

Article

# Algebraic Reflexivity of Non-Canonical Isometries on Lipschitz Spaces

Antonio Jiménez-Vargas \*  and María Isabel Ramírez 

Departamento de Matemáticas, Universidad de Almería, 04120 Almería, Spain; mramirez@ual.es

\* Correspondence: ajimenez@ual.es

**Abstract:** Let  $\text{Lip}([0, 1])$  be the Banach space of all Lipschitz complex-valued functions  $f$  on  $[0, 1]$ , equipped with one of the norms:  $\|f\|_{\sigma} = |f(0)| + \|f'\|_{L^{\infty}}$  or  $\|f\|_m = \max\{|f(0)|, \|f'\|_{L^{\infty}}\}$ , where  $\|\cdot\|_{L^{\infty}}$  denotes the essential supremum norm. It is known that the surjective linear isometries of such spaces are integral operators, rather than the more familiar weighted composition operators. In this paper, we describe the topological reflexive closure of the isometry group of  $\text{Lip}([0, 1])$ . Namely, we prove that every approximate local isometry of  $\text{Lip}([0, 1])$  can be represented as a sum of an elementary weighted composition operator and an integral operator. This description allows us to establish the algebraic reflexivity of the sets of surjective linear isometries, isometric reflections, and generalized bi-circular projections of  $\text{Lip}([0, 1])$ . Additionally, some complete characterizations of such reflections and projections are stated.

**Keywords:** algebraic reflexivity; topological reflexivity; isometry group; Lipschitz function; Gleason–Kahane–Żelazko theorem

**MSC:** 47B38; 47B33; 46B04



**Citation:** Jiménez-Vargas, A.; Ramírez, M.I. Algebraic Reflexivity of Non-Canonical Isometries on Lipschitz Spaces. *Mathematics* **2021**, *9*, 1635. <https://doi.org/10.3390/math9141635>

Academic Editor: Jan L. Cieřliński

Received: 22 June 2021

Accepted: 9 July 2021

Published: 11 July 2021

**Publisher's Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Copyright:** © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

A function  $f: [0, 1] \rightarrow \mathbb{C}$  is said to be Lipschitz if there exists a positive constant  $K$  such that

$$|f(x) - f(y)| \leq K|x - y|, \quad \forall x, y \in [0, 1].$$

The infimum of such constants  $K$  is called the Lipschitz constant of  $f$  and it is denoted by  $\text{Lip}(f)$ .

On the other hand, a measurable function  $f: [0, 1] \rightarrow \mathbb{C}$  is said to be essentially bounded if there is a positive constant  $K$  such that

$$\mu(\{x \in [0, 1]: |f(x)| > K\}) = 0,$$

where  $\mu$  denotes Lebesgue measure on the Borel subsets of  $[0, 1]$ . The infimum of such constants  $K$  is called the essential supremum of  $f$  and we denote it here by  $\|f\|_{L^{\infty}}$ .

Let  $\text{Lip}([0, 1])$  be the linear space of all complex-valued Lipschitz functions on  $[0, 1]$ . This space is closely related to the space  $L^{\infty}([0, 1])$  of all complex-valued essentially bounded measurable functions on  $[0, 1]$ .

Namely, the derivative map  $f \mapsto f'$  is an isometric isomorphism from the space  $\text{Lip}([0, 1])$  (with the Lipschitz seminorm  $\text{Lip}(\cdot)$ ) onto the space  $L^{\infty}([0, 1])$  (with the essential supremum norm  $\|\cdot\|_{L^{\infty}}$ ). On  $L^{\infty}([0, 1])$ , we take into account the usual convention about identifying functions equal almost everywhere.

The isometry group of  $\text{Lip}([0, 1])$  has been studied under the following equivalent norms:

$$\begin{aligned} \|f\|_{\Sigma} &= \|f\|_{\infty} + \|f'\|_{L^{\infty}}, \\ \|f\|_M &= \max\{\|f\|_{\infty}, \|f'\|_{L^{\infty}}\}, \\ \|f\|_{\sigma} &= |f(0)| + \|f'\|_{L^{\infty}}, \\ \|f\|_m &= \max\{|f(0)|, \|f'\|_{L^{\infty}}\}, \end{aligned}$$

where

$$\|f\|_{\infty} = \sup\{|f(x)| : x \in [0, 1]\} \quad (f \in \text{Lip}([0, 1])).$$

