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Results in Mathematics

Isometries and Approximate Local Isometries Between $\mathrm{AC}^p(X)$-Spaces

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Abstract. Let X and Y be compact subsets of $\mathbb{R}$ with at least two points. For $p \geq 1$, let $\mathrm{AC}^p(X)$ be the space of all absolutely continuous complex-valued functions f on X such that $f' \in L^p(X)$, with the norm $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_p$. We describe the topological reflexive closure of the set of linear isometries from $\mathrm{AC}^p(X)$ onto $\mathrm{AC}^p(Y)$. Using this description, we prove that such a set is algebraically reflexive and 2-algebraically reflexive. Moreover, as another application, it is shown that the sets of isometric reflections and generalized bi-circular projections of $\mathrm{AC}^p(X)$ are topologically reflexive and 2-topologically reflexive.

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1. Introduction

For normed spaces E and F, let $\mathcal{B}(E,F)$ be the set of all continuous linear operators of E to F. If $E = F$, we will write $\mathcal{B}(E)$ instead of $\mathcal{B}(E,E)$. Let $\mathcal{S}$ be a nonempty subset of $\mathcal{B}(E,F)$. Define

$$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \{T \in \mathcal{B}(E,F)\colon T(e) \in \mathcal{S}(e),\ \forall e \in E\},$$
$$\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \left\{T \in \mathcal{B}(E,F)\colon T(e) \in \overline{\mathcal{S}(e)},\ \forall e \in E\right\},$$

where $\mathcal{S}(e) = \{L(e)\colon L \in \mathcal{S}\}$ and $\overline{\mathcal{S}(e)}$ denotes its norm-closure in F. We say that the set $\mathcal{S}$ is algebraically reflexive (respectively, topologically reflexive) if $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \mathcal{S}$ (respectively, $\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \mathcal{S}$).

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The problem of the reflexivity of different classes of bounded linear operators has attracted a considerable attention in the last decades. Originally, the first investigations of this kind are due to Kadison, Larson and Sourour by considering the classes of derivations and automorphisms on algebras of operators [19,22,23]. With regard to isometries, Molnár in [25] began the study of the reflexivity of the group of all surjective linear isometries on the full operator algebra $\mathcal{B}(H)$ on a Hilbert space H. In [27], Molnár and Zalar established the reflexivity of the isometry groups of some important Banach spaces including the function space $C(X)$ for a first countable compact Hausdorff space X. Now, there are a lot of studies on the reflexivity of sets of surjective linear isometries between different function spaces (see, for example, [3–5,7,8,15,16,28]).

Moreover, motivated by the Kowalski–Słodowski theorem [21], Šemrl relaxed the linearity assumption for (approximate) local maps as explained below:

$$\begin{aligned} \text{2-ref}_{\text{alg}}(\mathcal{S}) &= \big\{\Delta \in F^E \colon \forall e,u \in E,\ \exists S_{e,u} \in \mathcal{S} \mid \\ &\qquad S_{e,u}(e) = \Delta(e),\ S_{e,u}(u) = \Delta(u)\big\}, \\ \text{2-ref}_{\text{top}}(\mathcal{S}) &= \big\{\Delta \in F^E \colon \forall e,u \in E,\ \exists \{S_{e,u,n}\}_{n\in\mathbb{N}} \subset \mathcal{S} \mid \\ &\qquad \lim_{n\to\infty} S_{e,u,n}(e) = \Delta(e),\ \lim_{n\to\infty} S_{e,u,n}(u) = \Delta(u)\Big\}, \end{aligned}$$

where F^E denotes the set of all maps of E to F. The set $\mathcal{S}$ is said to be 2-algebraically reflexive (respectively, 2-topologically reflexive) if $\text{2-ref}_{\text{alg}}(\mathcal{S}) = \mathcal{S}$ (respectively, $\text{2-ref}_{\text{top}}(\mathcal{S}) = \mathcal{S}$).

In [32], Šemrl dealt with the 2-algebraic reflexivity of the set of automorphisms and derivations on operator algebras. Molnár [26] initiated the study of 2-topological reflexivity of the isometry groups of certain C^*-algebras. Then, in the context of function spaces, Győry [11] obtained the first result on 2-algebraic reflexivity of the isometry group of $C_0(X)$, where $C_0(X)$ denotes the Banach space of all continuous complex-valued functions on a first countable, σ-compact Hausdorff space X vanishing at infinity. We also refer to [12,13,15,16,18,24] for some related results done on spaces of continuous scalar-valued functions.

In this paper, we are concerned with the reflexivity of sets of linear operators between $\text{AC}^p(X)$-spaces. Let us recall that for $p \geq 1$, $\text{AC}^p(X)$ is the space of all absolutely continuous functions $f\colon X \to \mathbb{C}$ such that $f' \in L^p(X)$, with the norm $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_p$. More precisely, in the main results of the paper we give a complete description of operators in $\mathcal{G}(\text{AC}^p(X), \text{AC}^p(Y))$ and $\text{ref}_{\text{top}}(\mathcal{G}(\text{AC}^p(X), \text{AC}^p(Y)))$, where $\mathcal{G}(\text{AC}^p(X), \text{AC}^p(Y))$ stands for the set of all surjective linear isometries from $\text{AC}^p(X)$ onto $\text{AC}^p(Y)$. Then we apply them to obtain the algebraic reflexivity and 2-algebraic reflexivity of $\mathcal{G}(\text{AC}^p(X), \text{AC}^p(Y))$. Moreover, as other consequences of the main results, it is shown that the sets of isometric reflections and generalized bi-circular projections of $\text{AC}^p(X)$ are topologically reflexive and 2-topologically reflexive.

We finally remark that the key tools, as in some previous works, are the Gleason–Kahane–Żelazko theorem and a spherical variant of the Kowalski–Słodkowski theorem given in [24].

2. Preliminaries

Let X be a compact subset of $\mathbb{R}$ with at least two points and let f be a complex-valued function on X. Let us recall that f has bounded variation if the total variation $\mathrm{V}(f)$ of f is finite, that is,

$$\mathrm{V}(f) = \sup\left\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| : n \in \mathbb{N},\, x_0, x_1, \ldots, x_n \in X,\right.$$
$$\left. x_0 < x_1 < \ldots < x_n\right\} < \infty.$$

Moreover, f is said to be absolutely continuous if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$$

for each finite family of non-overlapping open intervals $\{(a_i, b_i) \colon i = 1, \ldots, n\}$ with $a_i, b_i \in X$ for all $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^{n}(b_i - a_i) < \delta$. We denote the space of all continuous (respectively, absolutely continuous) functions on X by $C(X)$ (respectively, $\mathrm{AC}(X)$). It is easily seen that each function in $\mathrm{AC}(X)$ has bounded variation.

For $p \geq 1$, we let $\mathrm{AC}^p(X)$ denote the space of all functions $f \in \mathrm{AC}(X)$ such that $f' \in L^p(X)$, with the norm

$$\|f\|_\Sigma = \|f\|_\infty + \|f'\|_p,$$

where $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ and $\|f'\|_p = (\int_X |f'|^p)^{1/p} d\mu$ (μ is the Lebesgue measure on $\mathbb{R}$). Moreover, note that $\mathrm{AC}^p(X)$ is an algebra because for any $f, g \in \mathrm{AC}^p(X)$, we have $fg \in \mathrm{AC}(X)$ and

$$\|(fg)'\|_p \leq \|f'\|_p \|g\|_\infty + \|f\|_\infty \|g'\|_p < \infty.$$

Moreover, $\mathrm{AC}^p(X)$ is sup-norm dense in $C(X)$ by the Stone–Weierstrass theorem.

