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ORIGINAL PAPER

(p, σ)-Absolute continuity of Bloch maps

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Abstract

Motivated by new progress in the theory of ideals of Bloch maps, we introduce (p, σ)-absolutely continuous Bloch maps with $p \in [1, \infty)$ and $\sigma \in [0, 1)$ from the complex unit open disc $\mathbb{D}$ into a complex Banach space X. We prove a Pietsch domination/factorization theorem for such Bloch maps that provides a reformulation of some results on both absolutely continuous (multilinear) operators and Lipschitz operators. We also identify the spaces of (p, σ)-absolutely continuous Bloch zero-preserving maps from $\mathbb{D}$ into X^* under a suitable norm $\pi^{\mathcal{B}}_{p,\sigma}$ with the duals of the spaces of X-valued Bloch molecules on $\mathbb{D}$ equipped with the Bloch version of the (p^*, σ)-Chevet–Saphar tensor norms.

Keywords Summing operators · (p, σ)-Absolutely continuous operators · Vector-valued Bloch maps · Pietsch factorization/domination · Compact Bloch maps

Mathematics Subject Classification 30H30 · 46E15 · 46E40 · 47B10 · 47B38

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1 Introduction and preliminaries

For any Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the Banach space of all continuous linear operators from X into Y, under the operator norm. In particular, $\mathcal{L}(X, \mathbb{K})$ is denoted by X^*. As usual, B_X stands for the closed unit ball of X.

Recall that $T \in \mathcal{L}(X, Y)$ is called *p-summing* with $p \in [1, \infty)$ if there exists $C \geq 0$ so that

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} \left|x^*(x_i)\right|^p\right)^{\frac{1}{p}}$$

for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$. The infimum of such constants C is denoted by $\pi_p(T)$, and the Banach space of all p-summing operators from X into Y, under the norm π_p, by $\Pi_p(X, Y)$.

In the eighties, Matter considered the ideal of (p, σ)-absolutely continuous linear operators for any $p \in [1, \infty)$ and $\sigma \in [0, 1)$, with the aim of analysing super-reflexive Banach spaces, providing its main properties in the papers [13, 14].

Let us recall that a linear map $T : X \to Y$ is called (p, σ)-*absolutely continuous* for $p \in [1, \infty)$ and $\sigma \in [0, 1)$ if there exist a Banach space Z and a p-summing operator $S \in \Pi_p(X, Z)$ for which

$$\|T(x)\| \leq \|x\|^\sigma \|S(x)\|^{1-\sigma} \qquad (x \in X).$$

We set $\pi_{p,\sigma}(T) = \inf\{\pi_p(S)^{1-\sigma}\}$, where the infimum is taken over all Banach spaces Z and $S \in \Pi_p(X, Z)$ such that the above inequality holds. Let $\Pi_{p,\sigma}(X, Y)$ be the Banach space of all (p, σ)-absolutely continuous operators from X into Y, under the norm $\pi_{p,\sigma}$.

In the nineties, López Molina and Sánchez Pérez investigated on the factorization properties and the tensor norms related to these operator ideals in the papers [11, 12, 19]. Roughly speaking, the ideal of (p, σ)-absolutely continuous operators can be considered as an interpolating ideal between the p-summing operators and the continuous operators since

$$\Pi_p(X, Y) \subseteq \Pi_{p,\sigma}(X, Y) \subseteq \mathcal{L}(X, Y)$$

with

$$\|T\| \leq \pi_{p,\sigma}(T) \leq \pi_p(T) \qquad (T \in \Pi_p(X, Y)).$$

We refer the reader to the book [9] for a complete study on p-summing operators.

In the second decade of the twentieth century, Achour, Dahia, Rueda and Sánchez Pérez dealt with the factorization of both absolutely continuous polynomials and strongly (p, σ)-continuous multilinear operators in [1, 2]. Besides, Achour, Rueda and Yahi [3] extended these studies for Lipschitz maps from a metric space into a Banach space.

Our main purpose in this paper is to introduce and establish the most notable properties of a notion of (p, σ)-absolutely continuous Bloch map on the open unit disc $\mathbb{D} \subseteq \mathbb{C}$, in terms of the concept of p-summing Bloch map. From now on, unless otherwise stated, X will denote a complex Banach space.

If $\mathcal{H}(\mathbb{D}, X)$ represents the space of all holomorphic maps from $\mathbb{D}$ into X, a map $f \in \mathcal{H}(\mathbb{D}, X)$ is called *Bloch* if

$$\rho_{\mathcal{B}}(f) := \sup\left\{(1-|z|^2)\left\|f'(z)\right\| : z \in \mathbb{D}\right\} < \infty.$$

The linear space of all Bloch maps from $\mathbb{D}$ into X, under the *Bloch seminorm* $\rho_{\mathcal{B}}$, is denoted by $\mathcal{B}(\mathbb{D}, X)$. The *normalized Bloch space* $\widehat{\mathcal{B}}(\mathbb{D}, X)$ is the closed subspace of $\mathcal{B}(\mathbb{D}, X)$ formed by all those maps f for which $f(0) = 0$, under the *Bloch norm* $\rho_{\mathcal{B}}$. For simplicity, we write $\widehat{\mathcal{B}}(\mathbb{D})$ instead of $\widehat{\mathcal{B}}(\mathbb{D}, \mathbb{C})$. Numerous authors have studied these function spaces (see, for example, the monographs [4] for the complex-valued case, and [20] for the vector-valued case).

In a recent paper [6], the p-summability of operators was adapted to address the property of p-summability in the setting of Bloch maps, as follows.

For any $p \in [1, \infty)$, we say that a map $f \in \mathcal{H}(\mathbb{D}, X)$ is *p-summing Bloch* if there exists $C \geq 0$ such that for any n in $\mathbb{N}$, $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$ and $z_1, \ldots, z_n$ in $\mathbb{D}$, one has

$$\left(\sum_{i=1}^{n}|\lambda_i|^p\left\|f'(z_i)\right\|^p\right)^{\frac{1}{p}} \leq C \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}|\lambda_i|^p\left|g'(z_i)\right|^p\right)^{\frac{1}{p}}.$$

The infimum of the constants C for which this inequality holds, denoted by $\pi_p^{\mathcal{B}}$, defines a seminorm on the linear space $\Pi_p^{\mathcal{B}}(\mathbb{D}, X)$ of all p-absolutely continuous Bloch maps from $\mathbb{D}$ into X. Furthermore, this seminorm becomes a norm on the subspace $\Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ consisting of all those maps $f \in \Pi_p^{\mathcal{B}}(\mathbb{D}, X)$ so that $f(0) = 0$. A complete study on these spaces can be consulted in [6].

Now, we introduce the Bloch analogue of the notion of (p, σ)-absolutely continuous operator.

Definition 1.1 For any $p \in [1, \infty)$ and $\sigma \in [0, 1)$, we say that a map $f \in \mathcal{H}(\mathbb{D}, X)$ is (p, σ)-*absolutely continuous Bloch* if there exist a complex Banach space Y and a map $g \in \Pi_p^{\mathcal{B}}(\mathbb{D}, Y)$ such that

$$\left\|f'(z)\right\| \leq \left(\frac{1}{1-|z|^2}\right)^{\sigma}\left\|g'(z)\right\|^{1-\sigma} \qquad (z \in \mathbb{D}).$$

In such case, we put

$$\pi_{p,\sigma}^{\mathcal{B}}(f) = \inf\left\{\pi_p^{\mathcal{B}}(g)^{1-\sigma}\right\},$$

taking the infimum over all complex Banach spaces Y and all $g \in \Pi_p^{\mathcal{B}}(\mathbb{D}, Y)$ such that the above inequality holds. $\Pi_{p,\sigma}^{\mathcal{B}}(\mathbb{D}, X)$ stands for the linear space of all (p, σ)-

absolutely continuous Bloch maps $f:\mathbb{D}\to X$. The linear subspace of $\Pi^{\mathcal{B}}_{p,\sigma}(\mathbb{D},X)$ consisting of all those maps f for which $f(0)=0$ is denoted by $\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X)$.

We divide the contents of this paper into some sections. We start by showing that $(\Pi^{\mathcal{B}}_{p,0}(\mathbb{D},X),\pi^{\mathcal{B}}_{p,0})$ can be identified with $(\Pi^{\mathcal{B}}_{p}(\mathbb{D},X),\pi^{\mathcal{B}}_{p})$. For this reason, the results that we establish in this paper extend some obtained in [6]. In a clear parallel with the linear setting, the class $\Pi^{\mathcal{B}}_{p,\sigma}$ can be considered as an interpolating class between the classes $\Pi^{\mathcal{B}}_{p}$ and $\mathcal{B}$.

In Sects. 2 and 5, we prove that $[\Pi^{\widehat{\mathcal{B}}}_{p,\sigma},\pi^{\mathcal{B}}_{p,\sigma}]$ is an injective Banach normalized Bloch ideal. Sections 3 and 4 are devoted to both versions of Pietsch domination theorem and Pietsch factorization theorem for (p,σ)-absolutely continuous Bloch maps on $\mathbb{D}$. We also address the invariance of the space $(\Pi^{\mathcal{B}}_{p}(\mathbb{D},X),\pi^{\mathcal{B}}_{p})$ by Möbius transformations of $\mathbb{D}$. In Sect. 6, we introduce and analyse the so-called (p,σ)-*Chevet–Saphar Bloch norms* $d^{\widehat{\mathcal{B}}}_{p,\sigma}$ on the tensor product space $\mathcal{G}(\mathbb{D})\otimes X$, where $\mathcal{G}(\mathbb{D})$ is the Bloch-free Banach space. If $p^*=\infty$ for $p=1$, and $p^*=p/(p-1)$ for $1<p<\infty$, we show that $(\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X^*),\pi^{\mathcal{B}}_{p,\sigma})$ can be canonically identified with the dual of the completion of the space $\mathcal{G}(\mathbb{D})\otimes_{d^{\widehat{\mathcal{B}}}_{p^*,\sigma}} X$.

2 Banach structure

We begin with an easy result on interpolation which can be compared to [13, Proposition 3.3]. We will need the following class of Bloch functions. For each $z\in\mathbb{D}$, the map $f_z:\mathbb{D}\to\mathbb{C}$ defined by

$$f_z(w)=\frac{(1-|z|^2)w}{1-\overline{z}w}\qquad(w\in\mathbb{D}),$$

is in $\widehat{\mathcal{B}}(\mathbb{D})$ and $\rho_{\mathcal{B}}(f_z)=1=(1-|z|^2)f_z'(z)$ (see [10, Proposition 2.2]). Clearly, $f_z\in\Pi^{\widehat{\mathcal{B}}}_{p}(\mathbb{D},\mathbb{C})$ with $\pi^{\mathcal{B}}_{p}(f_z)\le 1$ for any $p\in[1,\infty)$.

