the explicit expression of these functions in these special cases are given. Thus as we will see in the next section, the program will allow the user to choose the operator. A flow diagram of the algorithm is given in Fig. 1.

# 3 Symbolic Computer Program with Examples

This section is devoted to showing the practical performance of the program. As we have commented in the previous section, the functions (8)-(26) (Steps 2-4 of the algorithm) are considerably simplified when we consider one of these three operators: $\frac{d}{dx}$, $\Delta$ or $\mathscr{D}_q$. For this reason, when we run the program, it asks us which operator we want to use. We can choose one of the three cases or the general operator $\mathscr{D}_{q,\omega}$. Thus running the program we find the first *dialog notebook*, see Fig. 2. Next, the program will request the information indicated in Step 1 of the algorithm. Before getting the coefficients $\sigma_{1,c,n}(x)$, $\sigma_{2,c,n}(x)$, and $\sigma_{3,c,n}(x)$ of the second-order differential/difference equation (3), we must enter a value for $n$ greater than 1. We remark that if we want to obtain these coefficients for other values of $n$ it is not necessary to run the program again, we only have to write $\sigma 1[n]$, $\sigma 2[n]$, and $\sigma 3[n]$ with $n \geq 2$ on the principal *Mathematica* notebook to get the corresponding coefficients. We will show an example.

We are going to illustrate how the program works with three examples. Many other examples could be made with only the knowledge of the information required in Step 1 of the algorithm.

## 3.1 The Jacobi case

We consider the inner product (1) with the Jacobi weight on $[-1,1]$, i.e. $\varrho(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha, \beta > -1$. Then, we have

$$(f,g)_S = \int_{-1}^{1} f(x)g(x)(1-x)^\alpha(1+x)^\beta dx + Mf^{(j)}(c)g^{(j)}(c). \tag{27}$$

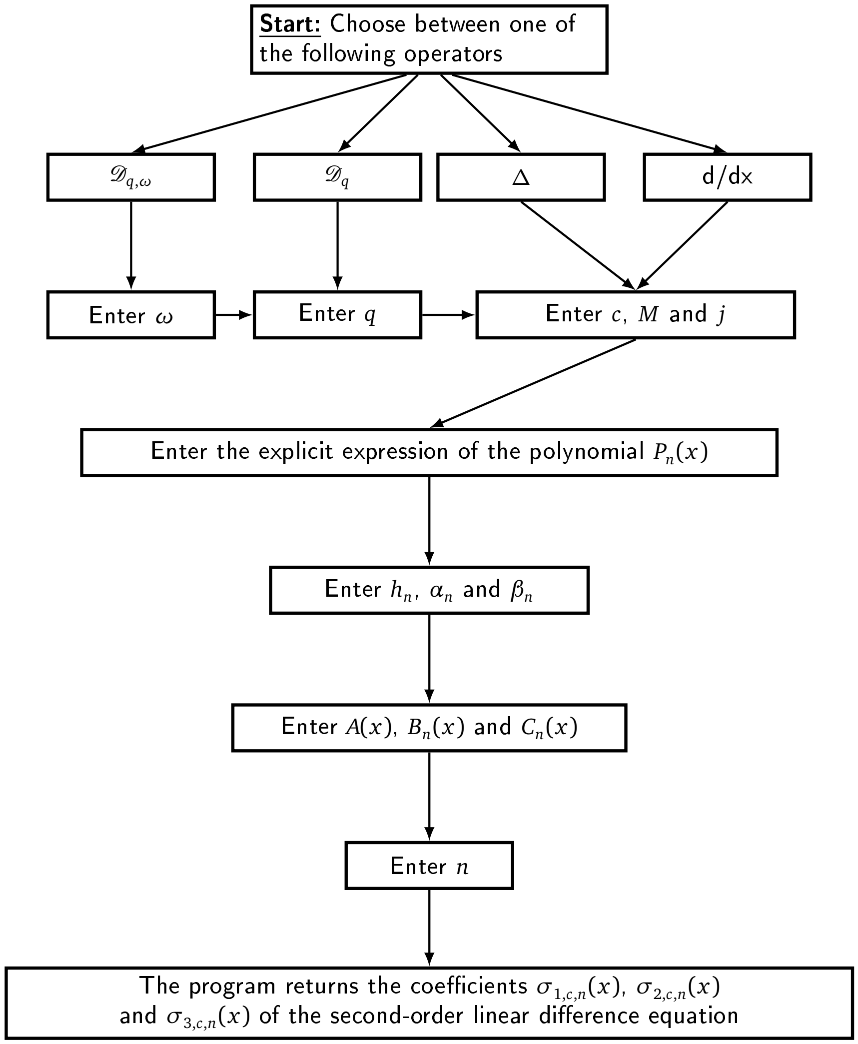


Figure 1: Flow diagram.