Indeed, for any  $f \in \text{Lip}([0, 1])$ , we have

$$\frac{1}{2}\|f\|_{\sigma} \leq \|f\|_m \leq \|f\|_M \leq \|f\|_{\sigma} \leq \|f\|_{\Sigma} \leq 2\|f\|_M.$$

Furthermore,  $(\text{Lip}([0, 1]), \|\cdot\|_{\Sigma})$  is a Banach algebra as it satisfies the Banach algebra law:

$$\|fg\|_{\Sigma} \leq \|f\|_{\Sigma}\|g\|_{\Sigma} \quad (f, g \in \text{Lip}([0, 1])),$$

but  $\text{Lip}([0, 1])$ , equipped with any of the other norms, is only a complete normed algebra in the sense that there exists a positive constant  $K$  (not necessarily equal to 1) such that

$$\|fg\| \leq K\|f\|\|g\| \quad (f, g \in \text{Lip}([0, 1])).$$

Surjective linear isometries of  $\text{Lip}([0, 1])$ , with both the  $\sigma$ -norm or the  $m$ -norm, were characterized as a sum of a weighted composition operator and an integral operator by Koshimizu [1,2], in contrast with the isometry groups of  $\text{Lip}([0, 1])$ , with both the  $\Sigma$ -norm or the  $M$ -norm, whose members have a canonical form in the sense that they can be represented as a weighted composition operator [3–6].

Indeed, the prominent part of the representation of the isometries on  $\text{Lip}([0, 1])$  with the  $\sigma$ -norm or the  $m$ -norm lies on the integral operator as the involved weighted composition operator is elementary in the sense that it is the evaluation at the point 0 multiplied by a unimodular constant.

To present our results, we recall the concepts of reflexivity studied in this paper. Let  $E$  be a Banach space,  $\mathcal{B}(E)$  be the space of all bounded linear operators from  $E$  into  $E$ , and  $\mathcal{S}$  be a nonempty subset of  $\mathcal{B}(E)$ . For each  $e \in E$ , let  $\mathcal{S}(e)$  be  $\{L(e) : L \in \mathcal{S}\}$  and let  $\overline{\mathcal{S}(e)}$  denote the norm-closure of  $\mathcal{S}(e)$  in  $E$ . Define the algebraic reflexive closure and the topological reflexive closure of  $\mathcal{S}$ , respectively, by

$$\begin{aligned} \text{ref}_{\text{alg}}(\mathcal{S}) &= \{T \in \mathcal{B}(E) : T(e) \in \mathcal{S}(e), \forall e \in E\}, \\ \text{ref}_{\text{top}}(\mathcal{S}) &= \left\{T \in \mathcal{B}(E) : T(e) \in \overline{\mathcal{S}(e)}, \forall e \in E\right\}. \end{aligned}$$

We say that  $\mathcal{S}$  is algebraically reflexive (topologically reflexive) if  $\text{ref}_{\text{alg}}(\mathcal{S}) = \mathcal{S}$  (respectively,  $\text{ref}_{\text{top}}(\mathcal{S}) = \mathcal{S}$ ).

In the case  $\mathcal{S} = \text{Iso}(E)$  (the set of all linear isometries from  $E$  onto  $E$ ), the elements of  $\text{ref}_{\text{alg}}(\mathcal{S})$  and  $\text{ref}_{\text{top}}(\mathcal{S})$  are known as local isometries and approximate local isometries of  $E$ , respectively.

The consideration of approximate local isometries instead of local isometries is more general and allows us to deal with the problems of topological reflexivity and algebraic reflexivity at the same time.

The study of algebraic and topological reflexivity of the sets of isometries, derivations, and automorphisms on operator algebras and function algebras is a classical problem which follows attracting the attention of numerous researchers. Molnár’s monograph [7] can give a complete account of these developments.

Reflexivity problems have been addressed on spaces of vector and scalar-valued Lipschitz functions defined on metric spaces, equipped with norms of type  $\|\cdot\|_\Sigma$  and  $\|\cdot\|_M$  [8–12]. In the last three references, the canonical form of the isometries of such spaces allowed the application of the Gleason–Kahane–Żelazko theorem (or of some of its generalizations [11,13]) in their arguments.

