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Weighted holomorphic mappings attaining their norms

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Abstract
Given an open subset U of $\mathbb{C}^n$, a weight v on U and a complex Banach space F, let $\mathcal{H}_v(U,F)$ denote the Banach space of all weighted holomorphic mappings $f: U \to F$, under the weighted supremum norm $\|f\|_v := \sup\{v(z)\|f(z)\| : z \in U\}$. We prove that the set of all mappings $f \in \mathcal{H}_v(U,F)$ that attain their weighted supremum norms is norm dense in $\mathcal{H}_v(U,F)$, provided that the closed unit ball of the little weighted holomorphic space $\mathcal{H}_{v_0}(U,F)$ is compact-open dense in the closed unit ball of $\mathcal{H}_v(U,F)$. We also prove a similar result for mappings $f \in \mathcal{H}_v(U,F)$ such that vf has a relatively compact range.

Keywords Holomorphic function · Norm attaining operator · Radon–Nikodým property

Mathematics Subject Classification 46B20 · 46B22 · 46J15

Introduction and preliminaries

The Bishop–Phelps Theorem [5] assures that the set of all continuous linear functionals attaining their norms on a Banach space E is norm dense in its dual Banach space E^*. The extension of this theorem for holomorphic mappings has attracted the attention of some authors.

Choi and Kim [11, Theorem 2.7] proved that a Banach space E with the Radon–Nikodým property satisfies that the set of norm attaining k-homogeneous polynomials on E is dense in the set of all k-homogeneous polynomials on E. Acosta et al. [1, Theorem 3.1] showed that on a complex Banach space E, the functions uniformly continuous on the closed unit ball and holomorphic on the open unit ball that attain their norms are norm dense, provided that E has the Radon–Nikodým property. Other results

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on norm attaining polynomials, bilinear forms and bounded holomorphic mappings can be consulted in the works of Aron et al. [2] and Carando and Mazzitelli [10].

We will address in this note several problems regarding the attainment of the norm of weighted holomorphic mappings defined on an open subset of $\mathbb{C}^n$. Such spaces have been studied by Rubel and Shields [16], Bierstedt and Summers [4], Bierstedt et al. [3], Boyd and Rueda [8, 9] and Gupta and Baweja [15], among many others.

Let U be an open subset of $\mathbb{C}^n$ and F be a complex Banach space. A *weight* v on U is a (strictly) positive continuous function on U. Let $\mathcal{H}(U, F)$ denote the space of all holomorphic mappings from U to F.

The *weighted space of holomorphic mappings* $\mathcal{H}_v(U, F)$ is the Banach space of all mappings $f \in \mathcal{H}(U, F)$ such that

$$\|f\|_v := \sup \{v(z) \|f(z)\| : z \in U\} < \infty,$$

endowed with the *weighted supremum norm* $\|\cdot\|_v$.

The *little weighted space of holomorphic mappings* $\mathcal{H}_{v_0}(U, F)$ is the norm closed linear subspace of $\mathcal{H}_v(U, F)$ formed by all mappings f so that vf vanishes at infinity on U, that is, for every $\varepsilon > 0$ there is a compact subset K of U such that $v(z) \|f(z)\| < \varepsilon$ for all $z \in U \setminus K$. We will write $\mathcal{H}_v(U)$ and $\mathcal{H}_{v_0}(U)$ instead of $\mathcal{H}_v(U, \mathbb{C})$ and $\mathcal{H}_{v_0}(U, \mathbb{C})$, respectively.

We say that a mapping $f \in \mathcal{H}_v(U, F)$ *attains its weighted supremum norm* if there exists a point $z \in U$ such that $v(z) \|f(z)\| = \|f\|_v$.

Our main result states that for every complex Banach space F, the set of all mappings $f \in \mathcal{H}_v(U, F)$ that attain their weighted supremum norms is norm dense in the space $\mathcal{H}_v(U, F)$, provided that the closed unit ball of $\mathcal{H}_{v_0}(U, F)$ is dense in the closed unit ball of $\mathcal{H}_v(U, F)$ with respect to the compact-open topology τ_0. A similar result for the space $\mathcal{H}_{v\mathcal{K}}(U, F)$ consisting of all mappings $f \in \mathcal{H}_v(U, F)$ so that vf has a relatively compact range is also established.

We must point out that it is not possible to obtain a Bishop–Phelps type theorem for any weighted space of holomorphic mappings $\mathcal{H}_v(U, F)$. For example, taking the weight $v(z) = 1$ for all $z \in \mathbb{D}$, note that every function $f \in \mathcal{H}_v(\mathbb{D})$ that attains its supremum norm is a constant function by the maximum modulus principle.

