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ORIGINAL PAPER

On local isometries between algebras of C(Y)-valued differentiable maps

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Abstract
Let K be either the real unit interval $[0, 1]$ or the complex unit circle $\mathbb{T}$ and let $C(Y)$ be the space of all complex-valued continuous functions on a compact Hausdorff space Y. We prove that the isometry group of the algebra $C^1(K, C(Y))$ of all $C(Y)$-valued continuously differentiable maps on K, equipped with the Σ-norm, is topologically reflexive and 2-topologically reflexive whenever the isometry group of $C(Y)$ is topologically reflexive.

Keywords Algebraic reflexivity · Topological reflexivity · Local isometry · 2-local isometry · Differentiable map

Mathematics Subject Classification Primary 46B04 · 47B48; Secondary 46E15 · 47B38

1 Introduction

Surjective linear isometries on the space $C^1([0, 1], E)$ of all continuously differentiable maps on the real unit interval $[0, 1]$ with values in a Banach space E, equipped with the Σ-norm:

$$\|F\|_\Sigma = \max_{x\in[0,1]} \|F(x)\|_E + \max_{x\in[0,1]} \left\|F'(x)\right\|_E \qquad \left(F \in C^1([0, 1], E)\right),$$

have been studied for some concrete Banach spaces E. For $E = \mathbb{C}$, such isometries were described first by Rao and Roy [28], and, later, by Jarosz and Pathak [12] within a more general study of the surjective isometries of some classic function spaces. Miura and Takagi [18] extended the result of Rao and Roy by characterizing surjective, not necessarily linear, isometries on $C^1([0, 1], \mathbb{C})$.

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Hatori and Oi [8] (see also [7]) proposed a unified approach to the study of isometries on Banach algebras of vector-valued maps by means of the notion of admissible quadruples and described isometries on Banach algebras $C^1(K, C(Y))$, where K is either $[0, 1]$ or $\mathbb{T}$ (the complex unit circle) and $C(Y)$ is the Banach algebra of all complex-valued continuous functions on a compact Hausdorff space Y with the supremum norm:

$$\|f\|_\infty = \max_{y\in Y} |f(y)| \qquad (f \in C(Y)).$$

Another usual norm considered on the linear space $C^1([0, 1], E)$ is the C-norm given by

$$\|F\|_C = \max_{x\in[0,1]} \left\{\|F(x)\|_E + \left\|F'(x)\right\|_E\right\} \qquad \left(F \in C^1([0, 1], E)\right).$$

Cambern [4] gave a representation for the surjective linear isometries of $C^1([0, 1], \mathbb{C})$ with the C-norm. Botelho and Jamison [2] extended this result to spaces $C^1([0, 1], H)$, where H is a finite-dimensional real Hilbert space. Recently, Ranjbar-Motlagh [27] has characterized the surjective linear isometries of $C^1([0, 1], E)$ whenever E is a strictly convex real Banach space.

Equipped with other norms, Li and Wang [15] investigated surjective linear isometries between spaces $C_0^{(n)}(\Omega, E)$, where Ω is an open subset of Euclidean space and E is a reflexive strictly convex space. The case in which Ω is an open subset of $\mathbb{R}$ and E is a strictly convex Banach space with dimension greater than 1 was addressed by Li et al. [13].

Reflexivity and 2-reflexivity of the set of surjective linear isometries between Banach spaces are properties that are closely related to the study of isometries. Algebraical and topological reflexivity of the group of surjective isometries on Banach spaces were introduced by Molnár and Zalar in [22]. The paper [3] by Cabello Sánchez and Molnár is concerned with the algebraical and topological reflexivity of the isometry group and the automorphism group of some important metric linear spaces and algebras.

The research on 2-local isometries between Banach spaces was initiated by Molnár [19], motivated by the paper [30] of Šemrl who obtained the first results on 2-local automorphisms and 2-local derivations between Banach algebras. The study of 2-local isometries of $C(X)$-spaces was raised by Molnár [20]. In [6], Győry gave a description of 2-local isometries of $C_0(L, \mathbb{C})$-spaces.

Fleming and Jamison proposed the research on these topics in their monograph [5]. The study of reflexivity and 2-reflexivity of the sets of isometries, derivations and automorphisms on operator algebras and function algebras is a problem which follows attracting the attention of numerous researchers.

We briefly recall these notions. For two Banach spaces E and F, let F^E be the set of all maps of E to F, $B(E, F)$ be the space of all continuous linear operators of E to F and $\mathrm{Iso}(E, F)$ be the set of all surjective linear isometries of E to F. When $E = F$, we write $\mathrm{Iso}(E)$ instead of $\mathrm{Iso}(E, E)$.

It is said that $\mathrm{Iso}(E, F)$ is algebraically reflexive (topologically reflexive) if

$$\mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(E, F)) = \mathrm{Iso}(E, F) \quad \text{(respectively, } \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(E, F)) = \mathrm{Iso}(E, F)),$$

where

$$\mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(E, F)) = \{T \in B(E, F) \colon \forall e \in E,\ \exists T_e \in \mathrm{Iso}(E, F) \mid T(e) = T_e(e)\}$$

and

$$\mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(E, F)) = \left\{T \in B(E, F) \colon \forall e \in E, \forall \varepsilon > 0\ \exists T_{e,\varepsilon} \in \mathrm{Iso}(E, F) \mid \left\|T(e) - T_{e,\varepsilon}(e)\right\| < \varepsilon\right\}.$$

The elements of $\mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(E, F))$ and $\mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(E, F))$ are called local isometries and approximate local isometries of E to F, respectively.

Besides, it is said that $\mathrm{Iso}(E, F)$ is 2-algebraically reflexive (2-topologically reflexive) whenever

$$2\text{-}\mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(E, F)) = \mathrm{Iso}(E, F) \quad (\text{respectively, } 2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(E, F)) = \mathrm{Iso}(E, F)),$$

where $2\text{-}\mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(E, F))$ and $2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(E, F))$ are the sets defined, respectively, by

$$\left\{\Delta \in F^E : \forall e, u \in E,\ \exists T_{e,u} \in \mathrm{Iso}(E, F) \mid \Delta(e) = T_{e,u}(e),\ \Delta(u) = T_{e,u}(u)\right\}$$

and

$$\left\{\Delta \in F^E : \forall e, u \in E,\ \forall \varepsilon > 0\ \exists T_{e,u,\varepsilon} \in \mathrm{Iso}(E, F) \mid \left\|\Delta(e) - T_{e,u,\varepsilon}(e)\right\| < \varepsilon,\ \left\|\Delta(u) - T_{e,u,\varepsilon}(u)\right\| < \varepsilon\right\}.$$

The members of $2\text{-}\mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(E, F))$ and $2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(E, F))$ are known as 2-local isometries and approximate 2-local isometries of E to F, respectively.

The main purpose of this article is to prove that the isometry group of the algebra $C^1(K, C(Y))$ for $K = [0, 1]$ or $K = \mathbb{T}$, equipped with the Σ-norm, is topologically reflexive and 2-topologically reflexive whenever the isometry group of $C(Y)$ is topologically reflexive.