This inner product was used in [12] with $j = 0$ (Jacobi-Krall polynomials) or in a more general framework in [9], where the authors obtained certain asymptotic properties of the orthogonal polynomials with respect to (27) with $c = 1$.

We denote by $\left\{P_n^{(\alpha,\beta)}\right\}_{n\geq 0}$ the sequence of monic Jacobi orthogonal polynomials with respect to the weight $\varrho(x)$. According to [8, Section 9.8]), these polynomials have the form

$$P_n^{(\alpha,\beta)}(x) = \frac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\ {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right), \tag{28}$$

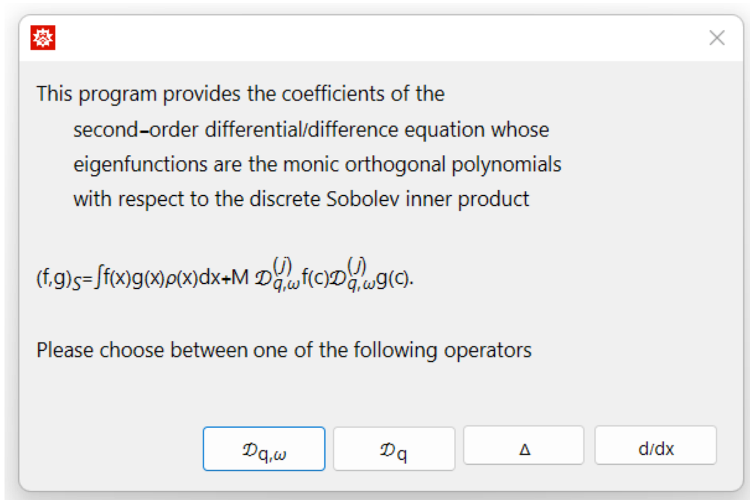


Figure 2: The first *dialog notebook*.

where $(a)_k$ denotes the Pochhammer symbol [8, Eq. (1.3.1)]

$$(a)_0 := 1 \quad \text{and} \quad (a)_k := \prod_{i=1}^{k}(a+i-1), \quad k = 1,2,3\ldots$$

and ${}_rF_s$ denotes the hypergeometric function [8, Eq. (1.4.1)]:

$${}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!}.$$

The coefficients of the three-term recurrence relation (6) are given by

$$\alpha_n = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \tag{29}$$

$$\beta_n = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)}, \tag{30}$$

see e.g. [2, Eqs. (2.15), (2.18)] or [3, Page 211].

The square of the norm of $P_n^{(\alpha,\beta)}(x)$ is

$$h_n = \frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+2)} \tag{31}$$

cf. [8, Section 9.8].

It follows from [2, Section 3] that

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{(n+r_n)P_n^{(\alpha,\beta)}(x) - \beta_n R_n P_{n-1}^{(\alpha,\beta)}(x)}{x-1} + \frac{\beta_n R_n P_{n-1}^{(\alpha,\beta)}(x) - r_n P_n^{(\alpha,\beta)}(x)}{x+1},$$

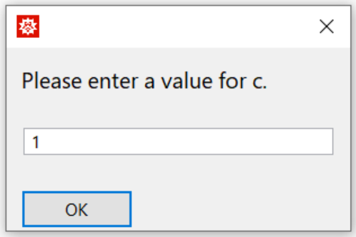


Figure 3: Entering a value for $c$.

where

$$r_n = r_n(\alpha, \beta) := \frac{\beta - \alpha - 2n - (\alpha + \beta + 2n + 2)\alpha_n}{4},$$
$$R_n = R_n(\alpha, \beta) := \frac{2n + \alpha + \beta + 1}{2}.$$

Therefore, we deduce that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the relationship (7) with

$$A(x) = x^2 - 1, \quad B_n(x) = (x+1)n + 2r_n, \quad C_n(x) = -2\beta_n R_n,$$

cf. [2, Eqs.(2.9), (2.12)].

To apply the algorithm and the constructed program, we choose concrete values for the parameters in the inner product (27). The choice for this example is

$$(f, g)_S = \int_{-1}^{1} f(x)g(x)(1-x)^2(1+x)^4 dx + 5f^{(3)}(1)g^{(3)}(1).$$

Thus we possess all the information required in Step 1 of the algorithm and therefore we can run the *Mathematica* program. Next, we describe in detail how to enter the data required in Step 1 into the program.

1. First, we choose the operator — Fig. 2. In this case, we click on d/dx.

2. We enter the values of the parameters $c$, $M$, and $j$. For this example, our choice is $c = 1$, $M = 5$, and $j = 3$. The information is entered in the corresponding *dialog notebooks* — cf. Figs. 3, 6, and 7, respectively. After entering any quantity we could want to modify it because we have made a mistake or for any other reason, so we have included another *dialog notebook* where the user must confirm if the entered value is valid — cf. Fig. 4. Otherwise, it can be modified in another *dialog notebook*. We show an example about this in Fig. 5.