Recently, Dantas and Medina have dealed with weighted holomorphic mappings that attain their norms in [12]. Our approach here is different, depends on the linearization of weighted holomorphic mappings and is influenced by the study done in [14] on similar questions in the setting of spaces of Lipschitz functions.

First, we introduce some basic notation. Given Banach spaces E and F, we denote by $\mathcal{L}(E, F)$ the Banach space of all bounded linear operators from E into F, equipped with the operator canonical norm, and by $\mathcal{K}(E, F)$ the norm closed subspace of $\mathcal{L}(E, F)$ consisting of all compact operators. As usual, B_E, S_E and $\mathrm{Ext}(B_E)$ represent the closed unit ball of E, the unit sphere of E and the set of extreme points of B_E, respectively. $\mathbb{T}$ stands for the set of all uni-modular complex numbers.

The key tool in our study is the canonical predual, denoted $\mathcal{G}_v(U)$, of the space $\mathcal{H}_v(U)$. Following [4], $\mathcal{G}_v(U)$ is the space of all linear functionals on $\mathcal{H}_v(U)$ whose restriction to $B_{\mathcal{H}_v(U)}$ is τ_0-continuous. The following result gathers the properties of $\mathcal{G}_v(U)$ that we will need later.

Theorem 0.1 [4, Theorem 1.1] [15, Theorem 3.1 and Proposition 5.1] *Let U be an open set of $\mathbb{C}^n$ and let v be a weight on U.*

(i) *$\mathcal{G}_v(U)$ is a Banach space with the norm induced from $\mathcal{H}_v(U)^*$, and the evaluation mapping $J_v\colon \mathcal{H}_v(U) \to \mathcal{G}_v(U)^*$, given by $J_v(f)(\phi) = \phi(f)$ for $\phi \in \mathcal{G}_v(U)$ and $f \in \mathcal{H}_v(U)$, is an isometric isomorphism.*
(ii) *The restriction mapping $R_v\colon \mathcal{G}_v(U) \to \mathcal{H}_{v_0}(U)^*$, given by $R_v(\phi) = \phi|_{\mathcal{H}_{v_0}(U)}$ for $\phi \in \mathcal{G}_v(U)$, is an isometric isomorphism if and only if $B_{\mathcal{H}_{v_0}(U)}$ is τ_0-dense in $B_{\mathcal{H}_v(U)}$.*
(iii) *The mapping $\Delta_v\colon U \to \mathcal{G}_v(U)$ defined by $\Delta_v(z) = \delta_z$, where $\delta_z(f) = f(z)$ for $f \in \mathcal{H}_v(U)$, belongs to $\mathcal{H}_v(U, \mathcal{G}_v(U))$ with $\|\Delta_v\|_v \leq 1$.*
(iv) *$\mathcal{G}_v(U)$ coincides with the norm closed linear hull of the set $\{\delta_z : z \in U\}$ in $\mathcal{H}_v(U)^*$.*
(v) *For every complex Banach space F and every mapping $f \in \mathcal{H}_v(U, F)$, there exists a unique operator $T_f \in \mathcal{L}(\mathcal{G}_v(U), F)$ such that $T_f \circ \Delta_v = f$.*
(vi) *The map $f \mapsto T_f$ is an isometric isomorphism from $\mathcal{H}_v(U, F)$ to $\mathcal{L}(\mathcal{G}_v(U), F)$ (resp., from $\mathcal{H}_{v\mathcal{K}}(U, F)$ to $\mathcal{K}(\mathcal{G}_v(U), F)$.* □

Recall that a Banach space E has the *Radon–Nikodým property* (RNP) provided for every measure space (Ω, Σ, μ) with $\mu(\Omega) < \infty$ and every μ-continuous measure $\nu\colon \Sigma \to E$ of finite variation, there is a Bochner integrable function $f\colon \Omega \to E$ such that $\nu(A) = \int_A f\, d\mu$ for all $A \in \Sigma$. The classical Dunford–Pettis Theorem (see [7, Theorem 4.1.3]) asserts that separable dual spaces have the (RNP).

A Banach space E has the *Krein–Milman property* (KMP) if every nonempty closed convex bounded subset of E has an extreme point. According to [7, Theorem 3.3.6], (RNP) implies (KMP), but the converse is still an open problem.