Given two semi-normed spaces (X,ρ_X) and (Y,ρ_Y), we will write $(X,\rho_X)\le(Y,\rho_Y)$ to indicate that $X\subseteq Y$ and $\rho_Y(x)\le\rho_X(x)$ for all $x\in X$.

Proposition 2.1 *If $p,q\in[1,\infty)$ with $p<q$ and $\sigma\in[0,1)$, then*

$$\begin{aligned}(\Pi^{\mathcal{B}}_{p,0}(\mathbb{D},X),\pi^{\mathcal{B}}_{p,0})=(\Pi^{\mathcal{B}}_{p}(\mathbb{D},X),\pi^{\mathcal{B}}_{p})&\le(\Pi^{\mathcal{B}}_{p,\sigma}(\mathbb{D},X),\pi^{\mathcal{B}}_{p,\sigma})\\&\le(\Pi^{\mathcal{B}}_{q,\sigma}(\mathbb{D},X),\pi^{\mathcal{B}}_{q,\sigma})\le(\mathcal{B}(\mathbb{D},X),\rho_{\mathcal{B}}).\end{aligned}$$

Proof If $f\in\Pi^{\mathcal{B}}_{p,0}(\mathbb{D},X)$, there is a map $g\in\Pi^{\mathcal{B}}_{p}(\mathbb{D},Y)$ for some complex Banach space Y such that $\|f'(z)\|\le\|g'(z)\|$ for all $z\in\mathbb{D}$. Given $n\in\mathbb{N}$, $\lambda_i\in\mathbb{C}$ and $z_i\in\mathbb{D}$ for all $i\in\{1,\ldots,n\}$, we get

$$\left(\sum_{i=1}^{n}|\lambda_i|^p\left\|f'(z_i)\right\|^p\right)^{\frac{1}{p}}\le\left(\sum_{i=1}^{n}|\lambda_i|^p\left\|g'(z_i)\right\|^p\right)^{\frac{1}{p}}$$

$$\leq \pi_p^{\mathcal{B}}(g) \sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} |\lambda_i|^p \left|h'(z_i)\right|^p\right)^{\frac{1}{p}},$$

hence $f \in \Pi_p^{\mathcal{B}}(\mathbb{D}, X)$ with $\pi_p^{\mathcal{B}}(f) \leq \pi_p^{\mathcal{B}}(g)$, and passing to the infimum over all such complex Banach spaces Y and all such maps g, one has $\rho_{\mathcal{B}}(f) \leq \pi_{p,0}^{\mathcal{B}}(f)$.

The inequality $(\Pi_p^{\mathcal{B}}(\mathbb{D}, X), \pi_p^{\mathcal{B}}) \leq (\Pi_{p,0}^{\mathcal{B}}(\mathbb{D}, X), \pi_{p,0}^{\mathcal{B}})$ is a particular case of the following. If $f \in \Pi_p^{\mathcal{B}}(\mathbb{D}, X)$, then

$$\begin{aligned}\left\|f'(z)\right\| &\leq \pi_p^{\mathcal{B}}(f) \sup_{g\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left|g'(z)\right| \leq \pi_p^{\mathcal{B}}(f)\frac{1}{1-|z|^2} \\ &= \pi_p^{\mathcal{B}}(f)\left(\frac{1}{1-|z|^2}\right)^{\sigma} \left|f_z'(z)\right|^{1-\sigma} \qquad (z\in\mathbb{D}).\end{aligned}$$

as for the second inequality we use the supremum is taken over $g's$ in $B_{\widehat{\mathcal{B}}(\mathbb{D})}$ and that $\rho_{\mathcal{B}}(g) \geq (1-|z|^2)\left\|g'(z)\right\|$. Hence $f \in \Pi_{p,\sigma}^{\mathcal{B}}(\mathbb{D}, X)$ with

$$\pi_{p,\sigma}^{\mathcal{B}}(f) \leq \pi_p^{\mathcal{B}}(\pi_p^{\mathcal{B}}(f)^{\frac{1}{1-\sigma}} f_z)^{1-\sigma} = \pi_p^{\mathcal{B}}(f)\pi_p^{\mathcal{B}}(f_z)^{1-\sigma} \leq \pi_p^{\mathcal{B}}(f).$$

If $f \in \Pi_{p,\sigma}^{\mathcal{B}}(\mathbb{D}, X)$, then $f \in \Pi_{q,\sigma}^{\mathcal{B}}(\mathbb{D}, X)$ with $\pi_{q,\sigma}^{\mathcal{B}}(f) \leq \pi_{p,\sigma}^{\mathcal{B}}(f)$ follows readily by applying [6, Proposition 1.1].

If $f \in \Pi_{q,\sigma}^{\mathcal{B}}(\mathbb{D}, X)$, we can take a complex Banach space Y and a map $g \in \Pi_q^{\mathcal{B}}(\mathbb{D}, Y)$ such that

$$\left\|f'(z)\right\| \leq \left(\frac{1}{1-|z|^2}\right)^{\sigma} \left\|g'(z)\right\|^{1-\sigma} \qquad (z\in\mathbb{D}).$$

It follows that

$$(1-|z|^2)\left\|f'(z)\right\| \leq \left((1-|z|^2)\left\|g'(z)\right\|\right)^{1-\sigma} \leq \rho_{\mathcal{B}}(g)^{1-\sigma} \qquad (z\in\mathbb{D}),$$

hence $f \in \mathcal{B}(\mathbb{D}, X)$ with $\rho_{\mathcal{B}}(f) \leq \rho_{\mathcal{B}}(g)^{1-\sigma}$, and taking infimum over all such complex Banach spaces Y and such maps g, we conclude that $\rho_{\mathcal{B}}(f) \leq \pi_{q,\sigma}^{\mathcal{B}}(f)$. □

The case $\sigma = 0$ in the next result follows from Proposition 2.1 and [6, Proposition 1.2]. In fact, we can adapt the proof of [6, Proposition 1.2] to yield a more general result.

Proposition 2.2 *$\left(\Pi_{p,\sigma}^{\widehat{\mathcal{B}}}(\mathbb{D}, X), \pi_{p,\sigma}^{\mathcal{B}}\right)$ is a Banach space for any $p \in [1,\infty)$ and $\sigma \in [0,1)$.*

Proof Assume that $\sigma \in (0,1)$. If $f \in \Pi_{p,\sigma}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ and $\pi_{p,\sigma}^{\mathcal{B}}(f) = 0$, then $\rho_{\mathcal{B}}(f) = 0$ by Proposition 2.1, and so $f = 0$. We now prove the triangle inequality. For $i = 1, 2$, consider $f_i \in \Pi_{p,\sigma}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$, a complex Banach space Y_i, and $g_i \in \Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D}, Y_i)$ such that

$$\left\| f_i'(z) \right\| \leq \left(\frac{1}{1-|z|^2} \right)^{\sigma} \left\| g_i'(z) \right\|_{Y_i}^{1-\sigma} \qquad (z \in \mathbb{D}).$$

Let Y be the ℓ_1-sum of Y_1 and Y_2, and let $I_i : Y_i \to Y$ be the canonical injection. The map $g = \sum_{i=1}^{2} \pi_p^{\mathcal{B}}(g_i)^{-\sigma}(I_i \circ g_i)$ belongs to $\Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D}, Y)$ and $\pi_p^{\mathcal{B}}(g) \leq \sum_{i=1}^{2} \pi_p^{\mathcal{B}}(g_i)^{1-\sigma}$. Using Holder's Inequality, we get

$$\begin{aligned}
\left\| \sum_{i=1}^{2} f_i'(z) \right\| &\leq \sum_{i=1}^{2} \left\| f_i'(z) \right\| \leq \left(\frac{1}{1-|z|^2} \right)^{\sigma} \sum_{i=1}^{2} \left\| g_i'(z) \right\|_{Y_i}^{1-\sigma} \\
&= \left(\frac{1}{1-|z|^2} \right)^{\sigma} \sum_{i=1}^{2} \left\| \pi_p^{\mathcal{B}}(g_i)^{-\sigma} g_i'(z) \right\|_{Y_i}^{1-\sigma} \pi_p^{\mathcal{B}}(g_i)^{\sigma(1-\sigma)} \\
&\leq \left(\frac{1}{1-|z|^2} \right)^{\sigma} \left(\sum_{i=1}^{2} \left\| \pi_p^{\mathcal{B}}(g_i)^{-\sigma} g_i'(z) \right\|_{Y_i} \right)^{1-\sigma} \left(\sum_{i=1}^{2} \pi_p^{\mathcal{B}}(g_i)^{1-\sigma} \right)^{\sigma} \\
&= \left(\frac{1}{1-|z|^2} \right)^{\sigma} \left\| g'(z) \right\|_{Y}^{1-\sigma} \left(\pi_p^{\mathcal{B}}(g_1)^{1-\sigma} + \pi_p^{\mathcal{B}}(g_2)^{1-\sigma} \right)^{\sigma} \qquad (z \in \mathbb{D}).
\end{aligned}$$

Thus $\sum_{i=1}^{2} f_i \in \Pi_{p,\sigma}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ with

$$\pi_{p,\sigma}^{\mathcal{B}} \left(\sum_{i=1}^{2} f_i \right) \leq \left(\sum_{i=1}^{2} \pi_p^{\mathcal{B}}(g_i)^{1-\sigma} \right)^{\sigma} \pi_p^{\mathcal{B}}(g)^{1-\sigma} \leq \sum_{i=1}^{2} \pi_p^{\mathcal{B}}(g_i)^{1-\sigma}.$$

Passing to the infimum over all such complex Banach spaces Y and such maps g_1 and g_2, we deduce that $\pi_{p,\sigma}^{\mathcal{B}}\left(\sum_{i=1}^{2} f_i\right) \leq \sum_{i=1}^{2} \pi_{p,\sigma}^{\mathcal{B}}(f_i)$.