   It is important to remark that the program checks whether the required parameters are admissible. For example, it is well known that the value of $j$ must be a nonnegative integer. Thus if we enter a non-admissible value, the program returns a warning message and requests a valid value. For example, if we enter $j = -3$ we obtain the warning shown in Fig. 8. This checkup is done for other data such as $M$, or $n$, among others.

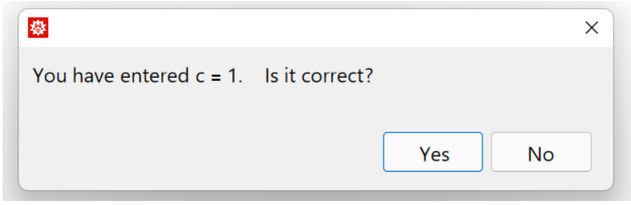


Figure 4: A *dialog notebook* requiring confirmation about the entered value.

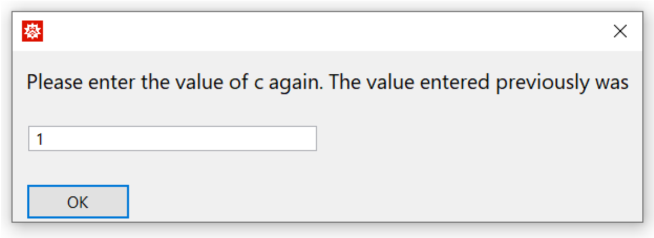


Figure 5: A *dialog notebook* asking for a new value for $c$. This box appears when clicking on "No" in the *dialog notebook* shown in Fig. 4.

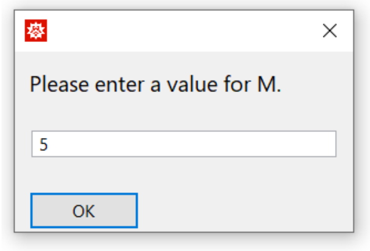


Figure 6: Entering a value for $M$.

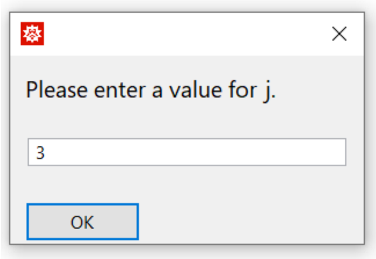


Figure 7: Entering a value for $j$.

3. According to formula (28), the explicit expression of the monic Jacobi orthogonal polynomials is:

$$P_n^{(2,4)}(x) = \frac{2^n (3)_n}{(n+7)_n} \, {}_2F_1\left( \begin{matrix} -n, n+7 \\ 3 \end{matrix} ; \frac{1-x}{2} \right).$$

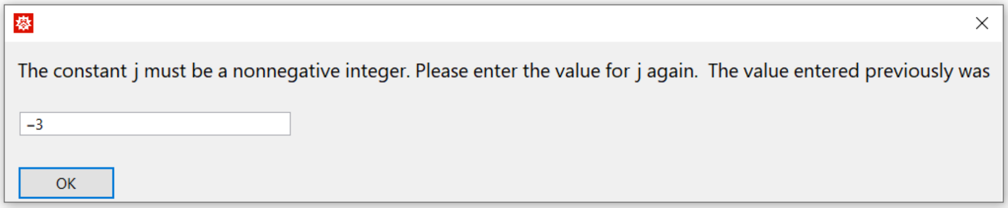


Figure 8: *Dialog notebook* when a non-admissible value of a parameter is entered.

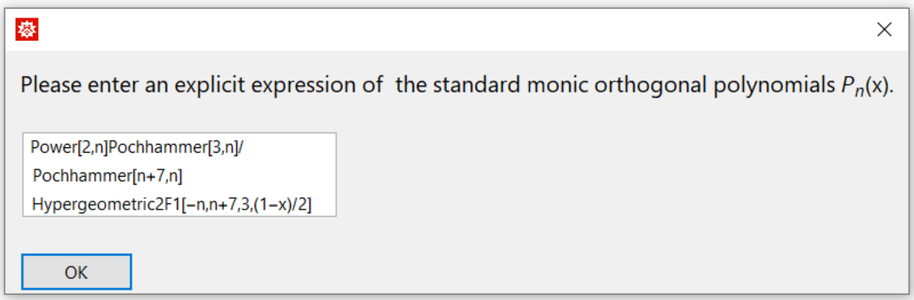


Figure 9: The explicit expression of the Jacobi polynomial $P_n^{(2,4)}(x)$.

Then, using the commands of *Mathematica*® *13.1.0*, we write the following expression into the corresponding *dialog notebook*:
`Power[2,n]Pochhammer[3,n]/Pochhammer[n+7,n]Hypergeometric2F1[-n,n+7,3,(1-x)/2]`
as it appears in Fig. 9.