From the assertions (i), (iii) and (iv) of Theorem 0.1, we deduce that $\mathcal{G}_v(U)$ is a separable Banach space. Moreover, in view of the assertion (ii), the aforementioned compact-open density condition imposed in the main results of this paper guarantees that $\mathcal{G}_v(U)$ is a dual Banach space. Therefore, $\mathcal{G}_v(U)$ has the (RNP) and this fact is very important in our arguments as happens in the main results of [1, 11, 14].

By [4, Corollary 1.2], the compact-open density condition assures that $\mathcal{H}_v(U)$ is canonically isometrically isomorphic to $\mathcal{H}_{v_0}(U)^{**}$. This problem known as the *Biduality Problem* was addressed first by Williams [17] and Rubel and Shields [16], afterwards by Bierstedt and Summers [4] and more recently by Boyd and Rueda [9]. As shown in Examples 2.1 and 2.2 of [4], the compact-open density condition is satisfied in the classical cases obtained in [16] (when $U = \mathbb{D}$ is the open unit disk in $\mathbb{C}$ and v is a radial weight on $\mathbb{D}$ that converges to 0 as z converges to the boundary of $\mathbb{D}$), and in [17] (when $U = \mathbb{C}$ and v is a radial weight that is rapidly decreasing at infinity).

1 The results

Given Banach spaces E and F, an operator $T \in \mathcal{L}(E, F)$ *attains its norm* at a point $x \in S_E$ if $\|T(x)\| = \|T\|$. As usual, $\mathrm{NA}(E, F)$ denotes the set of all bounded linear operators from E into F that attain their norms, and to simplify, $\mathrm{NA}(E)$ is written instead of $\mathrm{NA}(E, \mathbb{C})$.

The analogue of this concept can be introduced in the setting of weighted spaces of holomorphic functions as follows.

Definition 1.1 Let U be an open subset of $\mathbb{C}^n$, let v be a weight on U, let F be a complex Banach space and let $f \in \mathcal{H}_v(U, F)$.

(i) We say that f *attains its weighted supremum norm* if there exists a point $z \in U$ such that $v(z)\,\|f(z)\| = \|f\|_v$. We denote by $\mathcal{H}_{v\mathrm{NA}}(U, F)$ the set of all mappings $f \in \mathcal{H}_v(U, F)$ attaining their weighted supremum norms. In particular, we write $\mathcal{H}_{v\mathrm{NA}}(U)$ instead of $\mathcal{H}_{v\mathrm{NA}}(U, \mathbb{C})$.

(ii) We say that f *attains its weighted supremum norm on* $\mathcal{G}_v(U)$ if its linearization $T_f \in \mathcal{L}(\mathcal{G}_v(U), F)$ attains its operator canonical norm. We denote by $\mathcal{H}_{v\mathrm{NA}}(\mathcal{G}_v(U), F)$ the set of all mappings $f \in \mathcal{H}_v(U, F)$ that attain their weighted supremum norms on $\mathcal{G}_v(U)$. In addition, we write $\mathcal{H}_{v\mathrm{NA}}(\mathcal{G}(U))$ instead of $\mathcal{H}_{v\mathrm{NA}}(\mathcal{G}(U), \mathbb{C})$.

(iii) We say that f *attains its weighted supremum norm in the direction* $x \in F$ if $\|x\| = \|f\|_v$ and there exists a sequence (z_n) of points in U such that

$$\lim_{n\to+\infty} v(z_n)f(z_n) = x.$$

We begin with the following elementary observation.

Lemma 1.2 *Let U be an open subset of $\mathbb{C}^n$, let v be a weight on U, let F be a complex Banach space, let $f \in \mathcal{H}_v(U, F)$ and $z \in U$. Then f attains its weighted supremum norm at z if and only if f attains its weighted supremum norm on $\mathcal{G}_v(U)$ at $v(z)\delta_z$. In this case, f attains its weighted supremum norm in the direction $v(z)f(z)$.*

Proof If $v(z)\,\|f(z)\| = \|f\|_v$, we have $\|f\|_v = \|T_f\|$ and

$$v(z)\,\|f(z)\| = v(z)\,\left\|T_f \circ \Delta_v(z)\right\| = v(z)\,\left\|T_f(\delta_z)\right\| = \left\|T_f(v(z)\delta_z)\right\|$$

by Theorem 0.1(v), and thus $\|T_f(v(z)\delta_z)\| = \|T_f\|$. Conversely, if $\|T_f(v(z)\delta_z)\| = \|T_f\|$, we have $v(z)\,\|f(z)\| = \|f\|_v$ according to the above proof. The last statement follows easily. □

Our first aim is to obtain a version of Bishop–Phelps Theorem for weighted holomorphic complex-valued functions. In its proof, we will use the following fact that can be compared to Proposition 1 in [8].