In this paper, we deal with the reflexivity of the isometry group of  $\text{Lip}([0, 1])$ , equipped with the  $\sigma$ -norm or the  $m$ -norm. We provide the form of approximate local isometries of  $\text{Lip}([0, 1])$ . With the aid of this description, we prove that the isometry group of  $\text{Lip}([0, 1])$ , with each one of these norms, is algebraically reflexive.

Although the non-canonical form of the isometries on such spaces added initially a little more difficulty to the problem, another application of a spherical variant of Gleason–Kahane–Żelazko theorem, stated in [11], allows us to show that every approximate local isometry of  $\text{Lip}([0, 1])$  admits a representation as a sum of an elementary weighted composition operator and an integral operator. Compare this fact with [11], p. 250, where an example shows that the cited generalization of Gleason–Kahane–Żelazko theorem can not be applied when the isometry group is not canonical.

We prove also that the sets of isometric reflections and generalized bi-circular projections on  $\text{Lip}([0, 1])$  are algebraically reflexive. Our approach requires to characterize these types of maps on  $\text{Lip}([0, 1])$ , endowed with the  $\sigma$ -norm or the  $m$ -norm. This kind of projections was introduced in [14] and they have been characterized in various settings (see, for example, in [10,15] and the references therein).

## 2. Preliminaries

Let  $\text{Lip}([0, 1])$  denote the linear space of all complex-valued Lipschitz functions  $f$  on  $[0, 1]$ . This space is connected with the space  $L^\infty([0, 1])$  of all complex-valued essentially bounded measurable functions  $f$  on  $[0, 1]$ .

Namely, it is known (see, for example, Theorem 1.36 and Corollary 1.39 in [16]) that if  $f$  is function from  $[0, 1]$  to  $\mathbb{C}$ , then the following are equivalent:

- (i)  $f$  belongs to  $\text{Lip}([0, 1])$ .
- (ii)  $f$  is differentiable almost everywhere, its derivative belongs to  $L^\infty([0, 1])$  and

$$f(x) = f(0) + \int_0^x f'(t) dt \quad (x \in [0, 1]).$$

- (iii) There exists a function  $g \in L^\infty([0, 1])$  such that

$$f(x) = f(0) + \int_0^x g(t) dt \quad (x \in [0, 1]).$$

Moreover,  $\text{Lip}(f) = \|f'\|_{L^\infty} = \|g\|_{L^\infty}$ . Therefore, we can identify  $L^\infty([0, 1])$  with

$$\{f' : f \in \text{Lip}([0, 1])\}.$$

Throughout the paper, we will sometimes apply (without any explicit mention) the traditional convention in the space  $L^\infty([0, 1])$  about identifying functions equal almost everywhere on  $[0, 1]$ .

In what follows,  $\sigma(\cdot)$  denotes the spectrum. To simplify, we introduce the following notations:

$$\begin{aligned} \mathbb{T} &= \{z \in \mathbb{C} : |z| = 1\}, \\ L^\infty_{\mathbb{T}}([0, 1]) &= \{f \in L^\infty([0, 1]) : \sigma(f) \subseteq \mathbb{T}\}, \\ \text{Aut}(L^\infty([0, 1])) &= \{\Phi \text{ is a unital algebra automorphism of } L^\infty([0, 1])\}. \end{aligned}$$

From now on, the symbols  $\mathbf{1}$  and  $\iota$  stand for the function with the constant value 1 and the identity function on  $[0, 1]$ , respectively.

Koshimizu [1,2] gave the following characterizations of surjective linear isometries on the space  $\text{Lip}([0, 1])$ , equipped with the  $\sigma$ -norm or the  $m$ -norm given by

$$\begin{aligned} \|f\|_\sigma &= |f(0)| + \|f'\|_{L^\infty}, \\ \|f\|_m &= \max\{|f(0)|, \|f'\|_{L^\infty}\}, \end{aligned}$$

for  $f \in \text{Lip}([0, 1])$ . These descriptions provide a key tool in our study on the reflexivity of some subsets of linear maps on such spaces.