4. We provide the square of the norm of $P_n^{(2,4)}(x)$, i.e. in our case (31) gives

$$h_n = \frac{2^{2n+7}\Gamma(n+1)\Gamma(n+3)\Gamma(n+5)}{(n+7)_n\Gamma(2n+8)}.$$

   Using again the commands of *Mathematica*® *13.1.0*, we enter
   `(Power[2,2n+7]Gamma[n+3]Gamma[n+5]Gamma[n+1])/(Pochhammer[n+7,n]Gamma[2n+8])`
   as it appears in Fig. 10.

5. Now, it is necessary to provide the coefficients in the three-term recurrence relation (6). First, we begin with $\alpha_n$. For this example, from (29) we have $\alpha_n = \dfrac{3}{(n+3)(n+4)}$. Then, we enter the expression like in Fig. 11.

6. Next, we enter the coefficients $\beta_n$ in the three-term recurrence relation. For this case, from (30) we have $\beta_n = \dfrac{n(n+2)(n+4)(n+6)}{(n+3)^2(2n+5)(2n+7)}$, which is shown in Fig. 12.

It only remains to enter the functions $A(x)$, $B_n(x)$, and $C_n(x)$ satisfying the relationship (7).

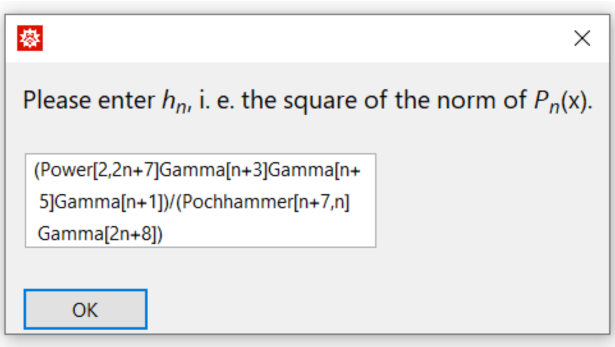


Figure 10: The square of the norm of $P_n^{(2,4)}(x)$.

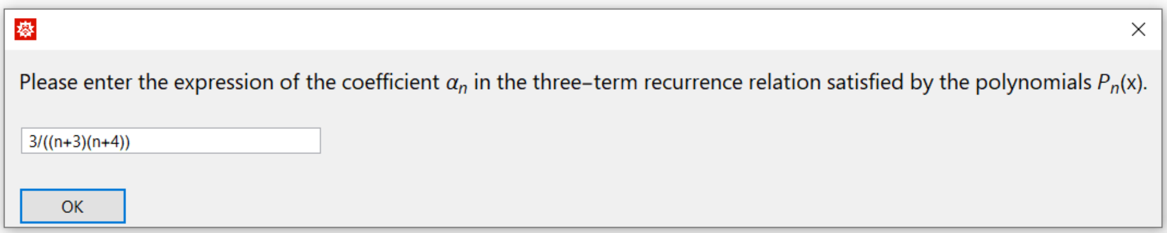


Figure 11: The coefficients $\alpha_n$ in the three-term recurrence relation (6).

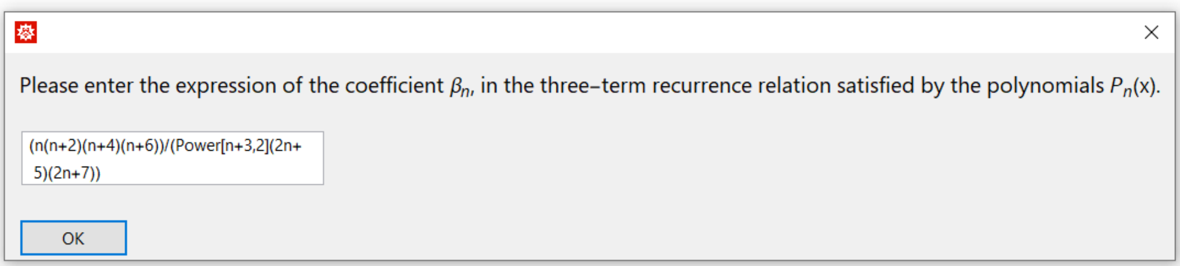


Figure 12: The coefficients $\beta_n$ in the three-term recurrence relation (6).

7. For this example, we know that $A(x) = x^2 - 1$. Then, Fig. 13 shows the corresponding *dialog notebook*.
8. In this case, $B_n(x) = \dfrac{n^2x + 3nx + n}{n+3}$, and it is entered as we can see in Fig. 14.
9. Finally, we enter $C_n(x) = -\dfrac{n(n+2)(n+4)(n+6)}{(n+3)^2(2n+5)}$ as it is shown in Fig. 15.
10. A last datum is necessary to run the program: we must enter a value for $n$. For this example, we have

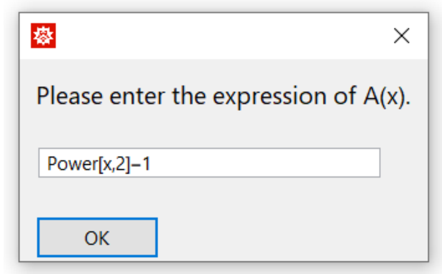


Figure 13: $A(x)$.