Apparently, this last condition is too restrictive since Molnár and Zalar [22] proved that the isometry group of $C(Y)$ is algebraically reflexive if Y is a first countable compact Hausdorff space, and Cabello-Sánchez and Molnár [3] gave an example where reflexivity may fail even if Y lacks first countability at only on point. However, the isometry group of $C(Y)$ is topologically reflexive whenever the homeomorphism group of Y is a finite group or a compact group, and it is known the abundance of such spaces Y in the literature (see [1, 10, 17]).

Our result finds a first motivation in the work [25] by Oi on the algebraic reflexivity of the isometry group of algebras of $C(Y)$-valued Lipschitz maps. The study of 2-local isometries and 2-local automorphisms without assuming linearity is a hard problem initiated by Molnár [21] in the algebra of all bounded linear operators on a Hilbert space. The 2-locality problem for surjective isometries on $C^1([0, 1], \mathbb{C})$, without assuming linearity, was addressed by Hatori and Oi [9]. The algebraic reflexivity of the isometry group of $C^1([0, 1], \mathbb{C})$ with the Σ-norm was stated first in [26]. For more results on this subject in the setting of Banach spaces of differentiable maps equipped with other norms, we refer the reader to [11, 16].

We have divided this paper into two sections. Section 2 gathers some known properties of $C(K, C(Y))$-algebras. The type BJ representation of the isometry group of $C^1(K, C(Y))$ with the Σ-norm, stated by Hatori and Oi [8], is essential in our arguments. We complete this section with descriptions of the maximal ideal space of the Banach algebra $C^1(K, C(Y))$ and of the algebra homomorphism group between $C^1(K, C(Y))$-algebras, which will be needed later. Section 3 contains the main results of this work. Using a spherical variant of the Gleason–Kahane–Żelazko theorem [14], the Gelfand theory and the Arzelá–Ascoli theorem, we prove first that the group $\mathrm{Iso}(C^1(K))$ is topologically reflexive. This fact joint to the Banach–Stone theorem are applied to state that $\mathrm{Iso}(C^1(K, C(Y)))$ is topologically reflexive whenever so is $\mathrm{Iso}(C(Y))$. As a consequence, but applying now a spherical variant of the Kowalski–Słodkowski theorem [14], we deduce that $\mathrm{Iso}(C^1(K, C(Y))$ is also 2-topologically reflexive under the same condition on $\mathrm{Iso}(C(Y))$. We finish with some observations about nice operators on Banach spaces that motivate new open problems.

2 Preliminaries

We first present the algebras of continuously differentiable maps which are studied in this paper. Given two Hausdorff spaces X and Z, denote

$$C(X, Z) = \{f \text{ is a continuous map of } X \text{ to } Z\},$$
$$\mathrm{Homeo}(X, Z) = \{f \text{ is a homeomorphism of } X \text{ onto } Z\}.$$

When X is a compact Hausdorff space and Z is a complex Banach space, we consider the linear space $C(X, Z)$ equipped with the supremum norm:

$$\|F\|_\infty = \sup_{x \in X} \|F(x)\|_Z \qquad (F \in C(X, Z)).$$

In the case $Z = \mathbb{C}$, we write $C(X)$ instead of $C(X, \mathbb{C})$. Given any set X, the symbols id_X and 1_X stand for the identity map on X and the function constantly 1 on X, respectively. For a metric space X, $\mathrm{Iso}(X)$ denotes the set of all isometries of X onto itself. All elements in $\mathrm{Iso}(X)$ are assumed to be linear when X has a vector space structure.

Let $F \in C(K, C(Y))$, where K is either the real unit interval $[0, 1]$ or the complex unit circle $\mathbb{T}$, and Y is a compact Hausdorff space. It is said that F is continuously differentiable if there exists a map $G \in C(K, C(Y))$ such that

$$\lim_{x \to x_0} \left\| \frac{F(x) - F(x_0)}{x - x_0} - G(x_0) \right\|_\infty = 0$$

for every $x_0 \in K$. We denote $F' = G$. Consider the set

$$C^1(K, C(Y)) = \{F \in C(K, C(Y)) \colon F \text{ is continuously differentiable}\}.$$

Given $f \in C^1(K)$ and $g \in C(Y)$, the map $f \otimes g \colon K \to C(Y)$, defined by

$$(f \otimes g)(x) = f(x)g \qquad (x \in K),$$

belongs to $C^1(K, C(Y))$ with $\|f \otimes g\|_\infty = \|f\|_\infty \|g\|_\infty$ and $\left\|(f \otimes g)'\right\|_\infty = \left\|f'\right\|_\infty \|g\|_\infty$, and therefore $\|f \otimes g\|_\Sigma = \|f\|_\Sigma \|g\|_\infty$.

The space $C^1(K, C(Y))$, equipped with the Σ-norm given by

$$\|F\|_\Sigma = \|F\|_\infty + \left\|F'\right\|_\infty \qquad \left(F \in C^1(K, C(Y))\right),$$

is a unital semisimple commutative Banach algebra with unit $1_K \otimes 1_Y$. If Y is a singleton, then $C(Y)$ is isometrically isomorphic to $\mathbb{C}$ and we write $C^1(K)$ instead of $C^1(K, C(Y))$.

Following to Hatori and Oi [8], we identify $C(K, C(Y))$ with $C(K \times Y)$ and assume that $C^1(K, C(Y))$ is a subalgebra of $C(K \times Y)$ by the correspondence:

$$(x \mapsto F(x)) \in C^1(K, C(Y)) \quad \leftrightarrow \quad ((x, y) \mapsto (F(x))(y)) \in C(K \times Y).$$

Taking into account that

$$\mathrm{Iso}([0, 1]) = \left\{\mathrm{id}_{[0,1]}, 1_{[0,1]} - \mathrm{id}_{[0,1]}\right\}$$

and

$$\mathrm{Iso}(\mathbb{T}) = \left\{\lambda \mathrm{id}_\mathbb{T}, \lambda \overline{\mathrm{id}_\mathbb{T}} \colon \lambda \in \mathbb{T}\right\},$$

where $\bar{\cdot}$ denotes the complex conjugation, we can gather in a unique statement the following representations, obtained and called of type BJ in [8], for the surjective linear isometries of the spaces $C^1(K, C(Y))$ endowed with the Σ-norm.

In what follows, given a function $\varphi\colon K\times Y\to K$, for each $y\in Y$, we denote by $\varphi^y\colon K\to K$ the function defined by $\varphi^y(x)=\varphi(x,y)$ for all $x\in K$.

Theorem 1 (see [8,Corollaries 18 and 19]) *Let K be either $[0,1]$ or $\mathbb{T}$ and let Y_1, Y_2 be compact Hausdorff spaces. A map U is a linear isometry of $C^1(K,C(Y_1))$ onto $C^1(K,C(Y_2))$ with respect to the Σ-norms if and only if there exist a function $h\in C(Y_2,\mathbb{T})$, a function $\varphi\in C(K\times Y_2,K)$ with $\varphi^y\in \mathrm{Iso}(K)$ for each $y\in Y_2$, and a map $\tau\in \mathrm{Homeo}(Y_2,Y_1)$ such that*

$$U(F)(x,y)=h(y)F(\varphi(x,y),\tau(y))\quad ((x,y)\in K\times Y_2),$$

for all $F\in C^1(K,C(Y_1))$. □

For our proofs, we also will need the following results. The first one shows that the maximal ideal space of $C^1(K,C(Y))$ can be identified with $K\times Y$. The second one provides a description of unital algebra homomorphisms between $C^1(K,C(Y))$-algebras.