The symbol 1_X stands for the function constantly 1 on X and id_X for the identity map of X. Given a normed space E, we denote by Id_E the identity operator of E. Also, $\mathbb{T}$ denotes the unit circle of $\mathbb{C}$.

We finish this section with some comments on the case $p = 1$.

Remarks 1. (1) Let $m_X = \min(X)$ and $M_X = \max(X)$. Since X is compact, $[m_X, M_X] \backslash X$ is an open subset of $\mathbb{R}$, whence $[m_X, M_X] \backslash X$ is the union of a countable number of disjoint open intervals. Let us recall that that each function $f \in \mathrm{AC}(X)$ has the unique extension F to $[m_X, M_X]$ formed by linearly interpolating on each of the open intervals in $[m_X, M_X] \setminus X$ (see the expression in equation (3.1) of [6], and also [29, Lemma 1.1]). Moreover, we have $F \in \mathrm{AC}([m_X, M_X])$ with $\|f\|_\infty = \|F\|_\infty$ and $\mathrm{V}(f) = \mathrm{V}(F)$. We know that F' exists a.e. on $[m_X, M_X]$ and $\mathrm{V}(F) = \int_{m_X}^{M_X} |F'| d\mu$. Then, taking into account that the set of all isolated points of X is a null set, indeed, a countable set (since it is a Lindelöf space), we deduce that f' exists a.e. on X, and so clearly we have $F' = f'$ a.e. on X and $f' \in L^1(X)$.

Hence we have $\mathrm{AC}^1(X) = \mathrm{AC}(X)$ and $\|f'\|_1 \leq \mathrm{V}(f)$ for each $f \in \mathrm{AC}(X)$. The latter inequality may be strict. For example, if $X = [0, 1] \cup \{2\}$, $\|(\mathrm{id}_X)'\|_1 = 1 < \mathrm{V}(\mathrm{id}_X) = 2$. Moreover, $(\mathrm{AC}^1([a, b]), \|\cdot\|_\Sigma) = (\mathrm{AC}([a, b]), \|\cdot\|_\infty + \mathrm{V}(\cdot))$.

(2) The definition of $\mathrm{AC}^1(X)$ as the set

$$\{f \in C(X) \colon f' \text{ exists a.e., } f' \in L^1(X)\}$$

in both [17, Example 7] or [30, page 188] does not coincide with the space $\mathrm{AC}(X)$ even if X is an interval. For example, the Cantor function is in $\mathrm{AC}^1([0, 1])$ but not in $\mathrm{AC}([0, 1])$. So it seems that the condition of absolute continuity has been dropped here (also, compare with [1, page 49] and [2,10,20]).

3. Results

In the sequel, unless explicitly stated, X and Y are two compact subsets of $\mathbb{R}$ with at least two points. We denote $m_X = \min(X)$ and $M_X = \max(X)$, and, similarly, $m_Y = \min(Y)$ and $M_Y = \max(Y)$.

Before stating the results, let us mention that $\mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y))$ stands for the set of all surjective linear isometries from $(\mathrm{AC}^p(X), \|\cdot\|_\Sigma)$ onto $(\mathrm{AC}^p(Y), \|\cdot\|_\Sigma)$. Further, for the case $X = Y$, we denote this set by $\mathcal{G}(\mathrm{AC}^p(X))$.

In the first main result of the paper we describe the surjective linear isometries in $\mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y))$. Before stating the theorem, let us bring a result of [17] which will be applied in our proof.

Theorem 1 [17, Theorem 4]. *Let A be a complex subspace of $C(X)$ (the space of all continuous complex-valued functions on a compact Hausdorff space X) such that*

(i) A is sup-norm dense in $C(X)$,
(ii) the norm on A is given by the formula $\|f\|_\infty + \|T_A(f)\|$ for all $f \in A$, where T_A is a linear map from A into a Banach space,
(iii) A contains the constant function 1_X and $T_A(1_X) = 0$.

Assume that B is a complex subspace of $C(Y)$ which satisfies the analogous assumptions (i)–(iii). Then any isometry T from A onto B with $T(1_X) = 1_Y$ is of the form

$$T(f) = f \circ \phi \qquad (f \in A),$$

where ϕ is a homeomorphism from Y onto X.

Theorem 2. *Let $T \in \mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y))$. Then there exist a (unique) unimodular function h in $\mathrm{AC}^p(Y)$ with $h' = 0$ a.e. on Y and a (unique) homeomorphism $\phi\colon Y \to X$ in $\mathrm{AC}^p(Y)$ such that*

$$T(f)(y) = h(y)f(\phi(y)) \qquad (f \in \mathrm{AC}^p(X),\ y \in Y).$$

Moreover, $h = T(1_X)$ and $\phi^{-1} = \overline{T^{-1}(1_Y)}T^{-1}(\mathrm{id}_Y) \in \mathrm{AC}^p(X)$.

Proof. First we assume that $p > 1$. We obtain the representation of T by considering the two cases as follows.

Case 1. $\mu(X) = \mu(Y) = 0$.

Clearly, $(\mathrm{AC}^p(X), \|\cdot\|_\Sigma) = (\mathrm{AC}(X), \|\cdot\|_\infty)$ and $(\mathrm{AC}^p(Y), \|\cdot\|_\Sigma) = (\mathrm{AC}(Y), \|\cdot\|_\infty)$. Since $\mathrm{AC}(X)$ and $\mathrm{AC}(Y)$ are sup-norm dense in $C(X)$ and $C(Y)$, respectively, from the Banach–Stone theorem it follows that there exist a continuous function $h\colon Y \to \mathbb{T}$ and a homeomorphism $\phi\colon Y \to X$ such that

$$T(f) = h \cdot f \circ \phi \qquad (f \in \mathrm{AC}^p(X)).$$

Since $T(1_X) = h$ and $\|T(1_X)\|_\infty = \|h\|_\infty = 1$, it follows that $h \in \mathrm{AC}^p(Y)$ with $h' = 0$ a.e. on Y.

Case 2. $\max\{\mu(X), \mu(Y)\} > 0$ (of course, after obtaining the representation of T it easily follows that $\mu(X) > 0$ if and only if $\mu(Y) > 0$ because an absolutely continuous homeomorphism between X and Y is established).