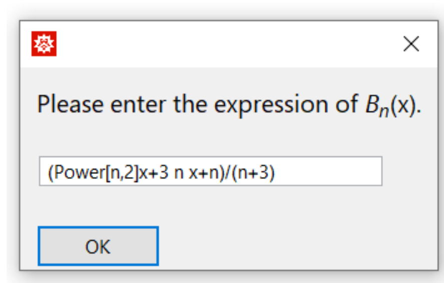


Figure 14: $B_n(x)$.

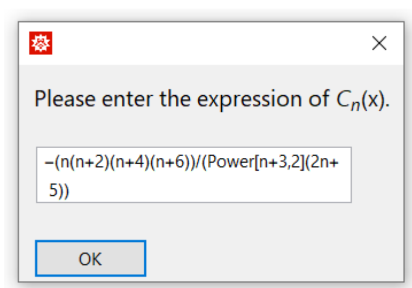


Figure 15: $C_n(x)$.

chosen $n = 6$ as we can see in Fig. 16.

We have already entered all the information required in Step 1 of the algorithm. Then, the program returns the coefficients $\sigma_{1,c,n}(x)$, $\sigma_{2,c,n}(x)$ and $\sigma_{3,c,n}(x)$ of the second-order differential equation (3) as we can see in Fig. 17.

As we have mentioned at the beginning of this section, we could want to obtain the expression of the coefficients of the second-order differential equation (3) for other values of $n$. Then, it is enough to enter $\sigma 1[n]$, $\sigma 2[n]$, and $\sigma 3[n]$ into the principal *Mathematica* notebook for a desired $n$ and the program returns them. An example with $n = 8$ is shown in Fig. 18.

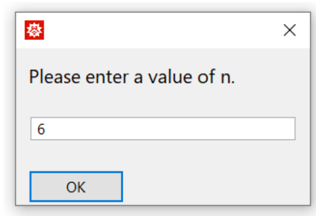


Figure 16: A value for $n$.

**The polynomial coefficient $\sigma_{1,1,6}$(x) in the second–order differential/difference equation is**

- (-1 + x) (1 + x) (83 518 558 283 529 145 823 + x (-344 860 488 738 182 625 392 + x (530 551 943 354 119 348 938 + x (-356 066 859 998 837 996 192 + 86 072 224 862 881 634 423 x))))

**The polynomial coefficient $\sigma_{2,1,6}$(x) in the second–order differential/difference equation is**

177 823 372 171 124 333 746 -
2 x (686 802 804 317 111 598 484 + x (-1 900 281 034 704 142 085 006 + x (2 468 837 306 497 522 017 344 + x (-1 516 228 645 131 889 348 441 + 344 288 899 451 526 537 692 x))))

**The polynomial coefficient $\sigma_{3,1,6}$(x) in the second–order differential/difference equation is**

78 (66 426 180 090 867 674 123 + x (-283 688 523 617 946 755 492 + x (457 461 083 025 655 471 038 + x (-327 350 107 547 794 839 692 + 86 072 224 862 881 634 423 x))))

Figure 17: The Jacobi case: Coefficients of the second-order differential equation (3) with $\alpha = 2$, $\beta = 4$, $c = 1$, $M = 5$, $j = 3$, and $n = 6$.

**σ1[8]**

- (-1 + x) (1 + x) (205 118 475 360 333 006 061 249 +
x (-831 007 710 172 139 258 389 996 + x (1 256 931 346 228 010 793 569 094 + x (-838 723 665 176 200 355 073 796 + 207 433 276 983 856 520 805 049 x))))

**σ2[8]**

420 770 759 451 473 246 267 498 - 2 x (1 656 634 488 217 869 571 546 592 +
x (-4 571 696 557 483 055 660 186 578 + x (5 860 799 619 931 566 654 381 772 + x (-3 566 823 216 309 045 077 027 033 + 829 733 107 935 426 083 220 196 x))))

**σ3[8]**

120 (176 431 321 049 942 883 683 509 +
x (-735 801 213 695 477 562 970 156 + x (1 151 656 936 305 720 252 340 434 + x (-800 057 921 443 626 418 098 436 + 207 433 276 983 856 520 805 049 x))))

Figure 18: An example of how to obtain $\sigma_{1,c,8}(x)$, $\sigma_{2,c,8}(x)$, and $\sigma_{3,c,8}(x)$ from the principal *Mathematica* notebook.