Let $\lambda \in \mathbb{C}$ and $f \in \Pi_{p,\sigma}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$. We have a complex Banach space Y and $g \in \Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D}, Y)$ such that

$$\left\| f'(z) \right\| \leq \left(\frac{1}{1-|z|^2} \right)^{\sigma} \left\| g'(z) \right\|^{1-\sigma} \qquad (z \in \mathbb{D}).$$

Therefore,

$$\left\| (\lambda f)'(z) \right\| \leq |\lambda| \left(\frac{1}{1-|z|^2} \right)^{\sigma} \left\| g'(z) \right\|^{1-\sigma} = \left(\frac{1}{1-|z|^2} \right)^{\sigma} \left\| \left(\lambda^{\frac{1}{1-\sigma}} g \right)'(z) \right\|^{1-\sigma} \qquad (z \in \mathbb{D}).$$

Since $\lambda^{\frac{1}{1-\sigma}} g \in \Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$, we have $\lambda f \in \Pi_{p,\sigma}^{\widehat{\mathcal{B}}}(\mathbb{D}, X)$ with $\pi_{p,\sigma}^{\mathcal{B}}(\lambda f) \leq \pi_p^{\mathcal{B}}\left(\lambda^{\frac{1}{1-\sigma}} g\right)^{1-\sigma} = |\lambda| \pi_p^{\mathcal{B}}(g)^{1-\sigma}$. For $\lambda = 0$, we obtain $\pi_{p,\sigma}^{\mathcal{B}}(\lambda f) = 0 = |\lambda| \pi_{p,\sigma}^{\mathcal{B}}(f)$. For $\lambda \neq 0$, we deduce that $\pi_{p,\sigma}^{\mathcal{B}}(\lambda f) \leq |\lambda| \pi_{p,\sigma}^{\mathcal{B}}(f)$. Hence $\pi_{p,\sigma}^{\mathcal{B}}(f) \leq$

$|\lambda|^{-1} \pi^{\mathcal{B}}_{p,\sigma}(\lambda f)$, then $|\lambda| \pi^{\mathcal{B}}_{p,\sigma}(f) \leq \pi^{\mathcal{B}}_{p,\sigma}(\lambda f)$, and thus $\pi^{\mathcal{B}}_{p,\sigma}(\lambda f) = |\lambda| \pi^{\mathcal{B}}_{p,\sigma}(f)$. So $(\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X), \pi^{\mathcal{B}}_{p,\sigma})$ is a complex normed space.

To prove its completeness, let (f_n) be a sequence in $\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ for which $\sum_{n=1}^{\infty} \pi^{\mathcal{B}}_{p,\sigma}(f_n) < \infty$. Since $\rho_{\mathcal{B}} \leq \pi^{\mathcal{B}}_{p,\sigma}$ on $\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ (by Proposition 2.1) and $\widehat{\mathcal{B}}(\mathbb{D}, X)$ with the norm $\rho_{\mathcal{B}}$ is a Banach space, there exists $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ such that $\sum_{n=1}^{\infty} f_n = f$ for $\rho_{\mathcal{B}}$. We will prove that $\sum_{n=1}^{\infty} f_n = f$ for $\pi^{\mathcal{B}}_{p,\sigma}$. Let $\varepsilon > 0$, and for each $n \in \mathbb{N}$, we can take a complex Banach space Y_n and a map $g_n \in \Pi^{\widehat{\mathcal{B}}}_{p}(\mathbb{D}, Y_n)$ for which

$$\left\| f_n'(z) \right\| \leq \left(\frac{1}{1 - |z|^2} \right)^{\sigma} \left\| g_n'(z) \right\|_{Y_n}^{1-\sigma} \qquad (z \in \mathbb{D}),$$

with

$$\pi^{\mathcal{B}}_{p}(g_n)^{1-\sigma} \leq \pi^{\mathcal{B}}_{p,\sigma}(f_n) + \frac{\varepsilon}{2^n}.$$

Then

$$\sum_{n=1}^{\infty} \pi^{\mathcal{B}}_{p}(g_n)^{1-\sigma} \leq \sum_{n=1}^{\infty} \pi^{\mathcal{B}}_{p,\sigma}(f_n) + \varepsilon.$$

Let $g = \sum_{n=1}^{\infty} \pi^{\mathcal{B}}_{p}(g_n)^{-\sigma} (I_n \circ g_n) \in \Pi^{\widehat{\mathcal{B}}}_{p}(\mathbb{D}, Y)$, where Y is the ℓ_1-sum of all Y_n and $I_n \colon Y_n \to Y$ is the canonical injection. Hence

$$\begin{aligned} \left\| f'(z) \right\| &\leq \sum_{n=1}^{\infty} \left\| f_n'(z) \right\| \leq \sum_{n=1}^{\infty} \left(\frac{1}{1 - |z|^2} \right)^{\sigma} \left\| g_n'(z) \right\|_{Y_n}^{1-\sigma} \\ &\leq \left(\frac{1}{1 - |z|^2} \right)^{\sigma} \left\| g'(z) \right\|_{Y}^{1-\sigma} \left(\sum_{n=1}^{\infty} \pi^{\mathcal{B}}_{p}(g_n)^{1-\sigma} \right)^{\sigma} \qquad (z \in \mathbb{D}). \end{aligned}$$

This implies that $f \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ with

$$\begin{aligned} \pi^{\mathcal{B}}_{p,\sigma}(f) &\leq \left(\sum_{n=1}^{\infty} \pi^{\mathcal{B}}_{p}(g_n)^{1-\sigma} \right)^{\sigma} \pi_p(g)^{1-\sigma} \\ &\leq \left(\sum_{n=1}^{\infty} \pi^{\mathcal{B}}_{p}(g_n)^{1-\sigma} \right)^{\sigma} \left(\sum_{n=1}^{\infty} \pi^{\mathcal{B}}_{p}(g_n)^{1-\sigma} \right)^{1-\sigma} \\ &= \sum_{n=1}^{\infty} \pi^{\mathcal{B}}_{p}(g_n)^{1-\sigma} \leq \sum_{n=1}^{\infty} \pi^{\mathcal{B}}_{p,\sigma}(f_n) + \varepsilon. \end{aligned}$$

Moreover, we have

$$\pi_{p,\sigma}^{\mathcal{B}}\left(f-\sum_{k=1}^{n} f_k\right)=\pi_{p,\sigma}^{\mathcal{B}}\left(\sum_{k=n+1}^{\infty} f_k\right)\leq \sum_{k=n+1}^{\infty}\pi_{p}^{\mathcal{B}}(f_k) \qquad (n\in\mathbb{N}),$$

and thus $\sum_{n=1}^{\infty} f_n = f$ for $\pi_{p,\sigma}^{\mathcal{B}}$. □

3 Pietsch domination

Our next result is a reformulation for (p,σ)-absolutely continuous Bloch maps of Pietsch domination theorem for (p,σ)-absolutely continuous operators stated by Matter in [13, Theorem 4.1]. However, to prove our result, we will apply an unified abstract version of the Pietsch domination theorem established by Pellegrino and Santos in [15, Theorem 3.1] (see also [5, 16]). Our proof is based on [6, Theorem 1.4 and Lemma 1.5].

Let us recall that $\widehat{\mathcal{B}}(\mathbb{D})$ is a dual Banach space (see, for example, [20]) and therefore we can consider this space equipped with its weak* topology. Let $\mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$ be the set of all Borel regular probability measures μ on $(B_{\widehat{\mathcal{B}}(\mathbb{D})}, w^*)$.

Given $\mu\in\mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$, $p\in[1,\infty)$ and $\sigma\in[0,1)$, consider the inclusion operators

$$I_{\infty,p/(1-\sigma)}\colon L_\infty(\mu)\to L_{p/(1-\sigma)}(\mu)$$

and

$$j_\infty\colon C(B_{\widehat{\mathcal{B}}(\mathbb{D})})\to L_\infty(\mu).$$

We will also use the map

$$\iota_{\mathbb{D}}\colon \mathbb{D}\to C(B_{\widehat{\mathcal{B}}(\mathbb{D})})$$

defined by

$$\iota_{\mathbb{D}}(z)(g)=g'(z) \qquad \left(g\in B_{\widehat{\mathcal{B}}(\mathbb{D})},\ z\in\mathbb{D}\right),$$

and, for a complex Banach space X, the isometric linear embedding

$$\iota_X\colon X\to \ell_\infty(B_{X^*})$$

given by

$$\iota_X(x)(x^*)=x^*(x) \qquad (x^*\in B_{X^*},\ x\in X).$$

Theorem 3.1 (Pietsch domination). *Let* $p\in[1,\infty)$, $\sigma\in[0,1)$ *and* $f\in\widehat{\mathcal{B}}(\mathbb{D},X)$. *The following are equivalent:*

(1) $f\in\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X)$.
(2) *There is a constant* $C\geq 0$ *and a measure* $\mu\in\mathcal{P}\left(B_{\widehat{\mathcal{B}}(\mathbb{D})}\right)$ *such that*

$$\left\|f'(z)\right\|\leq C\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\left(\frac{1}{1-|z|^2}\right)^{\sigma}\left|h'(z)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\mathrm{d}\mu(h)\right)^{\frac{1-\sigma}{p}}$$

for all $z\in\mathbb{D}$.
(3) *There is a constant* $C\geq 0$ *such that*

$$\left(\sum_{i=1}^{n}|\lambda_i|^{\frac{p}{1-\sigma}}\left\|f'(z_i)\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}\leq C\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|h'(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}$$

for all $n\in\mathbb{N}$, $\lambda_i\in\mathbb{C}$ *and* $z_i\in\mathbb{D}$ *for all* $i\in\{1,\ldots,n\}$.

Furthermore, the infimum of the constants $C\geq 0$ *in* (2) *(and in* (3)*) is* $\pi^{\mathcal{B}}_{p,\sigma}(f)$.

Proof (1) $\Rightarrow$ (2): If $f\in\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X)$, then there exist a complex Banach space Y and a map $g\in\Pi^{\widehat{\mathcal{B}}}_{p}(\mathbb{D},Y)$ such that

$$\left\|f'(z)\right\|\leq\left(\frac{1}{1-|z|^2}\right)^{\sigma}\|g(z)\|^{1-\sigma}\qquad(z\in\mathbb{D}).$$

By [6, Theorem 1.4], there is a measure $\mu\in\mathcal{P}\left(B_{\widehat{\mathcal{B}}(\mathbb{D})}\right)$ such that

$$\left\|g'(z)\right\|\leq\pi^{\mathcal{B}}_{p}(g)\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left|h'(z)\right|^{p}\mathrm{d}\mu(h)\right)^{\frac{1}{p}}\qquad(z\in\mathbb{D}),$$

and therefore

$$\begin{aligned}\left\|f'(z)\right\|&\leq\left(\frac{1}{1-|z|^2}\right)^{\sigma}\left\|g'(z)\right\|^{1-\sigma}\\&\leq\pi^{\mathcal{B}}_{p}(g)^{1-\sigma}\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\left(\frac{1}{1-|z|^2}\right)^{\sigma}\left|h'(z)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\mathrm{d}\mu(h)\right)^{\frac{1-\sigma}{p}}\quad(z\in\mathbb{D}).\end{aligned}$$

(2) $\Rightarrow$ (1): By [6, Lemma 1.5], there exists a map $k\in\widehat{\mathcal{B}}(\mathbb{D},L_\infty(\mu))$ with $\rho_{\mathcal{B}}(k)=1$ such that $k'=j_\infty\circ\iota_{\mathbb{D}}$. In fact, $k\in\Pi^{\widehat{\mathcal{B}}}_{p}(\mathbb{D},L_\infty(\mu))$ with $\pi^{\mathcal{B}}_{p}(k)=1$. By (2), we

can write

$$\begin{aligned}\left\| f'(z)\right\| &\leq C\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\left(\frac{1}{1-|z|^2}\right)^{\sigma}\left|h'(z)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\mathrm{d}\mu(h)\right)^{\frac{1-\sigma}{p}}\\ &=\left(\frac{1}{1-|z|^2}\right)^{\sigma}\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left|C^{\frac{1}{1-\sigma}}\left(I_{\infty,p}\circ j_{\infty}\circ \iota_{\mathbb{D}}\right)(z)\,(h)\right|^{p}\mathrm{d}\mu\,(h)\right)^{\frac{1-\sigma}{p}}\\ &=\left(\frac{1}{1-|z|^2}\right)^{\sigma}\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D}}}\left|C^{\frac{1}{1-\sigma}}\left(I_{\infty,p}\circ k\right)'(z)\,(h)\right|^{p}\mathrm{d}\mu\,(h)\right)^{\frac{1-\sigma}{p}}\\ &=\left(\frac{1}{1-|z|^2}\right)^{\sigma}\left\| g'(z)\right\|_{L_p(\mu)}^{1-\sigma},\end{aligned}$$

where $g=C^{\frac{1}{1-\sigma}}\left(I_{\infty,p}\circ k\right)\in \Pi_p^{\widehat{\mathcal{B}}}(\mathbb{D},L_p(\mu))$.