By an argument similar to [17, page 203] we prove that the function $|T(1_X)|$ is constant and unimodular. We first show that $|T(1_X)|$ is constant. Otherwise, there exists $y_0 \in Y$ such that $|T(1_X)(y_0)| < \|T(1_X)\|_\infty$. It is clear that for each $f \in \mathrm{AC}^p(X)$, there exists $\alpha \in \mathbb{T}$ such that $\|1_X + \alpha f\|_\infty = 1 + \|f\|_\infty$, which yields

$$\|1_X + \alpha f\|_\Sigma = \|1_X + \alpha f\|_\infty + \|f'\|_p = 1 + \|f\|_\infty + \|f'\|_p = 1 + \|f\|_\Sigma,$$

and so

$$\|T(1_X) + \alpha T(f)\|_\Sigma = 1 + \|T(f)\|_\Sigma = \|T(1_X)\|_\Sigma + \|T(f)\|_\Sigma$$

because T is an isometry. Since $\mathrm{AC}^p(Y)$ is sup-norm dense in $C(Y)$, we can choose a function $k \in \mathrm{AC}^p(Y)$ such that

$$\|k\|_\infty = \|T(1_X)\|_\infty - |T(1_X)(y_0)|,$$

and

$$|k(y)| \le \|T(1_X)\|_\infty - |T(1_X)(y)| + \frac{1}{2}\left(\|T(1_X)\|_\infty - |T(1_X)(y_0)|\right) \qquad (y \in Y).$$

For each $y \in Y$ we have

$$\begin{aligned}|T(1_X)(y)| + |k(y)| &\leq |T(1_X)(y)| + \|T(1_X)\|_\infty - |T(1_X)(y)| \\ &\quad + \frac{1}{2}\left(\|T(1_X)\|_\infty - |T(1_X)(y_0)|\right) \\ &= \|T(1_X)\|_\infty + \frac{1}{2}\left(\|T(1_X)\|_\infty - |T(1_X)(y_0)|\right) \\ &= \|T(1_X)\|_\infty + \frac{1}{2}\|k\|_\infty,\end{aligned}$$

and in consequence,

$$\|T(1_X) + \alpha k\|_\infty \leq \sup\left\{|T(1_X)(y)| + |k(y)| \colon y \in Y\right\} \leq \|T(1_X)\|_\infty + \frac{1}{2}\|k\|_\infty.$$

From the latter relation and taking $\alpha \in \mathbb{T}$ with

$$\|T(1_X) + \alpha k\|_\Sigma = 1 + \|k\|_\Sigma = \|T(1_X)\|_\Sigma + \|k\|_\Sigma,$$

it easily follows that

$$\begin{aligned}\|T(1_X) + \alpha k\|_\Sigma &= \|T(1_X) + \alpha k\|_\infty + \|T(1_X)' + \alpha k'\|_p \\ &\leq \|T(1_X)\|_\infty + \frac{1}{2}\|k\|_\infty + \|T(1_X)'\|_p + \|k'\|_p \\ &= \|T(1_X)\|_\Sigma + \|k\|_\Sigma - \frac{1}{2}\|k\|_\infty \\ &< \|T(1_X)\|_\Sigma + \|k\|_\Sigma,\end{aligned}$$

which is impossible. This contradiction shows that $|T(1_X)|$ is a constant function. Now we prove that $\|T(1_X)'\|_p = 0$. Clearly, it is valid if $\mu(Y) = 0$. So let us consider the case where $\mu(Y) > 0$. Since $\mu(Y) > 0$, the dimension of the subspace $\mathcal{L} = \{f' \colon f \in \mathrm{AC}^p(Y)\}$ of $L^p(Y)$ is strictly greater than 1. If $\|T(1_X)'\|_p \neq 0$, we can choose $g \in \mathrm{AC}^p(Y)$ such that $T(1_X)'$ and g' are not proportional because $\dim(\mathcal{L}) > 1$. Hence for each $\beta \in \mathbb{T}$, taking into account that $L^p(X)$ is strictly convex, we have

$$\begin{aligned}\|T(1_X) + \beta g\|_\Sigma &= \|T(1_X) + \beta g\|_\infty + \|T(1_X)' + \beta g'\|_p \\ &< \|T(1_X)\|_\infty + \|g\|_\infty + \|T(1_X)'\|_p + \|g'\|_p \\ &= \|T(1_X)\|_\Sigma + \|g\|_\Sigma = 1 + \|g\|_\Sigma,\end{aligned}$$

a contradiction. Hence $\|T(1_X)'\|_p = 0$, and so $1 = \|T(1_X)\|_\Sigma = \|T(1_X)\|_\infty$, which yields that the constant function $|T(1_X)|$ is unimodular. Therefore, $T(1_X)$ is a unimodular function with $\|T(1_X)'\|_p = 0$.

Let χ be a unimodular function in $\mathrm{AC}^p(X)$ with $\|\chi'\|_p = 0$. Hence $\chi' = 0$ a.e. on X. If $f \in \mathrm{AC}^p(X)$, then $\|f \cdot \chi\|_\infty = \|f\|_\infty$ and $(f \cdot \chi)' = f' \cdot \chi$ a.e. on X, which yields $\|(f \cdot \chi)'\|_p = \|f'\|_p$ and, in consequence, $\|f \cdot \chi\|_\Sigma = \|f\|_\Sigma$. Thus $T_\chi \colon \mathrm{AC}^p(X) \to \mathrm{AC}^p(X)$, defined by $T_\chi(f) = f \cdot \chi$, is a linear isometry. Moreover, T_χ is surjective because given $g \in \mathrm{AC}^p(X)$, similarly to above, one can see that $f = \overline{\chi} \cdot g \in \mathrm{AC}^p(X)$ and $T_\chi(f) = g$. Moreover, it is easy to see that

$(\mathrm{AC}^p(X), \|\cdot\|_\Sigma)$ satisfies the assumptions (i)–(iii) in Theorem 1. Now, taking into account that $T(1_X)$ is a unimodular function with $\|T(1_X)'\|_p = 0$, from Theorem 1 we infer that

$$T(f) = h \cdot f \circ \phi \qquad (f \in \mathrm{AC}^p(X)),$$

where $h = T(1_X)$ and $\phi\colon Y \to X$ is a homeomorphism. Note that $h \in \mathrm{AC}^p(Y)$ with $h' = 0$ a.e. on Y, as desired.

Now, suppose that $p = 1$. As in Case 2, we will see that $|T(1_X)|$ is a constant function. Our aim is to show that $\|T(1_X)'\|_1 = 0$, which finally implies that $T(1_X)$ is a unimodular function and so we can again obtain the representation of T from Theorem 1.

We assume, without loss of generality, that $\mu(Y) > 0$. First we prove the following claim.

Claim. For each $y \in Y$, we have

$$\int_{[m_Y, y]\cap Y} |T(1_X)'| d\mu = 0, \text{or} \int_{[y, M_Y]\cap Y} |T(1_X)'| d\mu = 0.$$

The claim is clearly valid if $y \in \{m_Y, M_Y\}$. So we suppose that $m_Y < y < M_Y$. Define

$$g_y(z) = \begin{cases} T(1_X)(z) & z \in [m_Y, y] \cap Y, \\ 2T(1_X)(y) - T(1_X)(z) & z \in (y, M_Y] \cap Y. \end{cases}$$

Clearly, $g_y \in \mathrm{AC}^1(Y)$. Take $f = T^{-1}(g_y)$. As above, there exists $\alpha \in \mathbb{T}$ such that

$$\|\alpha g_y + T(1_X)\|_\Sigma = \|\alpha T(f) + T(1_X)\|_\Sigma = \|T(f)\|_\Sigma + 1 = \|g_y\|_\Sigma + 1.$$