**Remark 3.1** *Notice that the program accepts the value $M = 0$ as a valid (or an admissible) value. In this case, the inner product (27) yields a standard inner product and the corresponding orthogonal polynomials are the classical Jacobi polynomials. Therefore, we have obtained the coefficients of the second-order differential equation of the classical Jacobi orthogonal polynomials with $\alpha = 2$, $\beta = 4$, and $n = 6$, cf. [8, Eq. (9.8.6)]. Thus running the program with $M = 0$, we obtain the results shown in Fig. 19.*

*This occurs in a similar way for all the cases when we take $M = 0$ in the inner product (1).*

## 3.2 The Meixner case

In this example we consider the forward operator $\Delta$. In particular, we take the Meixner weight function on $(0, +\infty)$, i.e. $\varrho(x) = \frac{(\beta)_x}{x!}\alpha^x$ with $\beta > 0$ and $0 < \alpha < 1$. Thus the inner product (1) is transformed into

**The polynomial coefficient $\sigma_{1,1,6}(x)$ in the second–order differential/difference equation is**

$1 - x^2$

**The polynomial coefficient $\sigma_{2,1,6}(x)$ in the second–order differential/difference equation is**

$2 - 8x$

**The polynomial coefficient $\sigma_{3,1,6}(x)$ in the second–order differential/difference equation is**

78

Figure 19: The Jacobi case: Coefficients of the classical second-order differential equation with $M = 0$, $\alpha = 2$, $\beta = 4$, and $n = 6$.

$$(f,g)_S = \sum_{x=0}^{\infty} f(x)g(x)\frac{(\beta)_x}{x!}\alpha^x + M\Delta^{(j)}f(c)\Delta^{(j)}g(c).$$

We denote by $\{M_n(x;\beta,\alpha)\}_{n\geq 0}$ the sequence of monic orthogonal polynomials with respect to (38). It is known [8, Section 9.10] that these polynomials have the form

$$M_n(x;\beta,\alpha) = (\beta)_n \left(\frac{\alpha}{\alpha-1}\right)^n {}_2F_1\left(\begin{matrix}-n,-x\\ \beta\end{matrix};1-\frac{1}{\alpha}\right). \tag{32}$$

Moreover, it follows from [8, Section 9.10] that

$$\alpha_n = \frac{n+\alpha(n+\beta)}{1-\alpha}, \tag{33}$$

$$\beta_n = \frac{\alpha n(n+\beta-1)}{(\alpha-1)^2}, \tag{34}$$

$$h_n = \frac{\alpha^n(\beta)_n n!}{(1-\alpha)^\beta(\alpha-1)^{2n}}. \tag{35}$$

On the other hand, according to [6, Eq. (6.4.15)], we have

$$\Delta\widetilde{M}_n(x;\beta,\alpha) = \frac{n}{\beta+x}\widetilde{M}_n(x;\beta,\alpha) - \frac{n}{\alpha(\beta+x)}\widetilde{M}_{n-1}(x;\beta,\alpha), \tag{36}$$

where $\widetilde{M}_n(x;\beta,\alpha) = \gamma_n x^n + \ldots$ are the Meixner orthogonal polynomials with the leading coefficient

$$\gamma_n = \frac{\left(1-\frac{1}{\alpha}\right)^n}{(\beta)_n}, \tag{37}$$

cf. [6, Eq. (6.1.7)]. Then, using (36) and (37), we get

$$(x+\beta)\Delta M_n(x;\beta,\alpha) = nM_n(x;\beta,\alpha) + \frac{n(n+\beta-1)}{1-\alpha}M_{n-1}(x;\beta,\alpha).$$

Thus the polynomials $M_n(x;\beta,\alpha)$ satisfy the relation (7) with

$$A(x) = x+\beta, \quad B_n(x) = n, \quad C_n(x) = \frac{n(n+\beta-1)}{1-\alpha}.$$

To run the program, we choose certain parameters in the inner product (38). In particular,

$$(f,g)_S = \sum_{x=0}^{\infty} f(x)g(x)\frac{(4)_x}{x!}\left(\frac{1}{3}\right)^x + 30\Delta^{(2)}f(0)\Delta^{(2)}g(0). \tag{38}$$

In fact, it clear that we can proceed similar to the Jacobi case — i.e. giving all the data required in Step 1 of the algorithm. However, in order to avoid unnecessary repetitions, we only outline the data entry process.