**Theorem 1** ([1,2], Theorem 1.2). *Let  $T$  be a linear operator of  $\text{Lip}([0, 1])$  to  $\text{Lip}([0, 1])$ . Then,  $T$  is a surjective isometry with respect to the  $\sigma$ -norm or the  $m$ -norm if and only if there exist a constant  $\lambda \in \mathbb{T}$ , a function  $\omega \in L^\infty_{\mathbb{T}}([0, 1])$  and a map  $\Phi \in \text{Aut}(L^\infty([0, 1]))$  such that*

$$T(f)(x) = \lambda f(0) + \int_0^x \omega(t)\Phi(f')(t) dt \quad (f \in \text{Lip}([0, 1]), x \in [0, 1]).$$

Notice that the algebraic structure of  $L^\infty([0, 1])$  appears involved in the description of such isometries. Let us recall that  $L^\infty([0, 1])$  is a unital commutative  $C^*$ -algebra whose maximal ideal space  $\mathcal{M}_{L^\infty}$  is extremally disconnected (see [17], p. 130).

We denote by  $C(\mathcal{M}_{L^\infty})$  the Banach algebra of all complex-valued continuous functions  $g$  on  $\mathcal{M}_{L^\infty}$  with the supremum norm:

$$\|g\|_{C(\mathcal{M}_{L^\infty})} = \sup\{|g(m)| : m \in \mathcal{M}_{L^\infty}\}.$$

If  $\widehat{f}$  denotes the Gelfand transform of  $f \in L^\infty([0, 1])$ , the Gelfand–Naimark theorem asserts that the Gelfand transform  $f \mapsto \widehat{f}$  is an isometric algebra  $*$ -isomorphism from  $L^\infty([0, 1])$  onto  $C(\mathcal{M}_{L^\infty})$ .

As every algebra automorphism of  $C(\mathcal{M}_{L^\infty})$  is isometric, it is clear that every algebra automorphism of  $L^\infty([0, 1])$  is a surjective linear isometry of  $L^\infty([0, 1])$ . Besides, the automorphism group of  $L^\infty([0, 1])$  is algebraically reflexive ([18], Corollary 1), but not topologically reflexive ([19], Theorem 5).

From Theorem 1, we deduce immediately the following.

**Corollary 1.** *Let  $T$  be a surjective linear isometry of  $(\text{Lip}([0, 1]), \|\cdot\|_\sigma)$  or  $(\text{Lip}([0, 1]), \|\cdot\|_m)$ . Then,  $|T(f)(0)| = |f(0)|$  and  $\|T(f)'\|_{L^\infty} = \|f'\|_{L^\infty}$  for all  $f \in \text{Lip}([0, 1])$ .*

**Proof.** In view of the expression of  $T$  as in Theorem 1, we infer that  $T(f)(0) = \lambda f(0)$  and  $T(f)' = \omega\Phi(f')$  a.e. on  $[0, 1]$  for all  $f \in \text{Lip}([0, 1])$ . Therefore,  $|T(f)(0)| = |f(0)|$  and

$$\|T(f)'\|_{L^\infty} = \|\omega\Phi(f')\|_{L^\infty} = \|\Phi(f')\|_{L^\infty} = \|f'\|_{L^\infty}. \quad \square$$

Regarding its Banach structure,  $L^\infty([0, 1])$  is isometrically isomorphic to the dual of the Banach space  $L^1([0, 1])$  of all complex-valued Lebesgue integrable functions on  $[0, 1]$  with the norm:

$$\|f\|_{L^1} = \int_0^1 |f(t)| dt \quad (f \in L^1([0, 1])),$$

and thus we may consider that  $L^\infty([0, 1])$  is equipped with the weak\* topology.

A result due to Sikorski and von Neumann (see in [18], Theorem 1) shows that every weak\* continuous algebra homomorphism  $\Phi: L^\infty([0, 1]) \rightarrow L^\infty([0, 1])$  has the form  $\Phi(f) = f \circ \phi$  for all  $f \in L^\infty([0, 1])$ , where  $\phi: [0, 1] \rightarrow [0, 1]$  is a measurable function. In particular, every algebra automorphism of  $L^\infty([0, 1])$  has this form as it is weak\* continuous by Lemmas 1 and 2 in [18] (observe that every algebra automorphism of  $L^\infty([0, 1])$  is a local algebra automorphism and it is continuous by Corollary 2.1.10 in [17]). This last fact can also be proven as in Theorem 1 of [20].

### 3. Results

In the rest of the paper, we will consider that the linear space  $\text{Lip}([0, 1])$  is equipped with the  $\sigma$ -norm or the  $m$ -norm. As the proofs of the results are similar for both norms, we only will prove them when  $\text{Lip}([0, 1])$  is provided with the  $\sigma$ -norm.

We first give a representation of the elements of the topological reflexive closure of the isometry group of  $\text{Lip}([0, 1])$ .