Hence

$$\begin{aligned} \|g_y\|_\Sigma + 1 = \|\alpha g_y + T(1_X)\|_\Sigma &= \|\alpha g_y + T(1_X)\|_\infty + \|\alpha g_y' + T(1_X)'\|_1, \\ &\le \|g_y\|_\infty + \|T(1_X)\|_\infty + \|g_y'\|_1 + \|T(1_X)'\|_1 \\ &= \|g_y\|_\Sigma + \|T(1_X)\|_\Sigma = \|g_y\|_\Sigma + 1, \end{aligned}$$

which yields

$$\|\alpha g_y + T(1_X)\|_\infty = \|g_y\|_\infty + \|T(1_X)\|_\infty$$

and

$$\|\alpha g_y' + T(1_X)'\|_1 = \|g_y'\|_1 + \|T(1_X)'\|_1.$$

From the second equation it follows that

$$\begin{aligned}\int_{[m_Y,y]\cap Y} |\alpha g_y' + T(1_X)'|d\mu + \int_{[y,M_Y]\cap Y} |\alpha g_y' + T(1_X)'|d\mu \\ = \int_{[m_Y,y]\cap Y} |g_y'|d\mu + \int_{[y,M_Y]\cap Y} |g_y'|d\mu + \int_{[m_Y,y]\cap Y} |T(1_X)'|d\mu \\ + \int_{[y,M_Y]\cap Y} |T(1_X)'|d\mu,\end{aligned}$$

which implies that

$$\int_{[m_Y,y]\cap Y} |\alpha g_y' + T(1_X)'|d\mu = \int_{[m_Y,y]\cap Y} |g_y'|d\mu + \int_{[m_Y,y]\cap Y} |T(1_X)'|d\mu$$

and

$$\int_{[y,M_Y]\cap Y} |\alpha g_y' + T(1_X)'|d\mu = \int_{[y,M_Y]\cap Y} |g_y'|d\mu + \int_{[y,M_Y]\cap Y} |T(1_X)'|d\mu.$$

From the definition of g_y we get

$$\int_{[m_Y,y]\cap Y} |\alpha+1||T(1_X)'|d\mu = 2\int_{[m_Y,y]\cap Y} |T(1_X)'|d\mu$$

and

$$\int_{[y,M_Y]\cap Y} |-\alpha+1||T(1_X)'|d\mu = 2\int_{[y,M_Y]\cap Y} |T(1_X)'|d\mu.$$

From the first equation it follows that $\alpha = 1$ if $\int_{[m_Y,y]\cap Y} |T(1_X)'|d\mu \neq 0$, and from the second one we conclude that $\alpha = -1$ if $\int_{[y,M_Y]\cap Y} |T(1_X)'|d\mu \neq 0$. Then at least one of the following equations holds

$$\int_{[m_Y,y]\cap Y} |T(1_X)'|d\mu = 0, \quad \text{or} \quad \int_{[y,M_Y]\cap Y} |T(1_X)'|d\mu = 0,$$

as claimed.

Now we prove that $\|T(1_X)'\|_1 = 0$. To the contrary, suppose that $\|T(1_X)'\|_1 > 0$. Take a positive scalar ϵ such that $\epsilon < \|T(1_X)'\|_1$. Since $T(1_X)' \in L^1(Y)$, there exists $\delta > 0$ such that $\int_E |T(1_X)'|d\mu < \epsilon$ holds for all measurable sets $E \subseteq Y$ with $\mu(E) < \delta$. We can choose a finite set $\{z_0, \cdots, z_n\}$ in the compact set Y such that $m_Y = z_0 < z_1 < \cdots < z_n = M_Y$, and $z_i - z_{i-1} < \delta$ for each i with $(z_{i-1}, z_i) \cap Y \neq \emptyset$ $(1 \leq i \leq n)$. Now from Claim it follows that there exists i, $1 \leq i \leq n$, such that $\int_Y |T(1_X)'|d\mu = \int_{[z_{i-1},z_i]\cap Y} |T(1_X)'|d\mu$. Meantime, from the choice of δ and the points z_i it follows that $\int_{[z_{i-1},z_i]\cap Y} |T(1_X)'|d\mu < \epsilon$. Therefore

$$\|T(1_X)'\|_1 = \int_{[z_{i-1},z_i]\cap Y} |T(1_X)'|d\mu < \epsilon,$$

a contradiction.

From the representation of T in all situations, it easily follows that $\phi = \overline{h}T(\mathrm{id}_X) \in \mathrm{AC}^p(Y)$. Furthermore, $\phi^{-1} \in \mathrm{AC}^p(X)$ since one can see that $T^{-1}\colon \mathrm{AC}^p(Y) \to \mathrm{AC}^p(X)$ has the form

$$\begin{aligned} T^{-1}(g)(x) &= T^{-1}(1_Y)(x)g(\phi^{-1}(x)) \\ &= \overline{h(\phi^{-1}(x))}g(\phi^{-1}(x)) \qquad (g \in \mathrm{AC}^p(Y),\ x \in X), \end{aligned}$$

and so $\phi^{-1} = \overline{T^{-1}(1_Y)}T^{-1}(\mathrm{id}_Y) \in \mathrm{AC}^p(X)$. □

Remarks 2. (1) From Theorem 2, we deduce that if T is in $\mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y))$, then T is an isometry with respect to the uniform norms, whence $\|T(f)\|_\infty = \|f\|_\infty$ and $\|T(f)'\|_p = \|f'\|_p$ for all $f \in \mathrm{AC}^p(X)$.

(2) In Theorem 2, h is not necessarily a constant function and ϕ is not necessarily monotonic (compare with [16, Theorem 1] and [14, Corollary 4.3]). For example, let $X = [0, 1] \cup \{2\}$ and define the surjective linear isometry $T\colon \mathrm{AC}^p(X) \to \mathrm{AC}^p(X)$ by

$$T(f)(x) = h(x)f(\phi(x)) \qquad (x \in X),$$

where $h(x) = -1$ if $x \in [0, 1]$ and $h(2) = 1$, and $\phi(x) = 1 - x$ if $x \in [0, 1]$ and $\phi(2) = 2$.

However, if X is an interval, then h is a constant function and ϕ is monotonic. To see it, assume that X is an interval, indeed, $X = [m_X, M_X]$ (and so $Y = [m_Y, M_Y]$). From the previous part, $\|h'\|_p = \|T(1_X)'\|_p = 0$. Hence

$$\mathrm{V}(h) = \|h'\|_1 \leq \|h'\|_p\, (M_X - m_X)^{1-1/p} = 0$$

by the Hölder's inequality, which yields $\mathrm{V}(h) = 0$, whence h is a constant function. Moreover, from the intermediate value theorem, one can see that the homeomorphism $\phi\colon [m_Y, M_Y] \to [m_X, M_X]$ is monotonic.

(3) Since $(\mathrm{AC}^1([0, 1]), \|\cdot\|_\Sigma) = (\mathrm{AC}([0, 1]), \|\cdot\|_\infty + \mathrm{V}(\cdot))$, Example 1 in [16] shows that the space $\mathcal{G}(\mathrm{AC}^1([0, 1]))$ is neither topologically reflexive nor 2-topologically reflexive. However, we will observe that the isometry groups of $\mathrm{AC}^p(X)$-spaces are algebraically reflexive and 2-algebraically reflexive.

Definition 3. Given a nonempty subset $\mathcal{S}$ of $\mathcal{B}(E, F)$, the elements of $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$ and $\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$ are defined as local $\mathcal{S}$-maps and approximate local $\mathcal{S}$-maps, respectively. Similarly, the elements of 2-$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$ and 2-$\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$ are referred to as 2-local $\mathcal{S}$-maps and approximate 2-local $\mathcal{S}$-maps, respectively.

We state the second main result of this paper which gives a complete description of approximate local isometries from $\mathrm{AC}^p(X)$ into $\mathrm{AC}^p(Y)$.

Theorem 4. *Let $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y)))$. Then T is an isometry of the form*

$$T(f)(y) = h(y)f(\phi(y)) \qquad (f \in \mathrm{AC}^p(X),\ y \in Y),$$

where h is a unimodular function in $\mathrm{AC}^p(Y)$ with $\|h'\|_p = 0$ and $\phi\colon Y \to X$ is a surjective function in $\mathrm{AC}^p(Y)$.