Lemma 1.3 *Let U be an open subset of $\mathbb{C}^n$ and let v be a weight on U. Suppose $B_{\mathcal{H}_{v_0}(U)}$ is τ_0-dense in $B_{\mathcal{H}_v(U)}$. Then every extreme point of $B_{\mathcal{G}_v(\mathbb{D})}$ is of the form $\lambda v(z)\delta_z$, where $\lambda \in \mathbb{T}$ and $z \in U$.*

Proof Let $\phi \in \mathrm{Ext}(B_{\mathcal{G}_v(\mathbb{D})})$. Then $R(\phi) \in \mathrm{Ext}(B_{\mathcal{H}_{v_0}(U)^*})$, where R is the isometric isomorphism from $\mathcal{G}_v(\mathbb{D})$ to $\mathcal{H}_{v_0}(U)^*$ given in Theorem 0.1(ii).

Let $C_0(U)$ denote the Banach space of all complex-valued continuous functions on U which vanish at infinity, under the supremum norm. Since the mapping $\Phi_0 \colon \mathcal{H}_{v_0}(U) \to C_0(U)$ given by

$$\Phi_0(f)(z) = v(z)f(z) \quad (z \in U,\ f \in \mathcal{H}_{v_0}(U)),$$

is an isometric linear embedding, there exists $y \in \mathrm{Ext}(B_{C_0(U)^*})$ such that $\Phi_0^*(y) = R(\phi)$, where $\Phi_0^*\colon C_0(U)^* \to \mathcal{H}_{v_0}(U)^*$ is the adjoint operator of Φ_0. By a generalization of the Arens–Kelley Theorem (see, for example, [13, Theorem 2.3.5]), it follows that $y = \lambda\delta_z$ for some $\lambda \in \mathbb{T}$ and $z \in U$. An easy verification yields

$$\begin{aligned}\phi(f) &= (R^{-1} \circ \Phi_0^*)(y)(f) = \lambda(R^{-1} \circ \Phi_0^*)(\delta_z)(f) = \lambda R^{-1}(\delta_z \circ \Phi_0)(f)\\ &= \lambda(\delta_z \circ \Phi_0)(f) = \lambda\Phi_0(f)(z) = \lambda v(z) f(z) = \lambda v(z)\delta_z(f)\end{aligned}$$

for all $f \in \mathcal{H}_{v_0}(U)$. Since $B_{\mathcal{H}_{v_0}(U)}$ is τ_0-dense in $B_{\mathcal{H}_v(U)}$, and both ϕ and δ_z are τ_0-continuous, we conclude that $\phi = \lambda v(z)\delta_z$. □

Theorem 1.4 *Let U be an open subset of $\mathbb{C}^n$ and let v be a weight on U. Suppose $B_{\mathcal{H}_{v_0}(U)}$ is τ_0-dense in $B_{\mathcal{H}_v(U)}$. Then every mapping $f \in \mathcal{H}_v(U)$ that attains its weighted supremum norm on $\mathcal{G}_v(U)$ attains its weighted supremum norm. In other words, $\mathcal{H}_{v\mathrm{NA}}(\mathcal{G}_v(U)) = \mathcal{H}_{v\mathrm{NA}}(U)$. Therefore, $\mathcal{H}_{v\mathrm{NA}}(U)$ is norm dense in $\mathcal{H}_v(U)$.*

Proof Clearly, the inclusion $\mathcal{H}_{v\mathrm{NA}}(U) \subseteq \mathcal{H}_{v\mathrm{NA}}(\mathcal{G}_v(U))$ follows from the necessity of Lemma 1.2. To prove the converse, let $f \in \mathcal{H}_{v\mathrm{NA}}(\mathcal{G}_v(U))$. By Theorem 0.1, $\mathcal{G}_v(U)$ is a separable dual Banach space and thus has the (KMP).