**Theorem 2.** *Every approximate local isometry  $T$  of  $\text{Lip}([0, 1])$  is a linear isometry having the form*

$$T(f)(x) = \lambda f(0) + \int_0^x \omega(t)\Phi(f')(t) dt \quad (f \in \text{Lip}([0, 1], x \in [0, 1]),$$

where  $\lambda \in \mathbb{T}$ ,  $\omega \in L^\infty_{\mathbb{T}}([0, 1])$  and  $\Phi$  is a unital algebra monomorphism of  $L^\infty([0, 1])$ .

**Proof.** Let  $T \in \text{ref}_{\text{top}}(\text{Iso}(\text{Lip}([0, 1])))$ . We establish some properties to prove the theorem.

**Property 1.** *For every  $f \in \text{Lip}([0, 1])$ , there are sequences  $\{\lambda_{f,n}\}_{n \in \mathbb{N}}$  in  $\mathbb{T}$ ,  $\{\omega_{f,n}\}_{n \in \mathbb{N}}$  in  $L^\infty_{\mathbb{T}}([0, 1])$  and  $\{\Phi_{f,n}\}_{n \in \mathbb{N}}$  in  $\text{Aut}(L^\infty([0, 1]))$  such that*

$$\lim_{n \rightarrow \infty} |\lambda_{f,n}f(0) - T(f)(0)| = 0$$

and

$$\lim_{n \rightarrow \infty} \|\omega_{f,n}\Phi_{f,n}(f') - T(f)'\|_{L^\infty} = 0.$$

Let  $f \in \text{Lip}([0, 1])$ . By hypothesis, there is a sequence  $\{T_{f,n}\}_{n \in \mathbb{N}}$  in  $\text{Iso}(\text{Lip}([0, 1]))$  such that

$$\lim_{n \rightarrow \infty} \|T_{f,n}(f) - T(f)\|_\sigma = 0.$$

This implies that

$$\lim_{n \rightarrow \infty} |T_{f,n}(f)(0) - T(f)(0)| = 0$$

and

$$\lim_{n \rightarrow \infty} \|T_{f,n}(f)' - T(f)'\|_{L^\infty} = 0.$$

By Theorem 1, for each  $n \in \mathbb{N}$ , there are a number  $\lambda_{f,n}$  in  $\mathbb{T}$  and maps  $\omega_{f,n}$  in  $L^\infty_{\mathbb{T}}([0, 1])$  and  $\Phi_{f,n}$  in  $\text{Aut}(L^\infty([0, 1]))$  such that

$$T_{f,n}(f)(x) = \lambda_{f,n}f(0) + \int_0^x \omega_{f,n}(t)\Phi_{f,n}(f')(t) dt \quad (x \in [0, 1]).$$

As  $T_{f,n}(f)(0) = \lambda_{f,n}f(0)$  and  $T_{f,n}(f)' = \omega_{f,n}\Phi_{f,n}(f')$  a.e. on  $[0, 1]$ , then Property 1 is fulfilled.

From on now, Property 1 will be frequently applied without any explicit mention along the paper.

**Property 2.** *It holds that  $\|T(f)\|_\sigma = \|f\|_\sigma$ ,  $|T(f)(0)| = |f(0)|$  and  $\|T(f)'\|_{L^\infty} = \|f'\|_{L^\infty}$  for all  $f \in \text{Lip}([0, 1])$ .*

Let  $f \in \text{Lip}([0, 1])$ . Therefore, there exists a sequence  $\{T_{f,n}\}_{n \in \mathbb{N}}$  in  $\text{Iso}(\text{Lip}([0, 1]))$  such that

$$\lim_{n \rightarrow \infty} \|T_{f,n}(f) - T(f)\|_\sigma = 0.$$

Clearly, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|T_{f,n}(f)\|_{\sigma} &= \|T(f)\|_{\sigma}, \\ \lim_{n \rightarrow \infty} |T_{f,n}(f)(0)| &= |T(f)(0)|, \\ \lim_{n \rightarrow \infty} \|T_{f,n}(f)'\|_{L^{\infty}} &= \|T(f)'\|_{L^{\infty}}. \end{aligned}$$

As  $\|T_{f,n}(f)\|_{\sigma} = \|f\|_{\sigma}$  for all  $n \in \mathbb{N}$  and, by Corollary 1,  $|T_{f,n}(f)(0)| = |f(0)|$  and  $\|T_{f,n}(f)'\|_{L^{\infty}} = \|f'\|_{L^{\infty}}$  for all  $n \in \mathbb{N}$ , we obtain the equalities of Property 2.