(2) $\Rightarrow$ (3): If (2) holds, then

$$\begin{aligned}&\left(\sum_{i=1}^{n}|\lambda_i|^{\frac{p}{1-\sigma}}\left\| f'(z_i)\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}\\ &\leq C\left(\sum_{i=1}^{n}\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|h'(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\mathrm{d}\mu\,(h)\right)^{\frac{1-\sigma}{p}}\\ &\leq C\left(\sum_{i=1}^{n}\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left(\frac{1}{1-|z_i|^2}\right)^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\mathrm{d}\mu\,(h)\right)^{\frac{1-\sigma}{p}}\\ &=C\left(\sum_{i=1}^{n}\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|f'_{z_i}(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\mathrm{d}\mu\,(h)\right)^{\frac{1-\sigma}{p}}\\ &\leq C\left(\sum_{i=1}^{n}\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|f'_{z_i}(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}\\ &\leq C\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|h'(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}\end{aligned}$$

for all $n\in\mathbb{N}$ $\lambda_i\in\mathbb{C}$ and $z_i\in\mathbb{D}$ for all $i\in\{1,\ldots,n\}$, and this proves (3).

(3) $\Rightarrow$ (2): Let $R: B_{\widehat{\mathcal{B}}(\mathbb{D})}\times(\mathbb{D}\times\mathbb{R})\times\mathbb{R}\to\Big[0,+\infty\Big[$ be given by

$$R\left(h,(z,\lambda),b\right)=|\lambda|\left(\frac{1}{1-|z|^2}\right)^{\sigma}\left|h'(z)\right|^{1-\sigma}|b|,$$

and let $S : \widehat{\mathcal{B}}(\mathbb{D}, X) \times (\mathbb{D} \times \mathbb{R}) \times \mathbb{R} \to \left[0, +\infty\right[$ be defined by

$$S\left(f, (z, \lambda), b\right) = |\lambda| \left\| f'(z) \right\| |b|.$$

Then f is R-S-abstract $p/(1-\sigma)$-summing (see definition in [15]) since

$$\begin{aligned}
\left(\sum_{i=1}^{n} S\left(f, (z_i, \lambda_i), b_i\right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} &= \left(\sum_{i=1}^{n} \left(|\lambda_i| \left\| f'(z_i) \right\| |b_i| \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\leq C \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} \left(|\lambda_i| \left(\frac{1}{1-|z_i|^2} \right)^{\sigma} \left| h'(z_i) \right|^{1-\sigma} |b_i| \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&= C \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} R\left(h, (z_i, \lambda_i), b_i\right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}.
\end{aligned}$$

Then, by [15, Theorem 3.1], there are $C > 0$ and $\mu \in \mathcal{P}\left(B_{\widehat{\mathcal{B}}(\mathbb{D})}\right)$ such that

$$S\left(f, (z, \lambda), b\right) \leq C \left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} R\left(h, (z, \lambda), b\right)^{\frac{p}{1-\sigma}} \mathrm{d}\mu(h) \right)^{\frac{1-\sigma}{p}}$$

for all $(z, \lambda) \in \mathbb{D} \times \mathbb{R}$ and $b \in \mathbb{R}$. In particular, we have

$$\left\| f'(z) \right\| \leq C \left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\left(\frac{1}{1-|z|^2} \right)^{\sigma} \left| h'(z) \right|^{1-\sigma} \right)^{\frac{p}{1-\sigma}} \mathrm{d}\mu(h) \right)^{\frac{1-\sigma}{p}} \qquad (z \in \mathbb{D}).$$

□

4 Pietsch factorization

We now present the analogue for (p, σ)-absolutely continuous Bloch maps of Pietsch factorization theorem for (p, σ)-summing operators. Its proof is based on those of [6, Theorem 1.6] and [7, Theorem 3.5].

Theorem 4.1 (Pietsch factorization). *Let* $p \in [1, \infty)$, $\sigma \in [0, 1)$ *and* $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$. *The following are equivalent:*

(1) $f \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$.

(2) *There exist a measure $\mu \in \mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$, a map $h \in \widehat{\mathcal{B}}(\mathbb{D}, L_\infty(\mu))$ and an operator $T \in \mathcal{L}(L_{p/(1-\sigma)}(\mu), \ell_\infty(B_{X^*}))$ such that the following diagram commutes:*

$$\begin{array}{ccccc} L_\infty(\mu) & & \xrightarrow{I_{\infty,p/(1-\sigma)}} & & L_{p/(1-\sigma)}(\mu) \\ \uparrow h' & & & & \downarrow T \\ \mathbb{D} & \xrightarrow{f'} & X & \xrightarrow{\iota_X} & \ell_\infty(B_{X^*}) \end{array}$$

Furthermore, $\pi^{\mathcal{B}}_{p,\sigma}(f) = \inf\{\|T\|\,\rho_{\mathcal{B}}(h)\}$, the infimum being extended over all such decompositions of $\iota_X \circ f'$ as above, and this infimum is attained.

Proof If (1) holds, then Theorem 3.1 provides a measure $\mu \in \mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$ such that

$$\left\|f'(z)\right\| \leq \pi^{\mathcal{B}}_{p,\sigma}(f)\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\left(\frac{1}{1-|z|^2}\right)^{\sigma}\left|g'(z)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}} \mathrm{d}\mu(g)\right)^{\frac{1-\sigma}{p}} \qquad (z \in \mathbb{D}).$$

By [6, Lemma 1.5], there exists a map $h \in \widehat{\mathcal{B}}(\mathbb{D}, L_\infty(\mu))$ with $\rho_{\mathcal{B}}(h) = 1$ such that $h' = j_\infty \circ \iota_{\mathbb{D}}$. Denote the closed linear subspace

$$S_{p/(1-\sigma)} := \overline{\mathrm{lin}}(I_{\infty,p/(1-\sigma)}(h'(\mathbb{D}))) \subseteq L_{p/(1-\sigma)}(\mu),$$

and define $T_0 \in \mathcal{L}(S_{p/(1-\sigma)}, \ell_\infty(B_{X^*}))$ by

$$T_0(I_{\infty,p/(1-\sigma)}(h'(z))) = \iota_X(f'(z)) \qquad (z \in \mathbb{D}).$$

Notice that $\|T_0\| \leq \pi^{\mathcal{B}}_{p,\sigma}(f)$ since

$$\begin{aligned} \left\|T_0\left(\sum_{i=1}^n \alpha_i I_{\infty,p/(1-\sigma)}(h'(z_i))\right)\right\|_\infty &= \left\|\sum_{i=1}^n \alpha_i T_0(I_{\infty,p/(1-\sigma)}(h'(z_i)))\right\|_\infty \\ &= \left\|\sum_{i=1}^n \alpha_i \iota_X(f'(z_i))\right\|_\infty \leq \sum_{i=1}^n |\alpha_i|\,\left\|\iota_X(f'(z_i))\right\|_\infty = \sum_{i=1}^n |\alpha_i|\,\left\|f'(z_i)\right\| \\ &\leq \pi^{\mathcal{B}}_{p,\sigma}(f)\sum_{i=1}^n |\alpha_i|\left(\int_{B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|g'(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}} \mathrm{d}\mu(g)\right)^{\frac{1-\sigma}{p}} \\ &\leq \pi^{\mathcal{B}}_{p,\sigma}(f)\sum_{i=1}^n \frac{|\alpha_i|}{1-|z_i|^2} \end{aligned}$$

and

$$\sum_{i=1}^n \frac{|\alpha_i|}{1-|z_i|^2} = \left|\sum_{i=1}^n \alpha_i \frac{\overline{\alpha_i}}{|\alpha_i|} f'_{z_i}(z_i)\right| = \sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left|\sum_{i=1}^n \alpha_i g'(z_i)\right| = \sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left|\sum_{i=1}^n \alpha_i \iota_{\mathbb{D}}(z_i)(g)\right|$$

$$
\begin{aligned}
&= \left\| \sum_{i=1}^{n} \alpha_i \iota_{\mathbb{D}}(z_i) \right\|_{\infty} = \left\| \sum_{i=1}^{n} \alpha_i j_\infty(\iota_{\mathbb{D}}(z_i)) \right\|_{L_\infty(\mu)} = \left\| \sum_{i=1}^{n} \alpha_i h'(z_i) \right\|_{L_\infty(\mu)} \\
&= \left\| I_{\infty, p/(1-\sigma)} \left(\sum_{i=1}^{n} \alpha_i h'(z_i) \right) \right\|_{L_{p/(1-\sigma)}} = \left\| \sum_{i=1}^{n} \alpha_i I_{\infty, p/(1-\sigma)}(h'(z_i)) \right\|_{L_{p/(1-\sigma)}}
\end{aligned}
$$

for any $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}^*$ and $z_i \in \mathbb{D}$ for all $i \in \{1, \ldots, n\}$. By the injectivity of the Banach space $\ell_\infty(B_{X^*})$ (see [9, p. 45]), there exists $T \in \mathcal{L}(L_{p/(1-\sigma)}(\mu), \ell_\infty(B_{X^*}))$ such that $T|_{S_{p/(1-\sigma)}} = T_0$ with $\|T\| = \|T_0\|$. This allows us to conclude that $\iota_X \circ f' = T \circ I_{\infty, p/(1-\sigma)} \circ h'$ with $\|T\| \, \rho_{\mathcal{B}}(h) \leq \pi^{\mathcal{B}}_{p,\sigma}(f)$.