Proof. We prove the result through several claims.

Claim 1. $\|T(f)\|_\Sigma = \|f\|_\Sigma$, $\|T(f)\|_\infty = \|f\|_\infty$ *and* $\|T(f)'\|_p = \|f'\|_p$ *for all* $f \in \mathrm{AC}^p(X)$.

Let $f \in \mathrm{AC}^p(X)$. Then there exists a sequence $\{T_{f,n}\}_{n\in\mathbb{N}}$ in $\mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y))$ such that $\lim_{n\to\infty} T_{f,n}(f) = T(f)$. It is clear that $\lim_{n\to\infty} \|T_{f,n}(f)\|_\Sigma = \|T(f)\|_\Sigma$, $\lim_{n\to\infty} \|T_{f,n}(f)\|_\infty = \|T(f)\|_\infty$ and $\lim_{n\to\infty} \|T_{f,n}(f)'\|_p = \|T(f)'\|_p$. Since $\|T_{f,n}(f)\|_\Sigma = \|f\|_\Sigma$, $\|T_{f,n}(f)\|_\infty = \|f\|_\infty$ and $\|T_{f,n}(f)'\|_p = \|f'\|_p$ for all $n \in \mathbb{N}$ by Remarks 2 (1), the claim holds.

Claim 2. *For every* $f \in \mathrm{AC}^p(X)$, *there exist a sequence* $\{h_{f,n} : Y \to \mathbb{T}\}_{n\in\mathbb{N}}$ *in* $\mathrm{AC}^p(Y)$ *with* $\|h'_{f,n}\|_p = 0$ *and a sequence of homeomorphisms* $\{\phi_{f,n} : Y \to X\}_{n\in\mathbb{N}}$ *in* $\mathrm{AC}^p(Y)$ *such that*

$$\lim_{n\to\infty} h_{f,n}(f \circ \phi_{f,n}) = T(f).$$

Since $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y)))$, the claim is a quick consequence of Theorem 2.

Claim 3. $h := T(1_X)$ *is a unimodular function in* $\mathrm{AC}^p(Y)$ *with* $\|h'\|_p = 0$.

Clearly, $T(1_X) \in \mathrm{AC}^p(Y)$. By Claim 2, there exist a sequence $\{h_{1_X,n}\}_{n\in\mathbb{N}}$ of unimodular functions in $\mathrm{AC}^p(Y)$ with $\|h'_{1_X,n}\|_p = 0$ for all $n \in \mathbb{N}$ such that

$$T(1_X) = \lim_{n\to\infty} h_{1_X,n}.$$

Since the convergence in the $\|\cdot\|_\Sigma$-norm implies pointwise convergence, for each $y \in Y$ we have $T(1_X)(y) = \lim_{n\to\infty} h_{1_X,n}(y)$ and thus

$$|T(1_X)(y)| = \lim_{n\to\infty} |h_{1_X,n}(y)| = 1.$$

That convergence also implies that

$$\|T(1_X)'\|_p = \lim_{n\to\infty} \|h'_{1_X,n}\|_p = 0.$$

This proves the claim.

Claim 4. $(\mathrm{AC}^p(X), \|\cdot\|_0 = \|\cdot\|_\Sigma + \mathrm{V}(\cdot))$ *is a Banach space.*

Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathrm{AC}^p(X), \|\cdot\|_0)$. In particular, $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(\mathrm{AC}(X), \|\cdot\|_\infty + \mathrm{V}(\cdot))$, and so there exists $f \in \mathrm{AC}(X)$ such that

$$\|f_n - f\|_\infty + \mathrm{V}(f_n - f) \to 0.$$

Moreover, $\{f'_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(L^p(X), \|\cdot\|_p)$, whence there exists $g \in L^p(X)$ such that $\|f'_n - g\|_p \to 0$, which implies that $\|f'_n - g\|_1 \to 0$ because

$$\|f'_n - g\|_1 \le \|f'_n - g\|_p \, \mu(X)^{1-1/p} \to 0$$

by the Hölder's inequality.

On the other hand, in view of Remarks 1.(1) we have that $\|f_n' - f'\|_1 \leq \mathrm{V}(f_n - f)$ for all $n \in \mathbb{N}$, and since $\mathrm{V}(f_n - f) \to 0$, it follows that $\|f_n' - f'\|_1 \to 0$. Thus we conclude that $g = f'$ a.e. on X. Hence $f \in \mathrm{AC}^p(X)$ and $\|f_n - f\|_0 \to 0$. Therefore $(\mathrm{AC}^p(X), \|\cdot\|_0)$ is a Banach space.

Claim 5. *There exists a norm $\|\cdot\|$, equivalent to the complete norm $\|\cdot\|_0$, which makes $\mathrm{AC}^p(X)$ a Banach algebra with maximal ideal space X.*

First note that if $\{f_n\}_{n\in\mathbb{N}}$ is a convergent sequence in $(\mathrm{AC}^p(X), \|\cdot\|_0)$ to $f \in \mathrm{AC}^p(X)$ and $g \in \mathrm{AC}^p(X)$, then $f_n g \to fg$ because

$$\begin{aligned}\|f_n g - fg\|_0 &\leq \|f_n - f\|_\infty \|g\|_\infty + \mathrm{V}(g)\|f_n - f\|_\infty \\ &\quad + \mathrm{V}(f_n - f)\|g\|_\infty + \|g\|_\infty \|f_n' - f'\|_p + \|f_n - f\|_\infty \|g'\|_p \to 0.\end{aligned}$$

Now, taking into account Claim 4, from Theorem 10.2 in [31] it follows that there is a norm $\|\cdot\|$, equivalent to the complete norm $\|\cdot\|_0$, which makes $\mathrm{AC}^p(X)$ a Banach algebra.

We show that if $f \in \mathrm{AC}^p(X)$ such that $\{x \in X : f(x) = 0\} = \emptyset$, then $1_X/f \in \mathrm{AC}^p(X)$. We note that $1_X/f \in \mathrm{AC}(X)$ and also $(1_X/f)' \in L^p(X)$ because

$$\|(1_X/f)'\|_p = \left(\int_X \frac{|f'|^p}{|f|^{2p}} d\mu\right)^{1/p} \leq \frac{1}{m^2}\|f'\|_p < \infty,$$

where $m = \min\{|f(x)| : x \in X\}$ is a positive scalar. Furthermore, it is obvious that $\mathrm{AC}^p(X)$ is self-adjoint. Now, from Proposition 4.1.5 (ii) in [9] we infer that the maximal ideal space of $\mathrm{AC}^p(X)$ is homeomorphic to X.

Similarly, there exists a norm $\|\cdot\|'$ which makes $\mathrm{AC}^p(Y)$ a Banach algebra and its maximal ideal space is homeomorphic to Y.

Claim 6. *For each $y \in Y$, the map $S_y : (\mathrm{AC}^p(X), \|\cdot\|) \to \mathbb{C}$ defined by*

$$S_y(f) = \overline{h(y)}T(f)(y) \qquad (f \in \mathrm{AC}^p(X)),$$

is a unital multiplicative linear functional.