Theorem 2 *Let K be either $[0,1]$ or $\mathbb{T}$ and Y be a compact Hausdorff space. Then the maximal ideal space of $C^1(K,C(Y))$ is homeomorphic to $K\times Y$.*

Proof For each $(x,y)\in K\times Y$, let $\chi_{(x,y)}$ denote the evaluation functional at (x,y), that is,

$$\chi_{(x,y)}(F)=F(x,y)=F(x)(y)\qquad \left(F\in C^1(K,C(Y))\right).$$

It is clear that $\chi_{(x,y)}$ belongs to the maximal ideal space $\mathcal{M}$ of $C^1(K,C(Y))$. The map $(x,y)\mapsto\chi_{(x,y)}$ is an embedding from $K\times Y$ into $\mathcal{M}$.

We next prove that this embedding is surjective. Here we apply an argument similar to the proof of Proposition 11 in [24]. Suppose that $\chi\in\mathcal{M}$ such that $\chi\neq\chi_{(x,y)}$ for all $(x,y)\in K\times Y$. Fix $x_0\in K$, and let $y\in Y$. Since $\chi\neq\chi_{(x_0,y)}$, there is a map $F_{(x_0,y)}\in C^1(K,C(Y))$ with $\chi_{(x_0,y)}(F_{(x_0,y)})=F_{(x_0,y)}(x_0)(y)\neq 0$ and $\chi(F_{(x_0,y)})=0$. Clearly, $Y=\cup_{y\in Y}V_y$, where

$$V_y=\left\{z\in Y\colon F_{(x_0,y)}(x_0)(z)\neq 0\right\}$$

is an open subset of Y. From the compactness of Y, we conclude that there exist $y_1,\ldots,y_n\in Y$ such that $Y=\cup_{i=1}^n V_{y_i}$. Note that if $z\in Y$, then $F_{(x_0,y_i)}(x_0)(z)\neq 0$ for some $i\in\{1,\ldots,n\}$. Since $C^1(K,C(Y))$ is self-adjoint, the map

$$H_{x_0}=\sum_{i=1}^n F_{(x_0,y_i)}\overline{F_{(x_0,y_i)}}$$

belongs to $C^1(K,C(Y))$, where

$$\overline{F_{(x_0,y_i)}}(x)(z)=\overline{F_{(x_0,y_i)}(x)(z)}\qquad (x\in K,\ z\in Y).$$

We have

$$\chi(H_{x_0})=\sum_{i=1}^n\chi(F_{(x_0,y_i)})\chi(\overline{F_{(x_0,y_i)}})=0$$

and $H_{x_0}(x_0)(z)>0$ for all $z\in Y$. Since $H_{x_0}(x_0)$ is a positive continuous function on the compact space Y, there exists $\delta_{x_0}>0$ such that $H_{x_0}(x_0)(z)\geq\delta_{x_0}$ for all $z\in Y$. Put

$$M_{x_0}=\max_{x\in K}\left\|H'_{x_0}(x)\right\|_\infty.$$

Next, by using the above functions, we define a map $G \in C^1(K, C(Y))$ such that $1/G \in C^1(K, C(Y))$. First consider the case where $K = [0, 1]$. Set

$$V_{x_0} = \left\{ x \in K \colon H_{x_0}(x)(z) \geq \frac{\delta_{x_0}}{2},\ \forall z \in Y \right\}.$$

Obviously, $x_0 \in V_{x_0}$. We claim that $x_0 \in \mathrm{Int}(V_{x_0})$. Let $z \in Y$. Since for each $x \in K$,

$$\left| \frac{H_{x_0}(t)(z) - H_{x_0}(x)(z)}{t - x} - H'_{x_0}(x)(z) \right| \leq \left\| \frac{H_{x_0}(t) - H_{x_0}(x)}{t - x} - H'_{x_0}(x) \right\|_\infty \longrightarrow_{t \to x} 0,$$

we infer that the function $\mathcal{P}_z H_{x_0} \colon K \to [0, +\infty)$, defined by

$$\mathcal{P}_z H_{x_0}(x) = H_{x_0}(x)(z) \qquad (x \in K),$$

is differentiable at x and its derivative is $H'_{x_0}(x)(z)$. Then, according to the mean value Theorem, for each $x \in K$, there exists $t_x \in K$ such that

$$\left| H_{x_0}(x)(z) - H_{x_0}(x_0)(z) \right| = \left| H'_{x_0}(t_x)(z)(x - x_0) \right| \leq M_{x_0} \left| x - x_0 \right|.$$

Now, if $x \in K$ with $|x - x_0| < \delta_{x_0}/3M_{x_0}$, it follows that

$$\begin{aligned} H_{x_0}(x)(z) &\geq H_{x_0}(x_0)(z) - |H_{x_0}(x)(z) - H_{x_0}(x_0)(z)| \\ &\geq \delta_{x_0} - M_{x_0}|x - x_0| > \delta_{x_0} - M_{x_0} \frac{\delta_{x_0}}{3M_{x_0}} \\ &= \frac{2\delta_{x_0}}{3} > \frac{\delta_{x_0}}{2}. \end{aligned}$$

This discussion yields $x_0 \in \mathrm{Int}(V_{x_0})$. From the above argument, $K = \cup_{x \in K} \mathrm{Int}(V_x)$. Then there exist $x_1, \ldots, x_m \in K$ such that $K = \cup_{j=1}^m \mathrm{Int}(V_{x_j})$ because K is compact. Hence, if $x \in K$, then $x \in V_{x_j}$ for some $j \in \{1, \ldots, m\}$, whence $H_{x_j}(x)(z) \geq \delta_{x_j}/2$ for all $z \in Y$. Define $G = \sum_{j=1}^m H_{x_j}$. It is clear that $G \in C^1(K, C(Y))$ and $G(x)(z) \geq \delta_0 > 0$ for all $x \in K$ and $z \in Y$, where $\delta_0 = \min\{\delta_{x_1}/2, \ldots, \delta_{x_m}/2\}$. Now, one can easily check that $1/G \in C^1(K, C(Y))$ with $(1/G)' = -G'/G^2$.

Now suppose that $K = \mathbb{T}$. Choose $t_0 \in [0, 2\pi]$ with $e^{it_0} = x_0$. Set

$$\mathcal{V}_{t_0} = \left\{ t \in [0, 2\pi] \colon H_{x_0}(e^{it})(z) \geq \frac{\delta_{x_0}}{2},\ \forall z \in Y \right\}.$$

Obviously, $t_0 \in \mathcal{V}_{t_0}$. Similarly to the previous case, we prove that $t_0 \in \mathrm{Int}(\mathcal{V}_{t_0})$. Let $z \in Y$. We define the function $\mathcal{P}_z \mathcal{H}_{t_0} \colon [0, 2\pi] \to [0, +\infty)$ by

$$\mathcal{P}_z \mathcal{H}_{t_0}(t) = H_{e^{it_0}}(e^{it})(z) = H_{x_0}(e^{it})(z) \qquad (t \in [0, 2\pi]).$$

Similarly to above, the function $\mathcal{P}_z \mathcal{H}_{t_0}$ is differentiable at each $t \in [0, 2\pi]$ and its derivative is $ie^{it} H'_{x_0}(e^{it})(z)$, and so from the mean value Theorem, we obtain

$$\left| \mathcal{P}_z \mathcal{H}_{t_0}(t) - \mathcal{P}_z \mathcal{H}_{t_0}(t_0) \right| \leq M_{x_0} \left| t - t_0 \right|.$$