Conversely, assume that $\iota_X \circ f' = T \circ I_{\infty, p/(1-\sigma)} \circ h'$ as in (2). We have

$$
\begin{aligned}
&\left(\sum_{i=1}^{n} |\lambda_i|^{\frac{p}{1-\sigma}} \left\| f'(z_i) \right\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} = \left(\sum_{i=1}^{n} |\lambda_i|^{\frac{p}{1-\sigma}} \left\| \iota_X(f'(z_i)) \right\|_\infty^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&= \left(\sum_{i=1}^{n} |\lambda_i|^{\frac{p}{1-\sigma}} \left\| T(I_{\infty, p/(1-\sigma)}(h'(z_i))) \right\|_\infty^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\leq \|T\| \left(\sum_{i=1}^{n} |\lambda_i|^{\frac{p}{1-\sigma}} \left\| I_{\infty, p/(1-\sigma)}(h'(z_i)) \right\|_{L_{p/(1-\sigma)}(\mu)}^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&= \|T\| \left(\sum_{i=1}^{n} |\lambda_i|^{\frac{p}{1-\sigma}} \left\| h'(z_i) \right\|_{L_\infty(\mu)}^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\leq \|T\| \, \rho_{\mathcal{B}}(h) \left(\sum_{i=1}^{n} \frac{|\lambda_i|^{\frac{p}{1-\sigma}}}{(1 - |z_i|^2)^{\frac{p}{1-\sigma}}} \right)^{\frac{1-\sigma}{p}} \\
&= \|T\| \, \rho_{\mathcal{B}}(h) \left(\sum_{i=1}^{n} \left(|\lambda_i| \left(\frac{1}{1 - |z_i|^2} \right)^{\sigma} \left| f'_{z_i}(z_i) \right|^{1-\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \\
&\leq \|T\| \, \rho_{\mathcal{B}}(h) \sup_{g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} \left(|\lambda_i| \left(\frac{1}{1 - |z_i|^2} \right)^{\sigma} \left| g'(z_i) \right|^{1-\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}
\end{aligned}
$$

for any $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$ and $z_i \in \mathbb{D}$ for all $i \in \{1, \ldots, n\}$. Hence $f \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ with $\pi^{\mathcal{B}}_{p,\sigma}(f) \leq \|T\| \, \rho_{\mathcal{B}}(h)$ by Theorem 3.1. □

We now relate (p, σ)-absolutely continuous Bloch maps with (weakly) compact Bloch maps which were introduced in [10].

Let us recall that the *Bloch range* of a function $f \in \mathcal{H}(\mathbb{D}, X)$, denoted by $\mathrm{rang}_{\mathcal{B}}(f)$, is the set

$$
\left\{ (1 - |z|^2) f'(z) \in X \colon z \in \mathbb{D} \right\}.
$$

A map $f \in \mathcal{H}(\mathbb{D}, X)$ is called *(weakly) compact Bloch* if $\text{rang}_{\mathcal{B}}(f)$ is a relatively (weakly) compact set in X.

Proposition 4.2 *If $p \in [1, \infty)$ and $\sigma \in [0, 1)$, then every (p, σ)-absolutely continuous Bloch map $f : \mathbb{D} \to X$ is weakly compact Bloch, and if in addition X is reflexive, then f is compact Bloch.*

Proof Let $f \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$. Hence Theorem 4.1 guarantees that

$$(\iota_X \circ f)' = \iota_X \circ f' = T \circ I_{\infty, p/(1-\sigma)} \circ h' = T \circ (I_{\infty, p/(1-\sigma)} \circ h)',$$

for some measure $\mu \in \mathcal{P}(B_{\widehat{\mathcal{B}}(\mathbb{D})})$, an operator $T \in \mathcal{L}(L_{p/(1-\sigma)}(\mu), \ell_\infty(B_{X^*}))$ and a map $h \in \widehat{\mathcal{B}}(\mathbb{D}, L_\infty(\mu))$. Assume first $p > 1$ and then the reflexivity of $L_{p/(1-\sigma)}(\mu)$ shows that $\iota_X \circ f \in \widehat{\mathcal{B}}(\mathbb{D}, \ell_\infty(B_{X^*}))$ is weakly compact Bloch by [10, Theorem 5.6]. Now, the equality $\text{rang}_{\mathcal{B}}(\iota_X \circ f) = \iota_X(\text{rang}_{\mathcal{B}}(f))$ yields that f is weakly compact Bloch. The case $p = 1$ follows from the previous case when $\sigma \in (0, 1)$ and from Proposition 2.1 and [6, Corollary 1.7] when $\sigma = 0$.

So we have proved that $\text{rang}_{\mathcal{B}}(f)$ is relatively weakly compact in X, and therefore relatively compact in X whenever X is reflexive. □

5 Injective Banach normalized Bloch ideal

Motivated by the theory of operator ideals between Banach spaces [17], the concept of a Banach normalized Bloch ideal on $\mathbb{D}$ was introduced in [10, Definition 5.11]. Proposition 1.2 in [6] asserts that $[\Pi^{\widehat{\mathcal{B}}}_p, \pi^{\mathcal{B}}_p]$ is an injective Banach normalized Bloch ideal for any $p \in [1, \infty)$. We now show that $[\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}, \pi^{\mathcal{B}}_{p,\sigma}]$ enjoys the same property using [10].

Proposition 5.1 $[\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}, \pi^{\mathcal{B}}_{p,\sigma}]$ *is an injective Banach normalized Bloch ideal for any $p \in [1, \infty)$ and $\sigma \in [0, 1)$.*

Proof Note that we only need to prove the case $\sigma \in (0, 1)$.

(N1): By Proposition 2.2, $(\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X), \pi^{\mathcal{B}}_{p,\sigma})$ is a Banach space with $\rho_{\mathcal{B}}(f) \leq \pi^{\mathcal{B}}_{p,\sigma}(f)$ for all $f \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$.

(N2): Let $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x \in X$. Let us recall that $g \cdot x \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ with $\rho_{\mathcal{B}}(g \cdot x) = \rho_{\mathcal{B}}(g) \|x\|$ by [10, Proposition 5.13]. Assume $g \neq 0$. For all $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$ and $z_i \in \mathbb{D}$ for all $i \in \{1, \ldots, n\}$, it holds

$$\left(\sum_{i=1}^n |\lambda_i|^{\frac{p}{1-\sigma}} \left\| (g \cdot x)'(z_i) \right\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} = \rho_{\mathcal{B}}(g) \|x\| \left(\sum_{i=1}^n |\lambda_i|^{\frac{p}{1-\sigma}} \left| \left(\frac{g}{\rho_{\mathcal{B}}(g)} \right)' (z_i) \right|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}$$

$$\leq \rho_{\mathcal{B}}(g) \|x\| \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\lambda_i| \left(\frac{1}{1-|z|^2} \right)^{\sigma} \left| h'(z_i) \right|^{1-\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}},$$

and so $g \cdot x \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ with $\pi^{\mathcal{B}}_{p,\sigma}(g \cdot x) \leq \rho_{\mathcal{B}}(g)\,\|x\|$. Since $\rho_{\mathcal{B}}(g)\,\|x\| = \rho_{\mathcal{B}}(g \cdot x) \leq \pi^{\mathcal{B}}_{p}(g \cdot x)$, we have $\pi^{\mathcal{B}}_{p,\sigma}(g \cdot x) = \rho_{\mathcal{B}}(g)\,\|x\|$.

(N3): Let $f \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$, $T \in \mathcal{L}(X, Y)$ and let $g : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $g(0) = 0$. The Pick–Schwarz Lemma assures that

$$(1 - |z|^2)|g'(z)| \leq 1 - |g(z)|^2 \qquad (z \in \mathbb{D}).$$

Let us recall that $T \circ f \circ g \in \widehat{\mathcal{B}}(\mathbb{D}, Y)$ by [10, Proposition 5.13]. We have

$$\begin{aligned}
&\left(\sum_{i=1}^{n} |\lambda_i|^{\frac{p}{1-\sigma}} \left\|(T \circ f \circ g)'(z_i)\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&= \left(\sum_{i=1}^{n} |\lambda_i|^{\frac{p}{1-\sigma}} \left\|T\left(f'(g(z_i))\, g'(z_i)\right)\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&\leq \|T\| \left(\sum_{i=1}^{n} \left(|\lambda_i| \left|g'(z_i)\right| \left\|f'(g(z_i))\right\|\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&\leq \|T\|\, \pi^{\mathcal{B}}_{p,\sigma}(f) \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} \left(|\lambda_i| \left|g'(z_i)\right| \left|h'(g(z_i))\right|^{1-\sigma} \left(\frac{1}{1 - |g(z_i)|^2}\right)^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&= \|T\|\, \pi^{\mathcal{B}}_{p,\sigma}(f) \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} \left(|\lambda_i| \left|g'(z_i)\, h'(g(z_i))\right|^{1-\sigma} \left(\frac{\left|g'(z_i)\right|}{1 - |g(z_i)|^2}\right)^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&\leq \|T\|\, \pi^{\mathcal{B}}_{p,\sigma}(f) \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} \left(|\lambda_i| \left(\frac{1}{1 - |z_i|^2}\right)^{\sigma} \left|(h \circ g)'(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&\leq \|T\|\, \pi^{\mathcal{B}}_{p,\sigma}(f) \sup_{k \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} \left(|\lambda_i| \left(\frac{1}{1 - |z_i|^2}\right)^{\sigma} \left|k'(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}.
\end{aligned}$$

where $\rho_{\mathcal{B}}(h \circ g) \leq \rho_{\mathcal{B}}(h)$ by [10, Proposition 3.6]. Therefore, $T \circ f \circ g \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ with $\pi^{\mathcal{B}}_{p,\sigma}(T \circ f \circ g) \leq \|T\|\, \pi^{\mathcal{B}}_{p,\sigma}(f)$.