Fix $y \in Y$. Clearly, S_y is linear and

$$S_y(1_X) = \overline{h(y)}T(1_X)(y) = \overline{h(y)}h(y) = |h(y)|^2 = 1$$

by Claim 3. To prove its multiplicativity, define $T_y : (\mathrm{AC}^p(X), \|\cdot\|) \to \mathbb{C}$ by

$$T_y(f) = T(f)(y) \qquad (f \in \mathrm{AC}^p(X)).$$

From the above claim, there is a positive scalar M such that $\|\cdot\|_0 \leq M\|\cdot\|$. Since T_y is linear and

$$|T_y(f)| = |T(f)(y)| \leq \|T(f)\|_\Sigma = \|f\|_\Sigma \leq \|f\|_0 \leq M\|f\|$$

for all $f \in \mathrm{AC}^p(X)$ by Claim 1, we infer that T_y is continuous. Take now $f \in \mathrm{AC}^p(X)$. By Claim 2, there exist a sequence $\{h_{f,n} : Y \to \mathbb{T}\}_{n\in\mathbb{N}}$ in $\mathrm{AC}^p(Y)$ and a sequence of homeomorphisms $\{\phi_{f,n} : Y \to X\}_{n\in\mathbb{N}}$ in $\mathrm{AC}^p(Y)$ such that

$$T(f) = \lim_{n\to\infty} h_{f,n}(f \circ \phi_{f,n}).$$

We can assume, without loss of generality, that the sequence $\{h_{f,n}(y)\}_{n\in\mathbb{N}}$ converges to $\lambda_{f,y} \in \mathbb{T}$. Therefore we have

$$T_y(f) = T(f)(y) = \lim_{n\to\infty} h_{f,n}(y)f(\phi_{f,n}(y)) = \lambda_{f,y} \lim_{n\to\infty} f(\phi_{f,n}(y)) \in \mathbb{T}\sigma(f),$$

since the spectrum $\sigma(f)$ of f is compact. Applying a spherical variant of the Gleason–Kahane–Żelazko theorem [24, Proposition 2.2], we conclude that $S_y = \overline{T_y(1_X)}T_y$ is multiplicative.

Claim 7. *There exists a surjective function $\phi\colon Y \to X$ in $\mathrm{AC}^p(Y)$ such that $T(f)(y) = h(y)f(\phi(y))$ for all $y \in Y$ and $f \in \mathrm{AC}^p(X)$.*

Using Claim 6, we deduce easily that the map $S\colon (\mathrm{AC}^p(X), \|\cdot\|) \to (\mathrm{AC}^p(Y), \|\cdot\|')$ defined by

$$S(f)(y) = \overline{h(y)}T(f)(y) \qquad (f \in \mathrm{AC}^p(X),\ y \in Y)$$

is a unital algebra homomorphism. From Gelfand theory (see, e.g., [9, Theorem 2.3.25]) and Claim 5 we conclude that S induces a continuous map $\phi\colon Y \to X$ such that

$$S(f)(y) = f(\phi(y)) \qquad (f \in \mathrm{AC}^p(X),\ y \in Y),$$

which implies that

$$T(f)(y) = h(y)f(\phi(y)) \qquad (f \in \mathrm{AC}^p(X),\ y \in Y).$$

Notice that $\phi = \overline{h}T(\mathrm{id}_X) \in \mathrm{AC}^p(Y)$. To show the surjectivity of ϕ, assume on the contrary that there exists $x_0 \in X \setminus \phi(Y)$. Since $\phi(Y)$ is compact, we can take a function $f \in \mathrm{AC}^p(X)$ such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in \phi(Y)$. Hence $T(f)(y) = h(y)f(\phi(y)) = 0$ for all $y \in Y$, a contradiction because T is injective by Claim 1. □

Now we prove that every local isometry from $\mathrm{AC}^p(X)$ into $\mathrm{AC}^p(Y)$ is a surjective linear isometry.

Corollary 1. *The set $\mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y))$ is algebraically reflexive.*

Proof. Let $T \in \mathrm{ref}_{\mathrm{alg}}(\mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y)))$. By Theorem 4 there exist a unimodular function $h \in \mathrm{AC}^p(Y)$ and a surjective function $\phi\colon Y \to X$ in $\mathrm{AC}^p(Y)$ such that

$$T(f)(y) = h(y)f(\phi(y)) \qquad (f \in \mathrm{AC}^p(X),\ y \in Y).$$

Define $f_0\colon X \to \mathbb{R}$ by $f_0(x) = x - m_X + 1$ for all $x \in X$. Clearly, $f_0 \in \mathrm{AC}^p(X)$, and then by Theorem 2 there exist a unimodular function $h_{f_0} \in \mathrm{AC}^p(Y)$ and a homeomorphism $\phi_{f_0}\colon Y \to X$ in $\mathrm{AC}^p(Y)$ such that

$$h(y)f_0(\phi(y)) = T(f_0)(y) = h_{f_0}(y)f_0(\phi_{f_0}(y)) \qquad (y \in Y).$$

Consequently,

$$h(y)(\phi(y) - m_X + 1) = T(f_0)(y) = h_{f_0}(y)(\phi_{f_0}(y) - m_X + 1) \qquad (y \in Y).$$

Now, taking into account that h, h_{f_0} are unimodular functions and f_0 is a positive function, it follows that $\phi(y) - m_X + 1 = \phi_{f_0}(y) - m_X + 1$ for all $y \in Y$. Hence $\phi = \phi_{f_0}$. Again, from the above relation we infer that $h = h_{f_0}$. Then for any $f \in \mathrm{AC}^p(X)$ we have

$$T(f)(y) = h(y)f(\phi(y)) = h_{f_0}(y)f(\phi_{f_0}(y)) = T_{f_0}(f)(y) \qquad (y \in Y).$$

Therefore $T = T_{f_0} \in \mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y))$. □

We also show that every (approximate) 2-local isometry between $\mathrm{AC}^p(X)$-spaces is an (approximate) local isometry.

Theorem 5. *(1) Every approximate 2-local isometry of* $\mathrm{AC}^p(X)$ *to* $\mathrm{AC}^p(Y)$ *is an approximate local isometry.*
(2) Every 2-local isometry of $\mathrm{AC}^p(X)$ *to* $\mathrm{AC}^p(Y)$ *is a local isometry.*

Proof. Let $\Delta \in 2\text{-ref}_{\text{top}}(\mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y)))$. We first prove that for each $y \in Y$, the functional $\Delta_y\colon (\mathrm{AC}^p(X), \|\cdot\|) \to \mathbb{C}$ defined by

$$\Delta_y(f) = \Delta(f)(y) \qquad (f \in \mathrm{AC}^p(X)),$$

is linear, where $\|\cdot\|$ is the norm presented in Claim 5. Since Δ is an approximate 2-local isometry, it is easily seen that Δ_y is 1-homogeneous. Now, let $f, g \in \mathrm{AC}^p(X)$ and take a sequence of unimodular functions $\{h_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathrm{AC}^p(Y)$ and a sequence of homeomorphisms $\{\phi_{f,g,n}\colon Y \to X\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} h_{f,g,n}(y)f(\phi_{f,g,n}(y)) = \Delta(f)(y),$$
$$\lim_{n\to\infty} h_{f,g,n}(y)g(\phi_{f,g,n}(y)) = \Delta(g)(y).$$

Thus

$$\Delta_y(f) - \Delta_y(g) = \lim_{n\to\infty} h_{f,g,n}(y)(f-g)(\phi_{f,g,n}(y)) \in \mathbb{T}\sigma(f-g).$$

Then by the spherical variant of the Kowalski–Słodkowski theorem [24], Δ_y is linear. Hence Δ is linear by the arbitrariness of y. Therefore $\Delta \in \text{ref}_{\text{top}}(\mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y)))$. This proves (1), and (2) is obtained with an analogous proof. □

From Theorem 5 and Corollary 1, we immediately obtain the following result, which shows that every 2-local isometry of $\mathrm{AC}^p(X)$ to $\mathrm{AC}^p(Y)$ is a surjective linear isometry.