We claim that T_f also attains its norm at an extreme point of $B_{\mathcal{G}_v(U)}$. In fact, assume $f \neq 0$ (if $f = 0$, then $T_f = 0$ and there is nothing to prove) and denote $S_f = T_f/\|T_f\|$. Clearly, the set $C_f = \{\phi \in B_{\mathcal{G}_v(U)}\colon S_f(\phi) = 1\}$ is a closed convex bounded nonempty subset of $\mathcal{G}_v(U)$, so there exists a point $\phi \in \mathrm{Ext}(C_f)$. We must show that $\phi \in \mathrm{Ext}(B_{\mathcal{G}_v(U)})$. Thus, suppose $\phi = t\phi_1 + (1-t)\phi_2$ where $\phi_1, \phi_2 \in B_{\mathcal{G}_v(U)}$ and $t \in (0, 1)$. Then $1 = S_f(\phi) = tS_f(\phi_1) + (1-t)S_f(\phi_2)$ with $|S_f(\phi_1)| \leq 1$ and $|S_f(\phi_2)| \leq 1$. It follows that $1 = S_f(\phi_1) = S_f(\phi_2)$ and thus $\phi_1, \phi_2 \in C_f$. Since $\phi \in \mathrm{Ext}(C_f)$, we deduce that $\phi = \phi_1 = \phi_2$. So $\phi \in \mathrm{Ext}(B_{\mathcal{G}_v(U)})$ and this proves our claim.

Now Lemma 1.3 guarantees that $\phi = \lambda v(z)\delta_z$ for some $\lambda \in \mathbb{T}$ and $z \in U$. Therefore, f attains its weighted supremum norm on $\mathcal{G}_v(U)$ at $v(z)\delta_z$, and then $f \in \mathcal{H}_{v\mathrm{NA}}(U)$ by Lemma 1.2.

For the last statement, take $f \in \mathcal{H}_v(U)$ and $\varepsilon > 0$. Then $T_f \in \mathcal{G}_v(U)^*$ by Theorem 0.1(v), and the Bishop–Phelps Theorem [5] provides us an operator $T \in \mathrm{NA}(\mathcal{G}_v(U))$ such that $\|T_f - T\| < \varepsilon$. Now, Theorem 0.1(vi) gives us a function $g \in \mathcal{H}_v(U)$ so that $T = T_g$ and thus $g \in \mathcal{H}_{v\mathrm{NA}}(\mathcal{G}_v(U))$. Therefore, $g \in \mathcal{H}_{v\mathrm{NA}}(U)$ with $\|f - g\| = \|T_{f-g}\| = \|T_f - T_g\| = \|T_f - T\| < \varepsilon$. □

Our next goal is to extend the preceding density result to the vector-valued case.

Theorem 1.5 *Let U be an open subset of $\mathbb{C}^n$, let v be a weight on U, let F be a complex Banach space, and $f \in \mathcal{H}_v(U, F)$. Suppose $B_{\mathcal{H}_{v_0}(U)}$ is τ_0-dense in $B_{\mathcal{H}_v(U)}$. Then f attains its weighted supremum norm if and only if f attains its weighted supremum norm on $\mathcal{G}_v(U)$. In this case, $\mathcal{H}_{v\mathrm{NA}}(\mathcal{G}_v(U), F) = \mathcal{H}_{v\mathrm{NA}}(U, F)$.*

Proof Necessity holds always, and its proof is immediate. To prove sufficiency, assume that there is a $\phi \in B_{\mathcal{G}_v(U)}$ such that $\|T_f(\phi)\| = \|f\|_v$. Then, by the Hahn–Banach

Theorem, there exists $x^* \in S_{F^*}$ such that $|x^*(T_f(\phi))| = \|T_f(\phi)\|$. Clearly, $x^* \circ f \in \mathcal{H}_v(U)$ with $\|x^* \circ f\|_v \leq \|f\|_v$. Since $x^* \circ T_f$ is in $\mathcal{G}_v(U)^*$ and satisfies that

$$(x^* \circ T_f) \circ \Delta_v = x^* \circ (T_f \circ \Delta_v) = x^* \circ f,$$

it follows that $T_{x^* \circ f} = x^* \circ T_f$ by Theorem 0.1(v). Hence

$$\left\|T_f(\phi)\right\| = \left|x^*(T_f(\phi))\right| \leq \left\|x^* \circ T_f\right\| = \left\|T_{x^* \circ f}\right\| = \left\|x^* \circ f\right\|_v ,$$

and thus $\|T_f(\phi)\| = \|x^* \circ f\|_v$. Therefore, $|T_{x^* \circ f}(\phi)| = \|x^* \circ f\|_v$ and this tells us that $x^* \circ f$ attains its weighted supremum norm on $\mathcal{G}_v(U)$ (at ϕ). Now, Theorem 1.4 provides us a point $z \in U$ such that $v(z)|(x^* \circ f)(z)| = \|x^* \circ f\|_v$. Therefore, $v(z)\,|x^*(f(z))| = \|f\|_v$, and since $v(z)\,\|f(z)\| \leq \|f\|_v$, it follows that $v(z)\,\|f(z)\| = \|f\|_v$, and so f attains its weighted supremum norm. □