(I): Let $f \in \widehat{\mathcal{B}}(\mathbb{D}, X)$ and let $\iota\colon X \to Y$ be a linear isometry so that $\iota \circ f \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, Y)$. We have

$$\begin{aligned}
\left(\sum_{i=1}^{n} |\lambda_i|^{\frac{p}{1-\sigma}} \left\|f'(z_i)\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} &= \left(\sum_{i=1}^{n} |\lambda_i|^{\frac{p}{1-\sigma}} \left\|\iota\left(f'(z_i)\right)\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&= \left(\sum_{i=1}^{n} |\lambda_i|^{\frac{p}{1-\sigma}} \left\|(\iota \circ f)'(z_i)\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}
\end{aligned}$$

$$\leq \pi^{\mathcal{B}}_{p,\sigma}(\iota \circ f) \sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} \left(|\lambda_i| \left(\frac{1}{1-|z_i|^2} \right)^{\sigma} \left| h'(z_i) \right|^{1-\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}$$

and thus $f \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X)$ with $\pi^{\mathcal{B}}_{p,\sigma}(f) \leq \pi^{\mathcal{B}}_{p,\sigma}(\iota \circ f)$. The reverse inequality follows from (N3). □

The *Möbius group of* $\mathbb{D}$, designated $\mathrm{Aut}(\mathbb{D})$, consists of all biholomorphic bijections from $\mathbb{D}$ onto itself. Let us recall that a linear space $\mathcal{A}(\mathbb{D}, X) \subseteq \mathcal{B}(\mathbb{D}, X)$, under a seminorm $\rho_{\mathcal{A}}$, is *Möbius-invariant* if: (i) there is $C > 0$ such that $\rho_{\mathcal{B}}(f) \leq C\rho_{\mathcal{A}}(f)$ for all $f \in \mathcal{A}(\mathbb{D}, X)$; and (ii) $f \circ \phi \in \mathcal{A}(\mathbb{D}, X)$ with $\rho_{\mathcal{A}}(f \circ \phi) = \rho_{\mathcal{A}}(f)$ for all $\phi \in \mathrm{Aut}(\mathbb{D})$ and $f \in \mathcal{A}(\mathbb{D}, X)$.

By Proposition 2.1, $(\Pi^{\mathcal{B}}_{p,\sigma}(\mathbb{D}, X), \pi^{\mathcal{B}}_{p,\sigma}) \leq (\mathcal{B}(\mathbb{D}, X), \rho_{\mathcal{B}})$. Moreover, by the proof of (N3) in Proposition 5.1, one has that if $f \in \Pi^{\mathcal{B}}_{p,\sigma}(\mathbb{D}, X)$ and $\phi \in \mathrm{Aut}(\mathbb{D})$, then $f \circ \phi \in \Pi^{\mathcal{B}}_{p,\sigma}(\mathbb{D}, X)$ with $\pi^{\mathcal{B}}_{p,\sigma}(f \circ \phi) \leq \pi^{\mathcal{B}}_{p,\sigma}(f)$, and this fact also yields $\pi^{\mathcal{B}}_{p,\sigma}(f) = \pi^{\mathcal{B}}_{p,\sigma}((f \circ \phi) \circ \phi^{-1}) \leq \pi^{\mathcal{B}}_{p,\sigma}(f \circ \phi)$. So we have stated the following result which extends [6, Proposition 1.3].

Corollary 5.2 *$(\Pi^{\mathcal{B}}_{p,\sigma}(\mathbb{D}, X), \pi^{\mathcal{B}}_{p,\sigma})$ is a Möbius-invariant space for $p \in [1, \infty)$ and $\sigma \in [0, 1)$.* □

6 Duality

With the aim of studying the duality of the spaces $\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X^*)$, we first introduce the Bloch analogues of the (p, σ)-Chevet–Saphar norms on the tensor product of two Banach spaces. We refer the reader to the references [8, 18] for a complete study on the theory of tensor product. As usual, for any linear spaces E and F, the tensor product $E \otimes F$ equipped with a norm α is denoted by $E \otimes_\alpha F$, and its completion by $E \widehat{\otimes}_\alpha F$.

Towards our aim, we recall some concepts and results of [10]. For each $z \in \mathbb{D}$, a *Bloch atom of* $\mathbb{D}$ is the functional $\gamma_z \in \widehat{\mathcal{B}}(\mathbb{D})^*$ given by $\gamma_z(f) = f'(z)$ for all $f \in \widehat{\mathcal{B}}(\mathbb{D})$. The named *Bloch molecules of* $\mathbb{D}$ are the elements of the space

$$\mathrm{lin}(\{\gamma_z : z \in \mathbb{D}\}) \subseteq \widehat{\mathcal{B}}(\mathbb{D})^*,$$

and the *Bloch-free Banach space of* $\mathbb{D}$ is the space

$$\mathcal{G}(\mathbb{D}) := \overline{\mathrm{lin}}(\{\gamma_z : z \in \mathbb{D}\}) \subseteq \widehat{\mathcal{B}}(\mathbb{D})^*.$$

The map $\Gamma : z \in \mathbb{D} \mapsto \gamma_z \in \mathcal{G}(\mathbb{D})$ is holomorphic with $\|\gamma_z\| = 1/(1-|z|^2)$ for all $z \in \mathbb{D}$.

Define now the space of *X-valued Bloch molecules of* $\mathbb{D}$ by setting

$$\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X := \mathrm{lin}\{\gamma_z \otimes x : z \in \mathbb{D},\ x \in X\} \subseteq \widehat{\mathcal{B}}(\mathbb{D}, X^*)^*,$$

where $\gamma_z \otimes x : \widehat{\mathcal{B}}(\mathbb{D}, X^*) \to \mathbb{C}$ is the functional given by

$$(\gamma_z \otimes x)(f) = \langle f'(z), x \rangle \qquad \left(f \in \widehat{\mathcal{B}}(\mathbb{D}, X^*)\right).$$

Plainly, each element $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ can be expressed as $\gamma = \sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i$ for some $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $z_i \in \mathbb{D}$ and $x_i \in X$ for $i = 1, \ldots, n$. Moreover,

$$\gamma(f) = \sum_{i=1}^n \lambda_i \langle f'(z_i), x_i \rangle \qquad (f \in \widehat{\mathcal{B}}(\mathbb{D}, X^*)).$$

The following family of norms contains the p-Chevet–Saphar Bloch norms on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ introduced in [6, Subsection 2.3].

Definition 6.1 Let $p \in (1, \infty)$ and $\sigma \in [0, 1)$. We define the (p, σ)-Chevet–Saphar Bloch norm $d_{p,\sigma}^{\widehat{\mathcal{B}}}$ on $\gamma \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ by

$$d_{1,\sigma}^{\widehat{\mathcal{B}}}(\gamma) = \inf \left\{ \left(\sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\max_{1 \le i \le n} \left(|\lambda_i| \left(\frac{1}{1 - |z_i|^2} \right)^{\sigma} \left| h'(z_i) \right|^{1-\sigma} \right) \right) \right) \times \left(\sum_{i=1}^n \|x_i\|^{\frac{1}{1-\sigma}} \right)^{1-\sigma} \right\},$$

$$d_{p,\sigma}^{\widehat{\mathcal{B}}}(\gamma) = \inf \left\{ \left(\sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\lambda_i| \left(\frac{1}{1 - |z_i|^2} \right)^{\sigma} \left| h'(z_i) \right|^{1-\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}} \right) \times \left(\sum_{i=1}^n \|x_i\|^{\frac{p^*}{p^*-(1-\sigma)}} \right)^{\frac{p^*-(1-\sigma)}{p^*}} \right\}$$

$$d_{\infty,\sigma}^{\widehat{\mathcal{B}}}(\gamma) = \inf \left\{ \left(\sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\lambda_i| \left(\frac{1}{1 - |z_i|^2} \right)^{\sigma} \left| h'(z_i) \right|^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \right)^{1-\sigma} \right) \times \left(\max_{1 \le i \le n} \|x_i\|^{\frac{1}{1-\sigma}} \right)^{1-\sigma} \right\},$$

the infimum being taken over all the representations of γ as $\sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i$.

The following result concerning Bloch reasonable crossnorms introduced in [6, Definition 2.5] is based on [6, Theorem 2.6].

Given a complex Banach space X, let us recall that a norm α on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$ is a *Bloch reasonable crossnorm* if it satisfies the two conditions: (i) $\alpha(\gamma_z \otimes x) \le \|\gamma_z\| \, \|x\|$ for all $z \in \mathbb{D}$ and $x \in X$; and (ii) Given $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^* \in X^*$, the linear functional $g \otimes x^* : \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X \to \mathbb{C}$ given by $(g \otimes x^*)(\gamma_z \otimes x) = g'(z) x^*(x)$ is bounded on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_\alpha X$ with $\|g \otimes x^*\| \le \rho_{\mathcal{B}}(g) \, \|x^*\|$.

Theorem 6.2 *$d^{\widehat{\mathcal{B}}}_{p,\sigma}$ is a Bloch reasonable crossnorm on* $\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$ *for $\sigma\in[0,1)$ and $p\in[1,\infty]$.*

Proof For $\sigma=0$ and $p\in[1,\infty]$, the result was stated in [6, Theorem 2.6]. We will prove it here for $\sigma\in(0,1)$ and $p\in(1,\infty)$. For $p\in\{1,\infty\}$, the proofs are similar.

Let $\gamma\in\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$ and let $\sum_{i=1}^n\lambda_i\gamma_{z_i}\otimes x_i$ be a representation of γ. Clearly, $d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma)\geq 0$. Given $\lambda\in\mathbb{C}$, it is immediate that

$$d^{\widehat{\mathcal{B}}}_{p,\sigma}(\lambda\gamma)\leq|\lambda|\left(\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^n\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|h'(z_i)\right|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\right)$$
$$\times\left(\sum_{i=1}^n\|x_i\|^{\frac{p^*}{p^*-(1-\sigma)}}\right)^{\frac{p^*-(1-\sigma)}{p^*}}.$$

From this inequality, we infer that $d^{\widehat{\mathcal{B}}}_{p,\sigma}(\lambda\gamma)=0=|\lambda|\,d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma)$ if $\lambda=0$, and that $d^{\widehat{\mathcal{B}}}_{p,\sigma}(\lambda\gamma)\leq|\lambda|\,d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma)$ if $\lambda\neq 0$. In this case, $d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma)=d^{\widehat{\mathcal{B}}}_{p,\sigma}(\lambda^{-1}(\lambda\gamma))\leq|\lambda^{-1}|d^{\widehat{\mathcal{B}}}_{p,\sigma}(\lambda\gamma)$, hence $|\lambda|\,d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma)\leq d^{\widehat{\mathcal{B}}}_{p,\sigma}(\lambda\gamma)$, and thus also $d^{\widehat{\mathcal{B}}}_{p,\sigma}(\lambda\gamma)=|\lambda|\,d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma)$.