Now, if $t \in [0, 2\pi]$ with $|t - t_0| < \delta_{x_0}/3M_{x_0}$, it follows that

$$H_{e^{it_0}}(e^{it})(z) = \mathcal{P}_z \mathcal{H}_{t_0}(t) > \frac{\delta_{x_0}}{2}.$$

Therefore, it is inferred that $t_0 \in \mathrm{Int}(\mathcal{V}_{t_0})$. Clearly, $[0, 2\pi] = \cup_{t \in [0, 2\pi]} \mathrm{Int}(\mathcal{V}_t)$ (note that $\mathcal{V}_0 = \mathcal{V}_{2\pi}$), and so there exist $t_1, \ldots, t_m \in [0, 2\pi]$ such that $[0, 2\pi] = \cup_{j=1}^m \mathrm{Int}(\mathcal{V}_{t_j})$. Define

$G = \sum_{j=1}^{m} H_{x_j}$, where $x_j = e^{it_j}$. Then $G \in C^1(K, C(Y))$, and if $x \in \mathbb{T}$, then there exists $j \in \{1, \ldots, m\}$ such that $t \in \mathcal{V}_{t_j}$ and $x = e^{it}$, which implies that

$$G(x)(z) \geq \mathcal{H}_{x_j}(e^{it})(z) \geq \delta_0 > 0$$

for all $z \in Y$, where $\delta_0 = \min\{\delta_{x_1}/2, \ldots, \delta_{x_m}/2\}$. Now, one can easily check that $1/G \in C^1(K, C(Y))$, as desired.

Finally, as observed in both cases, $G \in C^1(K, C(Y))$ and $\chi(G) = \sum_{j=1}^{m} \chi(H_{x_j}) = 0$. However, $1/G \in C^1(K, C(Y))$, and so $1 = \chi(G \cdot (1/G)) = \chi(G)\chi(1/G)$ which yields $\chi(G) \neq 0$, a contradiction. This completes the proof. □

Theorem 3 *Let K be either $[0, 1]$ or $\mathbb{T}$ and Y_1, Y_2 be compact Hausdorff spaces. If T is a unital algebra homomorphism of $C^1(K, C(Y_1))$ to $C^1(K, C(Y_2))$, then there exist a function $\varphi \in C(K \times Y_2, K)$ satisfying that $\varphi^y \in C^1(K)$ for each $y \in Y_2$, and a map $\tau \in C(K \times Y_2, Y_1)$ such that*

$$T(F)(x, y) = F(\varphi(x, y), \tau(x, y)) \quad ((x, y) \in K \times Y_2)$$

for all $F \in C^1(K, C(Y_1))$.

Proof Since T is a unital algebra homomorphism, it is easy that to see for each $(x, y) \in K \times Y_2$, the functional $T_{(x,y)}\colon C^1(K, C(Y_1)) \to \mathbb{C}$ defined by

$$T_{(x,y)}(F) = T(F)(x, y) = T(F)(x)(y) \qquad \left(F \in C^1(K, C(Y_1))\right),$$

belongs to the maximal ideal space of $C^1(K, C(Y_1))$. Then, taking into account Theorem 2, this allows us to define the functions $\varphi\colon K \times Y_2 \to K$ and $\tau\colon K \times Y_2 \to Y_1$ such that, for each $(x, y) \in K \times Y_2$, we have

$$T_{(x,y)}(F) = \chi_{(\varphi(x,y),\tau(x,y))}(F) \qquad \left(F \in C^1(K, C(Y_1))\right).$$

Hence it follows that

$$T(F)(x, y) = F(\varphi(x, y), \tau(x, y)) \qquad \left(F \in C^1(K, C(Y_1)),\ (x, y) \in K \times Y_2\right).$$

We now prove that τ and φ are continuous. For this purpose, assume that $\{(x_\alpha, y_\alpha)\}_\alpha$ is a net in $K \times Y_2$ converging to $(x_0, y_0) \in K \times Y_2$.

Firstly, for each $f \in C(Y_1)$, from the fact that $T(1_K \otimes f) \in C(K \times Y_2)$ we get

$$\lim_\alpha T(1_K \otimes f)(x_\alpha, y_\alpha) = T(1_K \otimes f)(x_0, y_0),$$

which taking into account the above relation, implies that

$$\lim_\alpha (1_K \otimes f)(\varphi(x_\alpha, y_\alpha), \tau(x_\alpha, y_\alpha)) = (1_K \otimes f)(\varphi(x_0, y_0), \tau(x_0, y_0)).$$

Consequently, $\lim_\alpha f(\tau(x_\alpha, y_\alpha)) = f(\tau(x_0, y_0))$ for each $f \in C(Y_1)$. From this argument we easily deduce that $\lim_\alpha \tau(x_\alpha, y_\alpha) = \tau(x_0, y_0)$, and so $\tau \in C(K \times Y_2, Y_1)$.

Secondly, since $T(\mathrm{id}_K \otimes 1_{Y_1}) \in C(K \times Y_2)$, it follows that

$$\lim_\alpha T(\mathrm{id}_K \otimes 1_{Y_1})(x_\alpha, y_\alpha) = T(\mathrm{id}_K \otimes 1_{Y_1})(x_0, y_0),$$

and thus

$$\lim_\alpha (\mathrm{id}_K \otimes 1_{Y_1})(\varphi(x_\alpha, y_\alpha), \tau(x_\alpha, y_\alpha)) = (\mathrm{id}_K \otimes 1_{Y_1})(\varphi(x_0, y_0), \tau(x_0, y_0)),$$

which yields $\lim_\alpha \varphi(x_\alpha, y_\alpha) = \varphi(x_0, y_0)$, as desired.

Finally, let $y \in Y_2$. For each $x \in K$, we have

$$\begin{aligned}\varphi^y(x) &= \varphi(x, y) \\ &= (\mathrm{id}_K \otimes 1_{Y_1})(\varphi(x, y), \tau(x, y)) \\ &= T(\mathrm{id}_K \otimes 1_{Y_1})(x, y) \\ &= T(\mathrm{id}_K \otimes 1_{Y_1})(x)(y).\end{aligned}$$

This implies that $\varphi^y \in C^1(K)$ since $(\varphi^y)'(x) = T(\mathrm{id}_K \otimes 1_{Y_1})'(x)(y)$ for all $x \in K$. □

3 Results

We first show that every approximate local isometry of $C^1(K)$ is a surjective isometry. By [11,Corollary 2], this fact is known for $C^1([0, 1])$ equipped with the norm:

$$\|f\|_C = \max_{x\in[0,1]} \left(|f(x)| + \left|f'(x)\right|\right) \qquad \left(f \in C^1([0, 1])\right).$$

Theorem 4 *For $K = [0, 1]$ or $K = \mathbb{T}$, the group $\mathrm{Iso}(C^1(K))$ is topologically reflexive.*

Proof We first prove that $\mathrm{Iso}(C^1(K))$ is algebraically reflexive. It holds for $K = [0, 1]$ (see the last paragraph in [26]) but, apparently, the property is unknown for $K = \mathbb{T}$. For that reason, we include here a proof that is valid for both cases.