1. In this case we click on the $\Delta$, the forward operator, in the corresponding *dialog notebook* cf. Fig. 2.
2. We enter consecutively the values $c=0$, $M=30$, and $j=2$.
3. According to (32), we have

   $$M_n(x;4,1/3)(x) = (4)_n\left(\frac{-1}{2}\right)^n {}_2F_1\left(\begin{matrix}-n,-x\\4\end{matrix};-2\right).$$

   We can use the *Mathematica*® commands to enter the above expression into the program `Power[-1/2,n]Pochhammer[4,n]Hypergeometric2F1[-n,-x,4,-2]`.
4. The square of the norm of $M_n(x;4,1/3)(x)$ is given by (35), i.e. $h_n = \dfrac{3^{n+4}n!(4)_n}{2^{2(n+2)}}$. We enter this term as `Power[3,n+4]n!Pochhammer[4,n]/Power[2,2(n+2)]`.
5. Next, we enter the coefficients of the three-term recurrence relation (6) given by (33-34). For this example, we have $\alpha_n = 2n+2$ and $\beta_n = \frac{3}{4}n(n+3)$.
6. Now, we enter consecutively the coefficients in the relationship (7). They are $A(x) = x+4$, $B_n(x) = n$, and $C_n(x) = \dfrac{3}{2}n(n+3)$.
7. Finally, we choose a value for $n$. In this example, we take $n=8$ and the program returns the coefficients of the second-order difference equation (3) as we can see in Fig. 20.

### 3.3 The Stieltjes-Wigert case

We consider the Stieltjes-Wigert weight function on the positive semiaxis:

$$\varrho(x) = (2\pi\ln(1/q))^{-1/2}\exp\left(\left(\ln\left(xq^{-1/2}\right)\right)^2/(2\ln q)\right), \quad 0<q<1.$$

Then, the inner product (1) is transformed into

$$\begin{aligned}(f,g)_S = &\int_0^{\infty} f(x)g(x)(2\pi\ln(1/q))^{-1/2}\exp\left(\left(\ln\left(xq^{-1/2}\right)\right)^2/(2\ln q)\right)dx\\ &+ M\mathscr{D}_q^{(j)}f(c)\mathscr{D}_q^{(j)}g(c).\end{aligned} \tag{39}$$

**The polynomial coefficient $\sigma_{1,0,8}$(x) in the second–order differential/difference equation is**

(5 + x) (4 764 300 800 + x (32 582 029 474 + x (-81 604 313 075 + 32 931 073 361 x)))

**The polynomial coefficient $\sigma_{2,0,8}$(x) in the second–order differential/difference equation is**

107 243 548 160 + 2 x (317 101 325 746 + x (-801 136 692 269 + (377 983 973 324 - 32 931 073 361 x) x))

**The polynomial coefficient $\sigma_{3,0,8}$(x) in the second–order differential/difference equation is**

16 (9 461 468 800 + x (38 971 733 410 + x (-89 091 713 779 + 32 931 073 361 x)))

Figure 20: The Meixner case: Coefficients of the second-order difference equation (3) with $\alpha = 1/3$, $\beta = 4$, $c = 0$, $M = 30$, $j = 2$, and $n = 8$.

We denote by $\{S_n(x;q)\}_{n\geq 0}$ the sequence of monic Stieltjes-Wigert orthogonal polynomials with respect to (39). The explicit expression of these polynomials is given in [7, Section 4] and [8, Section 14.27], so that

$$S_n(x;q) = q^{-n^2}(q;q)_n \sum_{k=0}^{n} \frac{(-1)^{n-k} q^{k^2} x^k}{(q;q)_k (q;q)_{n-k}}, \tag{40}$$

where $(a;q)_k$ is a $q$-analogue of the Pochhammer symbol $(a)_k$ — cf. [8, Eq. (1.8.3)],

$$(a;q)_0 := 1 \quad \text{and} \quad (a;q)_k := \prod_{i=1}^{k} \left(1 - a q^{i-1}\right), \quad k = 1,2,3,\ldots$$

It is well known [7, Section 4] that both the expression of the coefficients of the three-term recurrence relation for these polynomials and of the square of their norms have the form

$$\alpha_n = \left(1 + q - q^{n+1}\right) q^{-2n-1}, \tag{41}$$

$$\beta_n = (1 - q^n) q^{-4n+1}, \tag{42}$$

$$h_n = (q;q)_n q^{-2n^2-n}. \tag{43}$$

It follows from [7, Eqs. (4.7), (4.8)] that the monic Stieltjes-Wigert orthogonal polynomials satisfy the following equation:

$$\mathscr{D}_q S_n(x;q) = \frac{(1-q^n)(x+q^{-n})}{(1-q)x^2} S_n(x;q) + \frac{q^n}{(1-q)x^2}(1-q^n) q^{1-4n} S_{n-1}(x;q).$$

Therefore, the coefficients of (7) are

$$A(x) = (1-q)x^2, \quad B_n(x) = (1-q^n)\left(x + q^{-n}\right), \quad C_n(x) = q^{1-3n}(1-q^n).$$

Now, like in the previous cases, we choose the values for the parameters in (39). We take $c = -1$, $M = 6$, $q = 1/2$, and $j = 1$. So, we are considering the nonstandard inner product

$$(f,g)_S = \int_0^\infty f(x)g(x)(2\pi \ln(2)^{-1/2} \exp\left(\left(\ln\left(x(1/2)^{-1/2}\right)\right)^2 / (2\ln(1/2))\right) dx$$
$$+ 6\mathscr{D}_q^{(1)} f(-1) \mathscr{D}_q^{(1)} g(-1).$$

We are ready to provide the necessary data to run the program. Next, we summarize the process.