Let E and F be Banach spaces. An operator $T \in \mathcal{L}(E, F)$ is said to be *absolutely strongly exposing* if there is a point $x \in S_E$ such that for every sequence (x_n) in B_E such that $\lim_{n \to +\infty} \|T(x_n)\| = \|T\|$, there is a subsequence $(x_{n_k})_k$ that converges to x or $-x$. It is known (see [6]) that every absolutely strongly exposing operator T attains its norm at the point x.

In [6, Theorem 5], Bourgain proved that if E has the (RNP), then for every Banach space F, the set of all absolutely strongly exposing operators from E to F is a G_δ dense subset of $\mathcal{L}(E, F)$. Furthermore, if $T \in \mathcal{L}(E, F)$ and $\varepsilon > 0$, there is an absolutely strongly exposing operator $S \in \mathcal{L}(E, F)$ such that $\|T - S\| < \varepsilon$ and $T - S \in \mathcal{K}(E, F)$.

Corollary 1.6 *Let U be an open subset of $\mathbb{C}^n$, let v be a weight on U and let F be a complex Banach space. Assume $B_{\mathcal{H}_{v_0}(U)}$ is τ_0-dense in $B_{\mathcal{H}_v(U)}$. Then $\mathcal{H}_{v\mathrm{NA}}(U, F)$ is norm dense in $\mathcal{H}_v(U, F)$.*

Proof Let $\varepsilon > 0$ and $f \in \mathcal{H}_v(U, F)$. Hence $T_f \in \mathcal{L}(\mathcal{G}_v(U), F)$ by Theorem 0.1(v). By Bourgain's Theorem, there is an absolutely strongly exposing operator $T \in \mathcal{L}(\mathcal{G}_v(U), F)$ such that $\|T_f - T\| < \varepsilon$. Moreover, $T \in \mathrm{NA}(\mathcal{G}_v(U), F)$, therefore $T = T_g \in \mathcal{L}(\mathcal{G}_v(U), F)$ for some $g \in \mathcal{H}_v(U, F)$ by Theorem 0.1(vi), thus $g \in \mathcal{H}_{v\mathrm{NA}}(\mathcal{G}_v(U), F)$ and therefore $g \in \mathcal{H}_{v\mathrm{NA}}(U, F)$ by Theorem 1.5. Furthermore, $\|f - g\| = \|T_f - T_g\| = \|T_f - T\| < \varepsilon$. □

If $\mathcal{H}_{v\mathcal{K}\mathrm{NA}}(U, F)$ denotes the set of all mappings $f \in \mathcal{H}_{v\mathcal{K}}(U, F)$ that attain their weighted supremum norms, we will now show the norm density of $\mathcal{H}_{v\mathcal{K}\mathrm{NA}}(U, F)$ in $\mathcal{H}_{v\mathcal{K}}(U, F)$.

Theorem 1.7 *Let U be an open subset of $\mathbb{C}^n$, let v be a weight on U and let F be a complex Banach space. Assume $B_{\mathcal{H}_{v_0}(U)}$ is τ_0-dense in $B_{\mathcal{H}_v(U)}$. Then $\mathcal{H}_{v\mathcal{K}\mathrm{NA}}(U, F)$ is norm dense in $\mathcal{H}_{v\mathcal{K}}(U, F)$.*

Proof Let $\varepsilon > 0$ and $f \in \mathcal{H}_{v\mathcal{K}}(U, F)$. Then $T_f \in \mathcal{K}(\mathcal{G}_v(U), F)$ by Theorem 0.1(vi). By Bourgain's Theorem, we have an absolutely strongly exposing operator $T \in \mathcal{L}(\mathcal{G}_v(U), F)$ such that $\|T_f - T\| < \varepsilon$ and $T_f - T \in \mathcal{K}(\mathcal{G}_v(U), F)$. Therefore, $T \in \mathcal{K}(\mathcal{G}_v(U), F)$. Since in the proof of Corollary 1.6, it follows that $T = T_g$ for some $g \in \mathcal{H}_{v\mathcal{K}}(U, F) \cap \mathcal{H}_{v\mathcal{K}\mathrm{NA}}(U, F)$ with $\|f - g\| < \varepsilon$. □

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