Let $T \in \mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(C^1(K)))$. Hence, for each $f \in C^1(K)$, there exists $T_f \in \mathrm{Iso}(C^1(K))$ such that $T(f) = T_f(f)$, and therefore $\|T(f)\|_\Sigma = \|f\|_\Sigma$. By Theorem 1, there are $\lambda_f \in \mathbb{T}$ and $\phi_f \in \mathrm{Iso}(K)$ such that $T_f(f) = \lambda_f(f \circ \phi_f)$ and thus $\|T(f)\|_\infty = \|f\|_\infty$. Consequently, we also have that $\left\|T(f)'\right\|_\infty = \left\|f'\right\|_\infty$.

Since $\|T(1_K)\|_\infty = \|1_K\|_\infty = 1$ and $\left\|T(1_K)'\right\|_\infty = \left\|(1_K)'\right\|_\infty = 0$, we deduce that $T(1_K)$ is a constant function of K to $\mathbb{T}$ and therefore $T(1_K) = \lambda 1_K$ for some $\lambda \in \mathbb{T}$.

Clearly, for each $x \in K$, the functional $S_x : C^1(K) \to \mathbb{C}$ defined by

$$S_x(f) = \overline{\lambda} T(f)(x) \qquad (f \in C^1(K)),$$

is linear and unital. To show its multiplicativity, define the functional $T_x : C^1(K) \to \mathbb{C}$ by

$$T_x(f) = T(f)(x) \qquad (f \in C^1(K)).$$

Clearly, T_x is linear and continuous. Given any $f \in C^1(K)$, there exist $\lambda_f \in \mathbb{T}$ and $\phi_f \in \mathrm{Iso}(K)$ such that $T(f) = \lambda_f(f \circ \phi_f)$ and therefore

$$T_x(f) = T(f)(x) = \lambda_f f(\phi_f(x)) \in \mathbb{T}\sigma(f),$$

where $\sigma(f)$ denotes the spectrum of f. Applying [14,Proposition 2.2], which is a spherical variant of the Gleason–Kahane–Żelazko theorem, we conclude that $S_x = \overline{T_x(1_K)} T_x$ is multiplicative.

By above-proved, the map $S : C^1(K) \to C^1(K)$, defined by

$$S(f)(x) = S_x(f) = \overline{\lambda} T(f)(x) \qquad (f \in C^1(K),\ x \in K),$$

is a unital algebra homomorphism. Since $C^1(K)$ is a semisimple commutative Banach algebra and the maximal ideal space of $C^1(K)$ is homeomorphic to K, it is well known [29] that S is automatically continuous and induces a map $\phi \in C(K, K)$ such that $S(f) = f \circ \phi$ for all $f \in C^1(K)$, and whence $T(f) = \lambda(f \circ \phi)$ for all $f \in C^1(K)$. In fact, $\phi = \overline{\lambda} T(\mathrm{id}_K) \in C^1(K)$.

We now prove that ϕ is an isometry of K. Take any $f \in C^1(K)$. By hypothesis, we get that $T(f) = \lambda_f(f \circ \phi_f)$ for some $\lambda_f \in \mathbb{T}$ and $\phi_f \in \mathrm{Iso}(K)$. It follows that $\lambda(f \circ \phi) = \lambda_f(f \circ \phi_f)$. Hence $|f \circ \phi| = |f \circ \phi_f|$ and this implies that $\phi = \phi_f \in C^1(K)$ because $C^1(K)$ is strongly separating. This means that for any pair of distinct points $x_1, x_2 \in K$, there exists $f \in C^1(K)$ such that $|f(x_1)| \neq |f(x_2)|$. For example, one can take the function $f : K \to \mathbb{C}$ defined by $f(x) = (x - x_2)/(x_1 - x_2)$ for all $x \in K$.

From this expression of T as a weighted composition operator, we deduce that $T \in \mathrm{Iso}(C^1(K))$ by Theorem 1, and this proves that $\mathrm{Iso}(C^1(K))$ is algebraically reflexive.

We now prove that $\mathrm{Iso}(C^1(K))$ is topologically reflexive. For it, let $T \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(C^1(K)))$. By Theorem 1, for each $f \in C^1(K)$, we can take two sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ and $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Iso}(K)$ such that

$$\lim_{n\to\infty} \left\| \lambda_{f,n}(f \circ \phi_{f,n}) - T(f) \right\|_\Sigma = 0.$$

Since $\mathbb{T}$ is compact in $\mathbb{C}$, and $\mathrm{Iso}(K)$ is compact in $C(K)$ by the Arzelá–Ascoli theorem, we may take subsequences $\{\lambda_{f,n_k}\}_{k\in\mathbb{N}}$ and $\{\phi_{f,n_k}\}_{k\in\mathbb{N}}$ such that $\left|\lambda_{f,n_k} - \lambda_f\right| \to 0$ and $\left\|\phi_{f,n_k} - \phi_f\right\|_\infty \to 0$ when $k \to \infty$ for some $\lambda_f \in \mathbb{T}$ and $\phi_f \in \mathrm{Iso}(K)$. Now, it easily follows that $T(f)(x) = \lim_{k\to\infty} \lambda_{f,n_k} f(\phi_{f,n_k}(x)) = \lambda_f f(\phi_f(x))$ for each $x \in K$. Hence $T(f) = \lambda_f(f \circ \phi_f)$ and therefore T is in $\mathrm{ref}_{\mathrm{alg}}(\mathrm{Iso}(C^1(K)))$. Since $\mathrm{Iso}(C^1(K))$ is algebraically reflexive as first proved, we conclude that $T \in \mathrm{Iso}(C^1(K))$. □

We are ready to state the topological reflexivity of the isometry group of $C^1(K, C(Y))$-algebras under a convenient condition on the isometry group of $C(Y)$-algebras.

Theorem 5 *Let K be either $[0, 1]$ or $\mathbb{T}$ and let Y_1, Y_2 be compact Hausdorff spaces. Assume that $\mathrm{Iso}(C(Y_1), C(Y_2))$ is topologically reflexive. Then $\mathrm{Iso}(C^1(K, C(Y_1)), C^1(K, C(Y_2)))$ is topologically reflexive.*

Proof Let T be an approximate local isometry of $C^1(K, C(Y_1))$ to $C^1(K, C(Y_2))$. We are going to show that T has a representation of type BJ as in Theorem 1, and therefore T will be a linear isometry of $C^1(K, C(Y_1))$ onto $C^1(K, C(Y_2)))$.

Claim 1 $\|T(F)\|_\Sigma = \|F\|_\Sigma$, $\|T(F)\|_\infty = \|F\|_\infty$ *and* $\left\|T(F)'\right\|_\infty = \left\|F'\right\|_\infty$ *for* $F \in C^1(K, C(Y_1))$.