To prove the triangular inequality of $d^{\widehat{\mathcal{B}}}_{p,\sigma}$, let $\gamma_1,\gamma_2\in\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$ and $\varepsilon>0$. We can choose representations

$$\gamma_1=\sum_{i=1}^n\lambda_{1,i}\gamma_{z_{1,i}}\otimes x_{1,i},\qquad\gamma_2=\sum_{i=1}^m\lambda_{2,i}\gamma_{z_{2,i}}\otimes x_{2,i},$$

so that

$$\left(\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^n\left(\left|\lambda_{1,i}\right|\left(\frac{1}{1-\left|z_{1,i}\right|^2}\right)^{\sigma}\left|h'(z_{1,i})\right|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\right)\left(\sum_{i=1}^n\left\|x_{1,i}\right\|^{\frac{p^*}{p^*-(1-\sigma)}}\right)^{\frac{p^*-(1-\sigma)}{p^*}}$$

and

$$\left(\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^m\left(\left|\lambda_{2,i}\right|\left(\frac{1}{1-\left|z_{2,i}\right|^2}\right)^{\sigma}\left|h'(z_{2,i})\right|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\right)\left(\sum_{i=1}^m\left\|x_{2,i}\right\|^{\frac{p^*}{p^*-(1-\sigma)}}\right)^{\frac{p^*-(1-\sigma)}{p^*}}$$

are less or equal than $d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_1)+\varepsilon$ and $d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_2)+\varepsilon$, respectively.

For $r,s\in\mathbb{R}^+$ arbitrary, define

$$\lambda_{3,i}\gamma_{z_{3,i}}=\begin{cases}r^{-1}\lambda_{1,i}\gamma_{z_{1,i}} & i=1,\ldots,n,\\ s^{-1}\lambda_{2,i-n}\gamma_{z_{2,i-n}} & i=n+1,\ldots,n+m,\end{cases}$$

and

$$x_{3,i}=\begin{cases} rx_{1,i} & i=1,\dots,n,\\ sx_{2,i-n} & i=n+1,\dots,n+m.\end{cases}$$

Plainly, $\gamma_1+\gamma_2=\sum_{i=1}^{n+m}\lambda_{3,i}\gamma_{z_{3,i}}\otimes x_{3,i}$ and, in consequence, $d_{p,\sigma}^{\widehat{\mathcal{B}}}(\gamma_1+\gamma_2)$ is less or equal than

$$\left(\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n+m}\left(|\lambda_{3,i}|\left(\frac{1}{1-|z_{3,i}|^2}\right)^{\sigma}|h'(z_{3,i})|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\right)\times\left(\sum_{i=1}^{n+m}\|x_{3,i}\|^{\frac{p^*}{p^*-(1-\sigma)}}\right)^{\frac{p^*-(1-\sigma)}{p^*}}.$$

A simple computation produces

$$\begin{aligned}&\left(\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n+m}\left(|\lambda_{3,i}|\left(\frac{1}{1-|z_{3,i}|^2}\right)^{\sigma}|h'(z_{3,i})|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\right)^{\frac{p^*}{1-\sigma}}\\ &\le\left(r^{-1}\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}\left(|\lambda_{1,i}|\left(\frac{1}{1-|z_{1,i}|^2}\right)^{\sigma}|h'(z_{1,i})|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\right)^{\frac{p^*}{1-\sigma}}\\ &+\left(s^{-1}\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{m}\left(|\lambda_{2,i}|\left(\frac{1}{1-|z_{2,i}|^2}\right)^{\sigma}|h'(z_{2,i})|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\right)^{\frac{p^*}{1-\sigma}}\end{aligned}$$

and

$$\sum_{i=1}^{n+m}\|x_{3,i}\|^{\frac{p^*}{p^*-(1-\sigma)}}=r^{\frac{p^*}{p^*-(1-\sigma)}}\sum_{i=1}^{n}\|x_{1,i}\|^{\frac{p^*}{p^*-(1-\sigma)}}+s^{\frac{p^*}{p^*-(1-\sigma)}}\sum_{i=1}^{m}\|x_{2,i}\|^{\frac{p^*}{p^*-(1-\sigma)}}.$$

Since $p^*/(1-\sigma)>1$ and $(p^*/(1-\sigma))^*=p^*/(p^*-(1-\sigma))$, Young's Inequality gives us

$$d_{p,\sigma}^{\widehat{\mathcal{B}}}(\gamma_1+\gamma_2)\le\frac{1-\sigma}{p^*}\left(\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n+m}\left(|\lambda_{3,i}|\left(\frac{1}{1-|z_{3,i}|^2}\right)^{\sigma}|h'(z_{3,i})|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\right)^{\frac{p^*}{1-\sigma}}$$

$$
\begin{aligned}
&+\frac{p^*-(1-\sigma)}{p^*}\sum_{i=1}^{n+m}\left\|x_{3,i}\right\|^{\frac{p^*}{p^*-(1-\sigma)}}\\
&\le \frac{(1-\sigma)r^{-\frac{p^*}{1-\sigma}}}{p^*}\left(\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}\left(\left|\lambda_{1,i}\right|\left(\frac{1}{1-\left|z_{1,i}\right|^2}\right)^{\sigma}\left|h'(z_{1,i})\right|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\right)^{\frac{p^*}{1-\sigma}}\\
&+\frac{(p^*-(1-\sigma))r^{\frac{p^*}{p^*-(1-\sigma)}}}{p^*}\sum_{i=1}^{n}\left\|x_{1,i}\right\|^{\frac{p^*}{p^*-(1-\sigma)}}\\
&+\frac{(1-\sigma)s^{-\frac{p^*}{1-\sigma}}}{p^*}\left(\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{m}\left(\left|\lambda_{2,i}\right|\left(\frac{1}{1-\left|z_{2,i}\right|^2}\right)^{\sigma}\left|h'(z_{2,i})\right|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\right)^{\frac{p^*}{1-\sigma}}\\
&+\frac{(p^*-(1-\sigma))s^{\frac{p^*}{p^*-(1-\sigma)}}}{p^*}\sum_{i=1}^{m}\left\|x_{2,i}\right\|^{\frac{p^*}{p^*-(1-\sigma)}}.
\end{aligned}
$$

In particular, taking above

$$
r=(d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_1)+\varepsilon)^{-\frac{1-\sigma}{p^*}}\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}\left(\left|\lambda_{1,i}\right|\left(\frac{1}{1-\left|z_{1,i}\right|^2}\right)^{\sigma}\left|h'(z_{1,i})\right|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}},
$$

$$
s=(d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_2)+\varepsilon)^{-\frac{1-\sigma}{p^*}}\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{m}\left(\left|\lambda_{2,i}\right|\left(\frac{1}{1-\left|z_{2,i}\right|^2}\right)^{\sigma}\left|h'(z_{2,i})\right|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}},
$$

one obtains $d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_1+\gamma_2)\le d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_1)+d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_2)+2\varepsilon$, and the arbitrariness of ε yields

$$
d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_1+\gamma_2)\le d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_1)+d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_2).
$$

To conclude that $d^{\widehat{\mathcal{B}}}_{p,\sigma}$ is a norm, note that the Hölder's Inequality gives

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}\lambda_i\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}g'(z_i)^{1-\sigma}x^*(x_i)\right|\le\sum_{i=1}^{n}|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|g'(z_i)\right|^{1-\sigma}\|x_i\|\\
&\le\left(\sum_{i=1}^{n}\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|g'(z_i)\right|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\left(\sum_{i=1}^{n}\|x_i\|^{\frac{p^*}{p^*-(1-\sigma)}}\right)^{\frac{p^*-(1-\sigma)}{p^*}}\\
&\le\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|h'(z_i)\right|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}}\left(\sum_{i=1}^{n}\|x_i\|^{\frac{p^*}{p^*-(1-\sigma)}}\right)^{\frac{p^*-(1-\sigma)}{p^*}},
\end{aligned}
$$

whenever $g \in B_{\widehat{\mathcal{B}}(\mathbb{D})}$ and $x^* \in B_{X^*}$. Note that the value $\left|\sum_{i=1}^n \lambda_i g'(z_i)x^*(x_i)\right|$ is independent on the representation of γ seeing as

$$\sum_{i=1}^n \lambda_i g'(z_i)x^*(x_i) = \left(\sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i\right)(g \cdot x^*) = \gamma(g \cdot x^*),$$

and taking infimum over all representations of γ produces

$$\left|\sum_{i=1}^n \lambda_i g'(z_i)x^*(x_i)\right| \leq d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma) \qquad \left(g \in B_{\widehat{\mathcal{B}}(\mathbb{D})},\ x^* \in B_{X^*}\right).$$

Now, if $d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma) = 0$, the above inequality gives

$$\left(\sum_{i=1}^n \lambda_i x^*(x_i)\gamma_{z_i}\right)(g) = \sum_{i=1}^n \lambda_i x^*(x_i)g'(z_i) = 0 \qquad \left(g \in B_{\widehat{\mathcal{B}}(\mathbb{D})},\ x^* \in B_{X^*}\right).$$

For each $x^* \in B_{X^*}$, this implies that $\sum_{i=1}^n \lambda_i x^*(x_i)\gamma_{z_i} = 0$, and since $\Gamma(\mathbb{D})$ is linearly independent in $\mathcal{G}(\mathbb{D})$ (see [10, Remark 2.8]), we secure that $x^*(x_i)\lambda_i = 0$ for all $i \in \{1, \ldots, n\}$, hence $\lambda_i = 0$ for all $i \in \{1, \ldots, n\}$ since B_{X^*} separate points, and so $\gamma = \sum_{i=1}^n \lambda_i \gamma_{z_i} \otimes x_i = 0$.

To finish, we show that $d^{\widehat{\mathcal{B}}}_{p,\sigma}$ is a Bloch reasonable crossnorm on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$:

(i) Given $z \in \mathbb{D}$ and $x \in X$,

$$\begin{aligned} d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma_z \otimes x) &\leq \left(\sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\left(\frac{1}{1-|z|^2}\right)^{\sigma} \left|h'(z)\right|^{1-\sigma}\right)^{\frac{p^*}{1-\sigma}}\right)^{\frac{1-\sigma}{p^*}} \|x\| \\ &\leq \frac{\|x\|}{1-|z|^2} = \|\gamma_z\| \, \|x\| . \end{aligned}$$