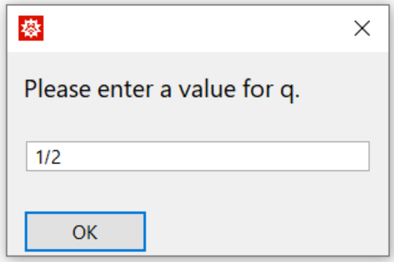


Figure 21: *Dialog notebook* where $q$ is required.

1. First, we choose the operator. In this case, we click on the $\mathscr{D}_q$ the *q-difference operator*, in the corresponding *dialog notebook* — cf. Fig. 2.

2. Furthermore, the algorithm needs the value of the parameter $q$ which is entered in a new *dialog notebook*. Then, we enter $q = 1/2$ as is shown in Fig. 21.

3. As in the previous cases, we enter consecutively the values $c = -1$, $M = 6$, and $j = 1$.

4. From (40),
$$S_n(x;1/2)(x) = 2^{n^2}\left(\frac{1}{2};\frac{1}{2}\right)_n \sum_{k=0}^{n} \frac{2^{-k^2}(-1)^{n-k}x^k}{\left(\frac{1}{2};\frac{1}{2}\right)_k\left(\frac{1}{2};\frac{1}{2}\right)_{n-k}}.$$
The *Mathematica*® commands help us enter this expression, so we can write in the *dialog notebook*:
```
Power[2,n n]QPochhammer[1/2,1/2,n]Sum[(Power[2,-k k]Power[-1,n-k]
Power[x,k])/(QPochhammer[1/2,1/2,k]QPochhammer[1/2,1/2,n-k]),{k,0,n}]
```

5. The square of the norm of $S_n(x;1/2)(x)$ is given by (43). For these concrete values it is $2^{2n^2+n}\left(\frac{1}{2};\frac{1}{2}\right)_n$.
We enter it into the program as:
```
Power[2, n (2 n + 1)] QPochhammer[1/2, 1/2, n]
```

6. The coefficients of the three-term recurrence relation (6) are given by (41-42). In this example, they are $\alpha_n = 2^n\,(3\,2^n - 1)$ and $\beta_n = 2^{3n-1}\,(2^n - 1)$.

7. Next, we enter consecutively the coefficients in the relationship (7). These functions are $A(x) = \frac{x^2}{2}$, $B_n(x) = \left(1-\left(\frac{1}{2}\right)^n\right)\left(x+\left(\frac{1}{2}\right)^{-n}\right)$, and $C_n(x) = \left(\frac{1}{2}\right)^{1-3n}\left(1-\left(\frac{1}{2}\right)^n\right)$.

8. Finally, we choose a value for $n$. On this occasion, we take $n = 4$ and the program returns the coefficients of the second-order $q$-difference equation (3) as it is shown in Fig. 22.

These three examples show how to run the program to obtain the coefficients of a second-order differential/difference equation satisfied by Sobolev-type orthogonal polynomials with respect to the inner product (1). We note that the program can be used with many other weight functions $\varrho(x)$.

**The polynomial coefficient $\sigma_{1,-1,4}(x)$ in the second–order differential/difference equation is**

$4x^2(4+x)(-346446889661702254+x(-262467093653839379+x(445094261902926628+291371494277939849x)))$

**The polynomial coefficient $\sigma_{2,-1,4}(x)$ in the second–order differential/difference equation is**

$-44345201876697888512-x(97528287591802055152+x(6732870483205436876+x(-37504062417843614372+x(1643085520812033959+6701544368392616527x))))$

**The polynomial coefficient $\sigma_{3,-1,4}(x)$ in the second–order differential/difference equation is**

$30(208419354249002000+x(-96625267291724892+x(-1099957691951480644+x(-127844571627425271+291371494277939849x))))$

Figure 22: The Stieltjes-Wigert case: Coefficients of the second-order difference equation (3) with $c=-1$, $M=6$, $j=1$, $q=1/2$, and $n=4$.


## Acknowledgements

The authors are partially supported by the Ministry of Science, Innovation, and Universities of Spain and the European Regional Development Fund (ERDF), grant PID2021-124472NB-I00. In addition, J.F.M.M and J.J.M.B. are partially supported by ERDF and Consejería de Economía, Conocimiento, Empresas y Universidad de la Junta de Andalucía (grant UAL18-FQM-B025-A), by the Research Group FQM-0229 (belonging to Campus of International Excellence CEIMAR) and by the research centre CDTIME of Universidad de Almería. Besides, G.F. acknowledges the support of the National Science Center (Poland) via grant OPUS2017/25/B/ST1/00931.


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