Let $F \in C^1(K, C(Y_1))$. Hence we have

$$\lim_{n\to\infty} \left\| U_{F,n}(F) - T(F) \right\|_\Sigma = 0$$

for some sequence $\{U_{F,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Iso}(C^1(K, C(Y_1)), C^1(K, C(Y_2)))$. Clearly,

$$\lim_{n\to\infty} \left\| U_{F,n}(F) \right\|_\Sigma = \|T(F)\|_\Sigma\,,$$
$$\lim_{n\to\infty} \left\| U_{F,n}(F) \right\|_\infty = \|T(F)\|_\infty\,,$$
$$\lim_{n\to\infty} \left\| U_{F,n}(F)' \right\|_\infty = \left\|T(F)'\right\|_\infty.$$

For each $n \in \mathbb{N}$, we can write

$$U_{F,n}(G) = h_{F,n} G(\varphi_{F,n}, \tau_{F,n}) \qquad \left(G \in C^1(K, C(Y_1))\right)$$

with $h_{F,n}$, $\varphi_{F,n}$ and $\tau_{F,n}$ being as in Theorem 1. An easy calculation yields

$$\begin{aligned}\left\|U_{F,n}(G)\right\|_\infty &= \sup_{(x,y)\in K\times Y_2}\left|h_{F,n}(y)G(\varphi_{F,n}(x,y),\tau_{F,n}(y))\right| \\ &= \sup_{(x,y)\in K\times Y_2}\left|G(\varphi_{F,n}(x,y),\tau_{F,n}(y))\right| \\ &= \sup_{y\in Y_2}\left(\sup_{x\in K}\left|G(\varphi^y_{F,n}(x),\tau_{F,n}(y))\right|\right) = \sup_{y\in Y_2}\left(\sup_{t\in K}\left|G(t,\tau_{F,n}(y))\right|\right) \\ &= \sup_{t\in K}\left(\sup_{y\in Y_2}\left|G(t,\tau_{F,n}(y))\right|\right) = \sup_{t\in K}\left(\sup_{w\in Y_1}|G(t,w)|\right) \\ &= \sup_{(t,w)\in K\times Y_1}|G(t,w)| = \|G\|_\infty .\end{aligned}$$

Since $\left\|U_{F,n}(G)\right\|_\Sigma = \|G\|_\Sigma$, we also have $\left\|U_{F,n}(G)'\right\|_\infty = \left\|G'\right\|_\infty$. Now the claim follows easily.

The following fact will be used repeatedly without any explicit mention in our proof.

Claim 2 *For every $F \in C^1(K, C(Y_1))$, there exist three sequences $\{h_{F,n}\}_{n\in\mathbb{N}}$ in $C(Y_2,\mathbb{T})$, $\{\varphi_{F,n}\}_{n\in\mathbb{N}}$ in $C(K\times Y_2, K)$ such that, for each $y \in Y_2$, $\varphi^y_{F,n} \in \mathrm{Iso}(K)$ for all $n\in\mathbb{N}$, and $\{\tau_{F,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Homeo}(Y_2, Y_1)$ satisfying that*

$$\lim_{n\to\infty}\left\|h_{F,n}F(\varphi_{F,n},\tau_{F,n}) - T(F)\right\|_\Sigma = 0.$$

Let $F \in C^1(K, C(Y_1))$. By hypothesis, we have

$$\lim_{n\to\infty}\left\|U_{F,n}(F) - T(F)\right\|_\Sigma = 0$$

for some sequence $\{U_{F,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Iso}(C^1(K, C(Y_1)), C^1(K, C(Y_2)))$. By Theorem 1, for each $n\in\mathbb{N}$, there are $h_{F,n}\in C(Y_2,\mathbb{T})$, $\varphi_{F,n}\in C(K\times Y_2, K)$ with $\varphi^y_{F,n}\in\mathrm{Iso}(K)$ for each $y\in Y_2$, and $\tau_{F,n}\in\mathrm{Homeo}(Y_2,Y_1)$ such that

$$U_{F,n}(G)(x,y) = h_{F,n}(y)G(\varphi_{F,n}(x,y),\tau_{F,n}(y)) \qquad ((x,y)\in K\times Y_2),$$

for all $G\in C^1(K, C(Y_1))$. Then the claim holds.

Claim 3 *There exists a function $h\in C(Y_2,\mathbb{T})$ such that $T(1_K\otimes 1_{Y_1}) = 1_K\otimes h$.*

Denote $F = 1_K\otimes 1_{Y_1}$. On a hand, we obtain that $\left\|T(F)'\right\|_\infty = \left\|F'\right\|_\infty = 0$ by Claim 1. Hence $T(F)$ is a constant function from K to $C(Y_2)$ and therefore $T(F) = 1_K\otimes h$ for some $h\in C(Y_2)$. On the other hand, we have

$$\lim_{n\to\infty}\left\|1_K\otimes h_{F,n} - T(F)\right\|_\Sigma = \lim_{n\to\infty}\left\|h_{F,n}F(\varphi_{F,n},\tau_{F,n}) - T(F)\right\|_\Sigma = 0.$$

It follows that

$$\lim_{n\to\infty}\left\|h_{F,n} - h\right\|_\infty = \lim_{n\to\infty}\left\|1_K\otimes(h_{F,n}-h)\right\|_\Sigma = \lim_{n\to\infty}\left\|1_K\otimes h_{F,n} - 1_K\otimes h\right\|_\Sigma = 0.$$

Since $\{h_{F,n}\}_{n\in\mathbb{N}}\subseteq C(Y_2,\mathbb{T})$, it follows that h is a unimodular function and this proves the claim.

Claim 4 *For each $(x,y)\in K\times Y_2$, the functional $S_{(x,y)}\colon C^1(K, C(Y_1))\to\mathbb{C}$ defined by*

$$S_{(x,y)}(F) = \overline{h(y)}T(F)(x,y) \quad (F\in C^1(K, C(Y_1))),$$

is linear, unital and multiplicative.

Fix $(x, y) \in K \times Y_2$. Clearly, $S_{(x,y)}$ is linear with $S_{(x,y)}(1_K \otimes 1_{Y_1}) = 1$ by Claim 3. To prove its multiplicativity, define $T_{(x,y)}\colon C^1(K, C(Y_1)) \to \mathbb{C}$ by

$$T_{(x,y)}(F) = T(F)(x, y) \qquad (F \in C^1(K, C(Y_1))).$$

Since $T_{(x,y)}$ is linear and

$$|T_{(x,y)}(F)| = |T(F)(x, y)| \le \|T(F)\|_\infty \le \|T(F)\|_\Sigma = \|F\|_\Sigma \qquad (F \in C^1(K, C(Y_1))),$$

we have that $T_{(x,y)}$ is continuous. Pick now any $F \in C^1(K, C(Y_1))$. We have

$$\lim_{n\to\infty} \left\|h_{F,n} F(\varphi_{F,n}, \tau_{F,n}) - T(F)\right\|_\Sigma = 0.$$

We infer that

$$T_{(x,y)}(F) = T(F)(x, y) = \lim_{n\to\infty} h_{F,n}(y) F(\varphi_{F,n}(x, y), \tau_{F,n}(y)) \in \mathbb{T}\sigma(F),$$

by Theorem 2. Finally, applying [14,Proposition 2.2] yields that $S_{(x,y)} = \overline{T_{(x,y)}(1_K \otimes 1_{Y_1})}\, T_{(x,y)}$ is multiplicative.

Claim 5 *There exist two maps $\varphi \in C(K \times Y_2, K)$, with $\varphi^y \in C^1(K)$ for each $y \in Y_2$, and $\tau \in C(K \times Y_2, Y_1)$ such that*

$$T(F)(x, y) = h(y) F(\varphi(x, y), \tau(x, y)) \quad ((x, y) \in K \times Y_2),$$

for all $F \in C^1(K, C(Y_1))$.