(ii) For any $g \in \widehat{\mathcal{B}}(\mathbb{D})$ and $x^* \in X^*$,

$$\begin{aligned} \left|(g \otimes x^*)(\gamma)\right| &= \left|\sum_{i=1}^n \lambda_i (g \otimes x^*)(\gamma_{z_i} \otimes x_i)\right| = \left|\sum_{i=1}^n \lambda_i g'(z_i)x^*(x_i)\right| \\ &\leq \sum_{i=1}^n |\lambda_i| \left|g'(z_i)\right| \left|x^*(x_i)\right| \leq \rho_{\mathcal{B}}(g) \left\|x^*\right\| \sum_{i=1}^n \frac{|\lambda_i|}{1-|z_i|^2} \|x_i\| \\ &= \rho_{\mathcal{B}}(g) \left\|x^*\right\| \sum_{i=1}^n |\lambda_i| \left|f'_{z_i}(z_i)\right| \|x_i\| = \rho_{\mathcal{B}}(g) \left\|x^*\right\| \sum_{i=1}^n |\lambda_i| \\ &\quad \times \left(\frac{1}{1-|z_i|^2}\right)^{\sigma} \left|f'_{z_i}(z_i)\right|^{1-\sigma} \|x_i\| \end{aligned}$$

$$\leq \rho_{\mathcal{B}}(g) \left\|x^*\right\| \left(\sum_{i=1}^{n} \left(|\lambda_i| \left(\frac{1}{1-|z_i|^2} \right)^{\sigma} \left| f'_{z_i}(z_i) \right|^{1-\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}}$$

$$\times \left(\sum_{i=1}^{n} \|x_i\|^{\frac{p^*}{p^*-(1-\sigma)}} \right)^{\frac{p^*-(1-\sigma)}{p^*}}$$

$$\leq \rho_{\mathcal{B}}(g) \left\|x^*\right\| \left(\sup_{h \in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^{n} \left(|\lambda_i| \left(\frac{1}{1-|z_i|^2} \right)^{\sigma} \left| h'(z_i) \right|^{1-\sigma} \right)^{\frac{p^*}{1-\sigma}} \right)^{\frac{1-\sigma}{p^*}} \right)$$

$$\times \left(\sum_{i=1}^{n} \|x_i\|^{\frac{p^*}{p^*-(1-\sigma)}} \right)^{\frac{p^*-(1-\sigma)}{p^*}}.$$

Passing to the infimum over all the representations of γ yields

$$\left|(g \otimes x^*)(\gamma)\right| \leq \rho_{\mathcal{B}}(g) \left\|x^*\right\| d^{\widehat{\mathcal{B}}}_{p,\sigma}(\gamma).$$

Therefore, $g \otimes x^* \in (\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_{d^{\widehat{\mathcal{B}}}_{p,\sigma}} X)^*$ and $\|g \otimes x^*\| \leq \rho_{\mathcal{B}}(g) \|x^*\|$.

□

We are now in a position to address the duality of the space of (p, σ)-absolutely continuous Bloch maps from $\mathbb{D}$ into the dual space X^* of a complex Banach space X. In the proof of the following result, we will make use of Proposition 2.1 and Theorem 3.1.

Theorem 6.3 *Let $p \in [1, \infty)$ and $\sigma \in [0, 1)$. Then $\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X^*)$ is isometrically isomorphic to $(\mathcal{G}(\mathbb{D})\widehat{\otimes}_{d^{\widehat{\mathcal{B}}}_{p^*,\sigma}} X)^*$, via the canonical pairing*

$$\Lambda(f)(\gamma) = \sum_{i=1}^{n} \lambda_i \left\langle f'(z_i), x_i \right\rangle$$

for all $f \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X^*)$ and $\gamma = \sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X$.

Proof For $\sigma = 0$, the result follows from Proposition 2.1 and [6, Theorem 2.8]. Assume $\sigma \in (0, 1)$. We are going to prove the case $1 < p < \infty$. The case $p = 1$ follows similarly.

Let $f \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X^*)$ and define the linear map $\Lambda_0(f) \colon \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X \to \mathbb{C}$ by setting

$$\Lambda_0(f)(\gamma) = \sum_{i=1}^{n} \lambda_i \left\langle f'(z_i), x_i \right\rangle \qquad \left(\gamma = \sum_{i=1}^{n} \lambda_i \gamma_{z_i} \otimes x_i \in \mathrm{lin}(\Gamma(\mathbb{D})) \otimes X\right).$$

Since $p/(1-\sigma) > 1$ and $(p/(1-\sigma))^* = p/(p-(1-\sigma))$, Hölder Inequality and Theorem 3.1 provide

$$
\begin{aligned}
|\Lambda_0(f)(\gamma)| &= \left|\sum_{i=1}^n \lambda_i \left\langle f'(z_i), x_i\right\rangle\right| \le \sum_{i=1}^n |\lambda_i| \left\| f'(z_i)\right\| \|x_i\| \\
&\le \left(\sum_{i=1}^n \left(|\lambda_i| \left\| f'(z_i)\right\|\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \left(\sum_{i=1}^n \|x_i\|^{\frac{p}{p-(1-\sigma)}}\right)^{\frac{p-(1-\sigma)}{p}} \\
&\le \pi^{\mathcal{B}}_{p,\sigma}(f) \sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}} \left(\sum_{i=1}^n \left(|\lambda_i| \left(\frac{1}{1-|z_i|^2}\right)^{\sigma} \left|h'(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&\left(\sum_{i=1}^n \|x_i\|^{\frac{p}{p-(1-\sigma)}}\right)^{\frac{p-(1-\sigma)}{p}}.
\end{aligned}
$$

Calculating the infimum on all the representations of γ yields

$$
|\Lambda_0(f)(\gamma)| \le \pi^{\mathcal{B}}_{p,\sigma}(f) d^{\widehat{\mathcal{B}}}_{p^*,\sigma}(\gamma).
$$

Hence $\Lambda_0(f)$ is continuous on $\mathrm{lin}(\Gamma(\mathbb{D})) \otimes_{d^{\widehat{\mathcal{B}}}_{p^*,\sigma}} X$ with $\|\Lambda_0(f)\| \le \pi^{\mathcal{B}}_{p,\sigma}(f)$.

Clearly, $\mathcal{G}(\mathbb{D}) \otimes X$ is a norm-dense linear subspace of $\mathcal{G}(\mathbb{D})\widehat{\otimes}_{d^{\widehat{\mathcal{B}}}_{p^*,\sigma}} X$ and therefore we can find a unique continuous map $\Lambda(f) \colon \mathcal{G}(\mathbb{D})\widehat{\otimes}_{d^{\widehat{\mathcal{B}}}_{p^*,\sigma}} X \to \mathbb{C}$ extending $\Lambda_0(f)$. Further, $\Lambda(f)$ is linear and $\|\Lambda(f)\| = \|\Lambda_0(f)\|$.

In this way, we define a map $\Lambda \colon \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X^*) \to (\mathcal{G}(\mathbb{D})\widehat{\otimes}_{d^{\widehat{\mathcal{B}}}_{p^*,\sigma}} X)^*$. By [6, Corollary 2.3], Λ is linear and injective since $\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X^*) \subseteq \widehat{\mathcal{B}}(\mathbb{D}, X^*)$. We now prove that Λ is a surjective isometry. For it, let $\varphi \in (\mathcal{G}(\mathbb{D})\widehat{\otimes}_{d^{\widehat{\mathcal{B}}}_{p^*,\sigma}} X)^*$ and define $F_\varphi \colon \mathbb{D} \to X^*$ by

$$
\left\langle F_\varphi(z), x\right\rangle = \varphi(\gamma_z \otimes x) \qquad (z \in \mathbb{D},\ x \in X).
$$

Apparently (see, for example, the proof of [6, Proposition 2.4]), $F_\varphi \in \mathcal{H}(\mathbb{D}, X^*)$ and $F_\varphi = f'_\varphi$ for a suitable map $f_\varphi \in \widehat{\mathcal{B}}(\mathbb{D}, X^*)$ with $\rho_{\mathcal{B}}(f_\varphi) \le \|\varphi\|$.

To prove that $f_\varphi \in \Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D}, X^*)$, let $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$ and $z_i \in \mathbb{D}$ for all $i \in \{1, \ldots, n\}$. Given $\varepsilon > 0$, for each $i \in \{1, \ldots, n\}$, we can find $x_i \in X$ with $\|x_i\| \le 1+\varepsilon$ so that

$$
\left\langle f'_\varphi(z_i), x_i\right\rangle = \left\| f'_\varphi(z_i)\right\|.
$$

Obviously, $T:\mathbb{C}^n\to\mathbb{C}$ defined by

$$T(t_1,\ldots,t_n)=\sum_{i=1}^{n}t_i\lambda_i\left\|f'_\varphi(z_i)\right\|,\qquad (t_1,\ldots,t_n)\in\mathbb{C}^n,$$

is in $(\mathbb{C}^n,||\cdot||_{(p/(1-\sigma))^*})^*$ and

$$\|T\|=\left(\sum_{i=1}^{n}\left(|\lambda_i|\left\|f'_\varphi(z_i)\right\|\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}.$$

If $||(t_1,\ldots,t_n)||_{(p/(1-\sigma))^*}\le 1$, we get

$$\begin{aligned}|T(t_1,\ldots,t_n)|&=\left|\varphi\left(\sum_{i=1}^{n}t_i\lambda_i\gamma_{z_i}\otimes x_i\right)\right|\le\|\varphi\|\,d^{\widehat{\mathcal{B}}}_{p^*,\sigma}\left(\sum_{i=1}^{n}\lambda_i\gamma_{z_i}\otimes t_ix_i\right)\\&\le\|\varphi\|\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|h'(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}\left(\sum_{i=1}^{n}\|t_ix_i\|^{\frac{p}{p-(1-\sigma)}}\right)^{\frac{p-(1-\sigma)}{p}}\\&\le(1+\varepsilon)\,\|\varphi\|\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|h'(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}},\end{aligned}$$

therefore

$$\begin{aligned}&\left(\sum_{i=1}^{n}\left(|\lambda_i|\left\|f'_\varphi(z_i)\right\|\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}\\&\quad\le\|\varphi\|\sup_{h\in B_{\widehat{\mathcal{B}}(\mathbb{D})}}\left(\sum_{i=1}^{n}\left(|\lambda_i|\left(\frac{1}{1-|z_i|^2}\right)^{\sigma}\left|h'(z_i)\right|^{1-\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}},\end{aligned}$$

and consequently Theorem 3.1 tells us that $f_\varphi\in\Pi^{\widehat{\mathcal{B}}}_{p,\sigma}(\mathbb{D},X^*)$ with $\pi^{\mathcal{B}}_{p,\sigma}(f_\varphi)\le\|\varphi\|$.

Now, for any $\gamma=\sum_{i=1}^{n}\lambda_i\gamma_{z_i}\otimes x_i\in\mathrm{lin}(\Gamma(\mathbb{D}))\otimes X$, one has

$$\Lambda(f_\varphi)(\gamma)=\sum_{i=1}^{n}\lambda_i\left\langle f'_\varphi(z_i),x_i\right\rangle=\sum_{i=1}^{n}\lambda_i\varphi(\gamma_{z_i}\otimes x_i)=\varphi\left(\sum_{i=1}^{n}\lambda_i\gamma_{z_i}\otimes x_i\right)=\varphi(\gamma),$$

and $\Lambda(f_\varphi)=\varphi$ on $\mathcal{G}(\mathbb{D})\widehat{\otimes}_{d^{\widehat{\mathcal{B}}}_{p^*,\sigma}}X$. Further, $\pi^{\mathcal{B}}_{p,\sigma}(f_\varphi)\le\left\|\Lambda(f_\varphi)\right\|$. This completes the proof. □

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Declarations

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