Corollary 2. *The set* $\mathcal{G}(\mathrm{AC}^p(X), \mathrm{AC}^p(Y))$ *is 2-algebraically reflexive.* □

We next study the topological reflexivity of other distinguished subsets of linear transformations of $\mathrm{AC}^p(X)$.

Let E be a Banach space. Let us recall that an isometric reflection of E is a linear isometry $T\colon E \to E$ with $T^2 = \mathrm{Id}_E$; and a generalized bi-circular projection of E is a linear projection $P\colon E \to E$ such that $P + \tau(\mathrm{Id}_E - P)$ is a linear surjective isometry for some $\tau \in \mathbb{T}$ with $\tau \neq 1$. Note that any isometric reflection of E is surjective. The symbols $\mathcal{G}^2(E)$ and $\mathrm{GBP}(E)$ stand for the sets of isometric reflections and generalized bi-circular projections of E, respectively.

The next theorem characterizes the isometric reflections on $\mathrm{AC}^p(X)$-spaces.

Theorem 6. *An isometry $T\colon \mathrm{AC}^p(X) \to \mathrm{AC}^p(X)$ is an isometric reflection if and only if there exist a function $h \in \mathrm{AC}^p(X)$ with $h(x) \in \{\pm 1\}$ for all $x \in X$, and a homeomorphism $\phi \in \mathrm{AC}^p(X)$ with $\phi^2(x) = x$ for all $x \in X$ such that*

$$T(f)(x) = h(x)f(\phi(x)) \qquad (f \in \mathrm{AC}^p(X),\ x \in X).$$

Proof. Let $T \in \mathcal{G}^2(\mathrm{AC}^p(X))$. By Theorem 2, there are a unimodular function h in $\mathrm{AC}^p(X)$ and a homeomorphism $\phi\colon X \to X$ in $\mathrm{AC}^p(X)$ such that

$$T(f)(x) = h(x)f(\phi(x)) \qquad (f \in \mathrm{AC}^p(X),\ x \in X).$$

Since $T^2 = \mathrm{Id}_{\mathrm{AC}^p(X)}$, it follows that

$$f(x) = T^2(f)(x) = T(T(f))(x) = [h(x)]^2 f(\phi^2(x)) \qquad (f \in \mathrm{AC}^p(X),\ x \in X).$$

Taking above $f = 1_X$, we deduce that $[h(x)]^2 = 1$ for all $x \in X$, and thus $h(x) \in \{\pm 1\}$ for all $x \in X$. Also, by considering $f = \mathrm{id}_X$, we obtain that $x = [h(x)]^2\phi^2(x) = \phi^2(x)$ for all $x \in X$.

Conversely, suppose that T has the form as in the statement. Then an easy verification yields

$$T^2(f)(x) = [h(x)]^2 f(\phi^2(x)) = f(x) \qquad (f \in \mathrm{AC}^p(X),\ x \in X).$$

Therefore $T \in \mathcal{G}^2(\mathrm{AC}^p(X))$, as desired. □

We can deduce that every approximate local isometric reflection of $\mathrm{AC}^p(X)$ is an isometric reflection.

Corollary 3. *The set $\mathcal{G}^2(\mathrm{AC}^p(X))$ is topologically reflexive.*

Proof. Let $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(\mathrm{AC}^p(X)))$. By Theorem 6, for $f \in \mathrm{AC}^p(X)$, we can take a sequence of unimodular functions $\{h_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{AC}^p(X)$ with $h_{f,n}(x) \in \{\pm 1\}$ for all $x \in X$, and a sequence of homeomorphisms $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ of X in $\mathrm{AC}^p(X)$ with $\phi_{f,n}^2 = \mathrm{id}_X$ satisfying

$$\lim_{n\to\infty} h_{f,n}(f \circ \phi_{f,n}) = T(f).$$

Obviously, $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}(\mathrm{AC}^p(X)))$ and, by Theorem 4, we can find a unimodular function $h \in \mathrm{AC}^p(X)$ and a surjective function $\phi\colon X \to X$ in $\mathrm{AC}^p(X)$ such that

$$T(f) = h(f \circ \phi) \qquad (f \in \mathrm{AC}^p(X)).$$

Hence $h = T(1_X) = \lim_{n\to\infty} h_{1_X,n}$ and since $h_{1_X,n}(x) \in \{\pm 1\}$ for all $n \in \mathbb{N}$ and $x \in X$, it is deduced easily that $h(x) \in \{\pm 1\}$ for all $x \in X$. Define $f_0(x) = x - m_X + 1$ for all $x \in X$. We have

$$h(f_0 \circ \phi) = T(f_0) = \lim_{n\to\infty} h_{f_0,n}(f_0 \circ \phi_{f_0,n}).$$

Since the convergence in the Σ-norm implies uniform convergence, and $f_0(x) > 0$, $h(x), h_{f_0,n}(x) \in \{\pm 1\}$ for all $x \in X$, we conclude that

$$\lim_{n\to\infty} \|f_0 \circ \phi_{f_0,n} - f_0 \circ \phi\|_\infty = 0.$$

Hence $\lim_{n\to\infty} \|\phi_{f_0,n} - \phi\|_\infty = 0$. Now, given $x \in X$ and $\epsilon > 0$, taking into account that ϕ is continuous and $\lim_{n\to\infty} \phi_{f_0,n}(x) = \phi(x)$, one can find $n_0 \in \mathbb{N}$ such that $\|\phi_{f_0,n} - \phi\|_\infty < \epsilon/2$ and $\left|\phi(\phi_{f_0,n}(x)) - \phi^2(x)\right| < \epsilon/2$ for all $n \geq n_0$. Then for any $n \geq n_0$, we have

$$\begin{aligned} \left|\phi^2_{f_0,n}(x) - \phi^2(x)\right| &\leq |\phi_{f_0,n}(\phi_{f_0,n}(x)) - \phi(\phi_{f_0,n}(x))| \\ &\quad + \left|\phi(\phi_{f_0,n}(x)) - \phi^2(x)\right| < \epsilon, \end{aligned}$$

which yields $\lim_{n\to\infty} \phi^2_{f_0,n}(x) = \phi^2(x)$. On the other hand, for each $n \in \mathbb{N}$ and $x \in X$ we have $\phi^2_{f_0,n}(x) = x$, which finally implies that $\phi^2(x) = x$. Therefore T is an isometry of the form $T(f) = h(f \circ \phi)$ with $\phi^2 = \mathrm{id}_X$ and $h^2 = 1_X$, and, in consequence, $T \in \mathcal{G}^2(\mathrm{AC}^p(X))$ by Theorem 6. □

The next theorem gives a complete description of generalized bi-circular projections on $\mathrm{AC}^p(X)$-spaces.