Using Claim 4, it is easily deduced that $S\colon C^1(K, C(Y_1)) \to C^1(K, C(Y_2))$, defined by

$$S(F)(x, y) = \overline{h(y)} T(F)(x, y) \qquad ((x, y) \in K \times Y_2,\ F \in C^1(K, C(Y_1))),$$

is a unital algebra homomorphism. By Theorem 3, there exist two maps $\varphi \in C(K \times Y_2, K)$, with $\varphi^y \in C^1(K)$ for each $y \in Y_2$, and $\tau \in C(K \times Y_2, Y_1)$ such that

$$S(F)(x, y) = F(\varphi(x, y), \tau(x, y)) \qquad ((x, y) \in K \times Y_2),$$

for all $F \in C^1(K, C(Y_1))$.

Claim 6 *For each $y \in Y_2$, $\varphi^y \in \mathrm{Iso}(K)$.*

Fix $y \in Y_2$ and define $T_y\colon C^1(K) \to C^1(K)$ by

$$T_y(f)(x) = T(f \otimes 1_{Y_1})(x, y) \qquad (x \in K,\ f \in C^1(K)).$$

By Claim 5, we have

$$T_y(f)(x) = h(y) f(\varphi(x, y)) = h(y) f(\varphi^y(x)) \qquad (x \in K,\ f \in C^1(K)).$$

Clearly, $T_y \in B(C^1(K), C^1(K))$ with $\left\|T_y(f)\right\|_\Sigma \le \|f\|_\Sigma$. For every $f \in C^1(K)$, there exist three sequences $\{h_{f\otimes 1_{Y_1},n}\}_{n\in\mathbb{N}}$ in $C(Y_2, \mathbb{T})$, $\{\varphi_{f\otimes 1_{Y_1},n}\}_{n\in\mathbb{N}}$ in $C(K \times Y_2, K)$ with $\varphi^y_{f\otimes 1_{Y_1},n} \in \mathrm{Iso}(K)$ for all $n \in \mathbb{N}$, and $\{\tau_{f\otimes 1_{Y_1},n}\}_{n\in\mathbb{N}}$ in $\mathrm{Homeo}(Y_2, Y_1)$ such that

$$\lim_{n\to\infty} \left\|h_{f\otimes 1_{Y_1},n}(f \otimes 1_{Y_1})(\varphi_{f\otimes 1_{Y_1},n}, \tau_{f\otimes 1_{Y_1},n}) - T(f \otimes 1_{Y_1})\right\|_\Sigma = 0.$$

Consequently, we obtain

$$\lim_{n\to\infty} \left\|h_{f\otimes 1_{Y_1},n}(y)(f \circ \varphi^y_{f\otimes 1_{Y_1},n}) - T_y(f)\right\|_\Sigma = 0.$$

For each $n \in \mathbb{N}$, define $T_{y,f,n} \colon C^1(K) \to C^1(K)$ by

$$T_{y,f,n}(g) = h_{f\otimes 1_{Y_1},n}(y)(g \circ \varphi^y_{f\otimes 1_{Y_1},n}) \qquad (g \in C^1(K)).$$

In the light of Theorem 1, $T_{y,f,n} \in \mathrm{Iso}(C^1(K))$ because $h_{f\otimes 1_{Y_1},n}(y) \in \mathbb{T}$ and $\varphi^y_{f\otimes 1_{Y_1},n} \in \mathrm{Iso}(K)$. Therefore $T_y \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(C^1(K)))$. Hence $T_y \in \mathrm{Iso}(C^1(K))$ by Theorem 4. By Theorem 1 again, we can find a number $\alpha_y \in \mathbb{T}$ and a map $\phi_y \in \mathrm{Iso}(K)$ such that

$$T_y(f)(x) = \alpha_y f(\phi_y(x)) \qquad (x \in K,\ f \in C^1(K)).$$

In addition, $\alpha_y = T_y(1_K)(x) = h(y)$ where x is any point in K, and thus

$$T_y(f)(x) = h(y)f(\phi_y(x)) \qquad (x \in K,\ f \in C^1(K)).$$

Therefore we can write

$$h(y)f(\varphi^y(x)) = T_y(f)(x) = h(y)f(\phi_y(x)) \qquad (x \in K,\ f \in C^1(K)).$$

Since $C^1(K)$ separates the points of K, we conclude that

$$\varphi^y(x) = \phi_y(x) \qquad (x \in K)$$

and so $\varphi^y = \phi_y \in \mathrm{Iso}(K)$, as required.

Claim 7 *There exists a map* $\beta \in \mathrm{Homeo}(Y_2, Y_1)$ *such that*

$$\beta(y) = \tau(x, y) \quad (y \in Y_2),$$

where x is any point of K.

Let $x \in K$ be fixed and define $T_x \colon C(Y_1) \to C(Y_2)$ by

$$T_x(g)(y) = T(1_K \otimes g)(x, y) \qquad (y \in Y_2,\ g \in C(Y_1)).$$

Claim 5 yields

$$T_x(g)(y) = h(y)g(\tau(x, y)) \qquad (y \in Y_2,\ g \in C(Y_1)).$$

Clearly, $T_x \in B(C(Y_1), C(Y_2))$ with $\|T_x(g)\|_\infty \le \|g\|_\infty$ for all $g \in C(Y_1)$. For each $g \in C(Y_1)$, we have three sequences $\{h_{1_K\otimes g,n}\}_{n\in\mathbb{N}}$ in $C(Y_2, \mathbb{T})$, $\{\varphi_{1_K\otimes g,n}\}_{n\in\mathbb{N}}$ in $C(K \times Y_2, K)$ such that, for each $y \in Y$, $\varphi^y_{1_K\otimes g,n} \in \mathrm{Iso}(K)$ for all $n \in \mathbb{N}$, and $\{\tau_{1_K\otimes g,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Homeo}(Y_2, Y_1)$ for which

$$\lim_{n\to\infty} \left\| h_{1_K\otimes g,n}(1_K \otimes g)(\varphi_{1_K\otimes g,n}, \tau_{1_K\otimes g,n}) - T(1_K \otimes g) \right\|_\Sigma = 0.$$

Therefore we have

$$\lim_{n\to\infty} \left\| h_{1_K\otimes g,n}(g \circ \tau_{1_K\otimes g,n}) - T_x(g) \right\|_\infty = 0.$$

For each $n \in \mathbb{N}$, define $T_{g,n} \colon C(Y_1) \to C(Y_2)$ by

$$T_{g,n}(f) = h_{1_K\otimes g,n}(f \circ \tau_{1_K\otimes g,n}) \qquad (f \in C(Y_1)).$$

Notice that $h_{1_K\otimes g,n} \in C(Y_2, \mathbb{T})$ and $\tau_{1_K\otimes g,n} \in \mathrm{Homeo}(Y_2, Y_1)$. Hence $T_{g,n} \in \mathrm{Iso}(C(Y_1), C(Y_2))$. Therefore $T_x \in \mathrm{ref}_{\mathrm{top}}(\mathrm{Iso}(C(Y_1), C(Y_2)))$. Hence $T_x \in \mathrm{Iso}(C(Y_1), C(Y_2))$ by the hypothesis of the theorem. Observe that $T_{g,n}(f)$ does not depend on x for any

$f \in C(Y_1)$. Now, the Banach–Stone theorem gives a function $\alpha \in C(Y_2, \mathbb{T})$ and a map $\beta \in \mathrm{Homeo}(Y_2, Y_1)$ such that

$$T_x(g)(y) = \alpha(y)g(\beta(y)) \qquad (y \in Y_2,\ g \in C(Y_1)).$$

In fact, $\alpha(y) = T_x(1_{Y_1})(y) = h(y)$ for all $y \in Y_2$, and therefore

$$T_x(g)(y) = h(y)g(\beta(y)) \qquad (y \in Y_2,\ g \in C(Y_1)).$$

Consequently, we have

$$h(y)g(\beta(y)) = T_x(g)(y) = h(y)g(\tau(x, y)) \qquad (g \in C(Y_1),\ y \in Y_2).$$

Since $C(Y_1)$ separates the points of Y_1, we infer that

$$\beta(y) = \tau(x, y) \qquad (y \in Y_2).$$

This proves Claim 7 and the proof of Theorem 5 is finished. □

Notice that Theorem 5 contains Theorem 4 as a special case (consider the case that Y_1 and Y_2 are singletons). Note, however, that Theorem 4 is used in its proof.

We next apply Theorem 5 to study the 2-topological reflexivity of the set of surjective linear isometries between algebras $C^1(K, C(Y))$.

Corollary 1 *Let Y_1, Y_2 be compact Hausdorff spaces and suppose that* $\mathrm{Iso}(C(Y_1), C(Y_2))$ *is topologically reflexive. Then* $\mathrm{Iso}(C^1(K, C(Y_1)), C^1(K, C(Y_2)))$ *is 2-topologically reflexive where K is either* $[0, 1]$ *or* $\mathbb{T}$.

Proof Let Δ be an approximate 2-local isometry of $C^1(K, C(Y_1))$ to $C^1(K, C(Y_2))$. We claim that given $(x, y) \in K \times Y_2$, the functional $\Delta_{(x,y)} : C^1(K, C(Y_1)) \to \mathbb{C}$, defined by

$$\Delta_{(x,y)}(F) = \Delta(F)(x, y) \qquad (F \in C^1(K, C(Y_1))),$$

is linear. To prove this claim, by Proposition 3.2 in [14], which is a spherical variant of the Kowalski–Słodkowski theorem (see also Theorem 2.2 in [26]), it is sufficient to check that $\Delta_{(x,y)}$ satisfies the following properties:

a) $\Delta_{(x,y)}$ is 1-homogeneous, that is, $\Delta_{(x,y)}(\beta F) = \beta\Delta_{(x,y)}(F)$ for all $F \in C^1(K, C(Y_1))$ and $\beta \in \mathbb{C}$.
b) $\Delta_{(x,y)}(F) - \Delta_{(x,y)}(G) \in \mathbb{T}\sigma(F - G)$ for all $F, G \in C^1(K, C(Y_1))$.

To prove a), let $F \in C^1(K, C(Y_1))$ and $\beta \in \mathbb{C}$. Hence there exists a sequence $\{T_{F,\beta F,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Iso}(C^1(K, C(Y_1)), C^1(K, C(Y_2)))$ such that

$$\lim_{n\to\infty} \left\|T_{F,\beta F,n}(F) - \Delta(F)\right\|_\Sigma = 0,$$
$$\lim_{n\to\infty} \left\|T_{F,\beta F,n}(\beta F) - \Delta(\beta F)\right\|_\Sigma = 0,$$

and since $T_{F,\beta F,n}(\beta F) = \beta T_{F,\beta F,n}(F)$ for all $n \in \mathbb{N}$, we deduce that

$$\Delta(\beta F) = \beta\Delta(F).$$

To check b), let $F, G \in C^1(K, C(Y_1))$. We can take three sequences $\{h_{F,G,n}\}_{n\in\mathbb{N}}$ in $C(Y_2, \mathbb{T})$, $\{\varphi_{F,G,n}\}_{n\in\mathbb{N}}$ in $C(K \times Y_2, K)$ and $\{\tau_{F,G,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Homeo}(Y_2, Y_1)$ satisfying that

$$\lim_{n\to\infty} \left\|h_{F,G,n}F(\varphi_{F,G,n}, \tau_{F,G,n}) - \Delta(F)\right\|_\Sigma = 0,$$
$$\lim_{n\to\infty} \left\|h_{F,G,n}G(\varphi_{F,G,n}, \tau_{F,G,n}) - \Delta(G)\right\|_\Sigma = 0.$$

Since

$$h_{F,G,n}(y)F(\varphi_{F,G,n}(x,y),\tau_{F,G,n}(y)) - h_{F,G,n}(y)G(\varphi_{F,G,n}(x,y),\tau_{F,G,n}(y))$$
$$= h_{F,G,n}(y)(F-G)(\varphi_{F,G,n}(x,y),\tau_{F,G,n}(y)) \in \mathbb{T}\sigma(F-G)$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \left| h_{F,G,n}(y)(F-G)(\varphi_{F,G,n}(x,y),\tau_{F,G,n}(y)) - \left[\Delta_{(x,y)}(F) - \Delta_{(x,y)}(G)\right]\right| = 0,$$

we get that

$$\Delta_{(x,y)}(F) - \Delta_{(x,y)}(G) \in \overline{\mathbb{T}\sigma(F-G)} = \mathbb{T}\sigma(F-G).$$

This proves our claim. By the arbitrariness of (x, y), we infer that Δ is linear. Consequently, Δ is an approximate local isometry of $C^1(K, C(Y_1))$ to $C^1(K, C(Y_2))$, hence Δ is a linear isometry of $C^1(K, C(Y_1))$ onto $C^1(K, C(Y_2))$ by Theorem 5, and the proof of the corollary is finished. □

Remark 1 Let Y_1, Y_2 be compact Hausdorff spaces, and let K be either $[0, 1]$ or $\mathbb{T}$. Note that from the proofs of Theorem 5 and Corollary 1 it follows that if $\mathrm{Iso}(C(Y_1), C(Y_2))$ is algebraically reflexive, then $\mathrm{Iso}(C^1(K, C(Y_1)), C^1(K, C(Y_2)))$ is algebraically and 2-algebraically reflexive.

We close the paper with an application on nice operators. Let $T: E \to F$ be a continuous linear operator between Banach spaces and $T^*: F^* \to E^*$ be its adjoint operator. Let $\mathrm{Ext}(B_E)$ denote the set of all extreme points of the unit closed ball B_E of E. Then T is said to be nice if $T^*(\mathrm{Ext}(B_{F^*})) \subseteq \mathrm{Ext}(B_{E^*})$. Surjective linear isometries are nice operators but the converse is not certain in general.

In [23], the authors show that $C^1([0, 1])$ is an example of an infinite-dimensional Banach space for which each nice operator on $C^1([0, 1])$ is an isometric isomorphism. As an immediate consequence of Theorem 4 and Corollary 1, we deduce the following.

Corollary 2 *The set of all nice operators on $C^1([0, 1])$ is topologically reflexive and 2-topologically reflexive.*

The search of a Banach–Stone type representation for nice isomorphisms has been approached by some authors (see Chapter 7 in [5]). It is a natural question to ask what can be stated on the algebraic and topological reflexivity (and 2-reflexivity) of the sets of nice isomorphisms of the classical Banach spaces.

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