Theorem 7. *A map $P\colon \mathrm{AC}^p(X) \to \mathrm{AC}^p(X)$ is a generalized bi-circular projection if and only if $P = (1/2)(\mathrm{Id}_{\mathrm{AC}^p(X)} + T)$ for a unique $T \in \mathcal{G}^2(\mathrm{AC}^p(X))$.*

Proof. The proof of the sufficiency is easy. To prove the necessity, assume that $P \in \mathrm{GBP}(\mathrm{AC}^p(X))$. Then $T := P + \tau(\mathrm{Id}_{\mathrm{AC}^p(X)} - P) \in \mathcal{G}(\mathrm{AC}^p(X))$ for some $\tau \in \mathbb{T} \setminus \{1\}$. By Theorem 2, we can find a unimodular function $h \in \mathrm{AC}^p(X)$ and a homeomorphism $\phi\colon X \to X$ in $\mathrm{AC}^p(X)$ such that

$$[P + \tau(\mathrm{Id}_{\mathrm{AC}^p(X)} - P)](f)(x) = h(x)f(\phi(x)) \qquad (f \in \mathrm{AC}^p(X),\ x \in X).$$

Moreover, $h = T(1_X)$ and $\phi^{-1} \in \mathrm{AC}^p(X)$. Then

$$P(f)(x) = (1-\tau)^{-1}[-\tau f(x) + h(x)f(\phi(x))] \qquad (f \in \mathrm{AC}^p(X),\ x \in X).$$

Since $P^2 = P$, we have the following equation:

$$\tau f(x) - (\tau+1)h(x)f(\phi(x)) + h(x)^2 f(\phi^2(x)) = 0 \qquad (f \in \mathrm{AC}^p(X),\ x \in X).$$

Suppose that there exists $x_0 \in X$ such that $x_0 \neq \phi(x_0)$ and $x_0 \neq \phi^2(x_0)$. Take a function $f_0 \in \mathrm{AC}^p(X)$ such that $f_0(x_0) = 1$ and $f_0(\phi(x_0)) = 0 = f_0(\phi^2(x_0))$. Observe that taking $f = f_0$ and $x = x_0$ in the equation above, we obtain $\tau = 0$, a contradiction. Hence $\phi(x) = x$ or $\phi^2(x) = x$ for all $x \in X$. In any case we conclude that $\phi^2 = \mathrm{id}_X$.

We now distinguish two cases. If $\phi \neq \mathrm{id}_X$, choose $x_0 \in X$ such that $x_0 \neq \phi(x_0)$ and consider $g \in \mathrm{AC}^p(X)$ such that $g(x_0) = 1$ and $g(\phi(x_0)) = 0$. Substituting now in the above equation, first $f = g$ and $x = x_0$, and after $f = 1_X$ and any x, we infer that $\tau + [h(x_0)]^2 = 0$ and $\tau - (\tau+1)h(x) + [h(x)]^2 = 0$

for all $x \in X$, respectively. Hence $\tau = -1$ and $[h(x)]^2 = 1$ for all $x \in X$. Thus $h(x) \in \{-1, 1\}$ for all $x \in X$ and the formula of P yields

$$P(f)(x) = \frac{1}{2}\left[f(x) + h(x)f(\phi(x))\right] \qquad (f \in \mathrm{AC}^p(X),\ x \in X).$$

Therefore $P = (1/2)(\mathrm{Id}_{\mathrm{AC}^p(X)} + T)$, where $T \in \mathcal{G}^2(\mathrm{AC}^p(X))$ by Theorem 6.

In the another case, if $\phi = \mathrm{id}_X$, taking $f = 1_X$ in the equation we get

$$[h(x) - \tau][h(x) - 1] = [h(x)]^2 - (\tau + 1)h(x) + \tau = 0 \qquad (x \in X).$$

Since $\tau \neq 1$, from the above relation it follows that for each $x \in X$, either $h(x) = 1$, or $h(x) = \tau$. If $h(x) = 1$ for all $x \in X$, then we have $P = \mathrm{Id}_{\mathrm{AC}^p(X)}$, and so P is of the desired form. Now, assume that $h(x) = \tau$ for some $x \in X$. Hence, for any $f \in \mathrm{AC}^p(X)$, we obtain

$$P(f)(x) = \begin{cases} f(x) & \text{if } h(x) = 1, \\ \\ 0 & \text{if } h(x) = \tau. \end{cases}$$

So by taking $\mathrm{h}\colon X \to \{1, -1\}$ by $\mathrm{h}(x) = 1$ if $h(x) = 1$ and $\mathrm{h}(x) = -1$ if $h(x) = \tau$, we have $\mathrm{h} \in \mathrm{AC}^p(X)$,

$$\mathcal{T}(f) = \mathrm{h}f \qquad (f \in \mathrm{AC}^p(X))$$

belongs to $\mathcal{G}^2(\mathrm{AC}^p(X))$ by Theorem 6 and also $P = (1/2)(\mathrm{Id}_{\mathrm{AC}^p(X)} + \mathcal{T})$, as desired. □

Corollary 4. *The set* $\mathrm{GBP}(\mathrm{AC}^p(X))$ *is topologically reflexive.*

Proof. Let $P \in \mathrm{ref}_{\mathrm{top}}(\mathrm{GBP}(\mathrm{AC}^p(X)))$. By Theorem 7, for each $f \in \mathrm{AC}^p(X)$ there exists a sequence $\{T_{f,n}\}_{n\in\mathbb{N}}$ in $\mathcal{G}^2(\mathrm{AC}^p(X))$ such that

$$P(f) = \lim_{n\to\infty} \frac{1}{2}\left[(\mathrm{Id}_{\mathrm{AC}^p(X)} + T_{f,n})(f)\right],$$

whence $(2P - \mathrm{Id}_{\mathrm{AC}^p(X)})(f) = \lim_{n\to\infty} T_{f,n}(f)$, which shows that $2P - \mathrm{Id}_{\mathrm{AC}^p(X)} \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(\mathrm{AC}^p(X)))$. Thus $2P - \mathrm{Id}_{\mathrm{AC}^p(X)} \in \mathcal{G}^2(\mathrm{AC}^p(X))$ by Corollary 3 and therefore $P \in \mathrm{GBP}(\mathrm{AC}^p(X))$ according to Theorem 7. □

Finally, we show the 2-topological reflexivity of the set of generalized bi-circular projections of $\mathrm{AC}^p(X)$-spaces.

Corollary 5. *The set* $\mathrm{GBP}(\mathrm{AC}^p(X))$ *is 2-topologically reflexive.*

Proof. Let $\Delta \in 2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathrm{GBP}(\mathrm{AC}^p(X)))$. According to Theorem 7, for any $f, g \in \mathrm{AC}^p(X)$, there exists a sequence $\{T_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathcal{G}^2(\mathrm{AC}^p(X))$ such that

$$\Delta(f) = \lim_{n\to\infty} \frac{1}{2}\left[(\mathrm{Id}_{\mathrm{AC}^p(X)} + T_{f,g,n})(f)\right],$$

$$\Delta(g) = \lim_{n\to\infty} \frac{1}{2}\left[(\mathrm{Id}_{\mathrm{AC}^p(X)} + T_{f,g,n})(g)\right].$$

Hence, for every $f, g \in \mathrm{AC}^p(X)$, we have

$$2\Delta(f) - f = \lim_{n\to\infty} T_{f,g,n}(f),$$
$$2\Delta(g) - g = \lim_{n\to\infty} T_{f,g,n}(g),$$

and this says that $2\Delta - \mathrm{Id}_{\mathrm{AC}^p(X)} \in$ 2-ref$_{\mathrm{top}}(\mathcal{G}^2(\mathrm{AC}^p(X)))$. Hence $2\Delta - \mathrm{Id}_{\mathrm{AC}^p(X)} \in \mathcal{G}^2(\mathrm{AC}^p(X))$ by Corollary 3. Therefore, from Theorem 7 we conclude that $\Delta \in \mathrm{GBP}(\mathrm{AC}^p(X))$. $\square$

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