**Property 3.** *There exists a number  $\lambda \in \mathbb{T}$  such that  $T(f)(0) = \lambda f(0)$  for all  $f \in \text{Lip}([0, 1])$ .*

Property 2 yields  $|T(\mathbf{1})(0)| = 1$ . Take  $\lambda = T(\mathbf{1})(0)$  and define the functional  $T_0: \text{Lip}([0, 1]) \rightarrow \mathbb{C}$  by

$$T_0(f) = \overline{\lambda}T(f)(0) \quad (f \in \text{Lip}([0, 1])).$$

Clearly,  $T_0$  is linear and unital. Let us recall that  $(\text{Lip}([0, 1]), \|\cdot\|_{\sigma})$  is a complete normed algebra, but it is not a Banach algebra. To see that  $T_0$  is multiplicative, define  $S_0$  from  $\text{Lip}([0, 1])$  to  $\mathbb{C}$  by

$$S_0(f) = T(f)(0) \quad (f \in \text{Lip}([0, 1])).$$

As  $S_0$  is linear and

$$|S_0(f)| = |T(f)(0)| \leq \|T(f)\|_{\sigma} = \|f\|_{\sigma} \leq \|f\|_{\Sigma}$$

for all  $f \in \text{Lip}([0, 1])$ , then  $S_0$  is continuous on the unital complex Banach algebra  $(\text{Lip}([0, 1]), \|\cdot\|_{\Sigma})$ . Pick  $f \in \text{Lip}([0, 1])$  and take a sequence  $\{\lambda_{f,n}\}_{n \in \mathbb{N}}$  in  $\mathbb{T}$  such that

$$\lim_{n \rightarrow \infty} |\lambda_{f,n}f(0) - T(f)(0)| = 0.$$

As  $\lambda_{f,n}f(0) \in \mathbb{T}\sigma(f)$  for all  $n \in \mathbb{N}$ , it follows that

$$S_0(f) = T(f)(0) \in \overline{\mathbb{T}\sigma(f)} = \mathbb{T}\sigma(f).$$

Applying a spherical variant of the Gleason–Kahane–Żelazko theorem, stated in Proposition 2.2 of [11], we conclude that  $T_0 = \overline{S_0(\mathbf{1})}S_0$  is multiplicative. It is easy to see that if  $f \in \text{Lip}([0, 1])$  with  $\{x \in [0, 1]: f(x) = 0\} = \emptyset$ , then  $\mathbf{1}/f \in \text{Lip}([0, 1])$ . Furthermore, it is obvious that the unital Banach function algebra  $(\text{Lip}([0, 1]), \|\cdot\|_{\sigma})$  is self-adjoint. Now, from Proposition 4.1.5 (ii) in [21] we infer that the maximal ideal space of that algebra is homeomorphic to  $[0, 1]$ .

Therefore, there exists a point  $x \in [0, 1]$  such that  $T_0(f) = f(x)$  for all  $f \in \text{Lip}([0, 1])$ , and thus  $T(f)(0) = \lambda f(x)$  for all  $f \in \text{Lip}([0, 1])$ . This implies that  $x = 0$  because  $T(\iota)(0) = 0$  by Property 2. Thus,  $T(f)(0) = \lambda f(0)$  for all  $f \in \text{Lip}([0, 1])$ .

**Property 4.** *The function  $\omega := T(\iota)'$  belongs to  $L^{\infty}_{\mathbb{T}}([0, 1])$ .*

Clearly,  $\omega \in L^{\infty}([0, 1])$ . We may take sequences  $\{\lambda_{\iota,n}\}_{n \in \mathbb{N}}$  in  $\mathbb{T}$ ,  $\{\omega_{\iota,n}\}_{n \in \mathbb{N}}$  in  $L^{\infty}_{\mathbb{T}}([0, 1])$  and  $\{\Phi_{\iota,n}\}_{n \in \mathbb{N}}$  in  $\text{Aut}(L^{\infty}([0, 1]))$  satisfying that

$$\lim_{n \rightarrow \infty} \|\omega_{\iota,n}\Phi_{\iota,n}(\iota') - \omega\|_{L^{\infty}} = 0.$$

As  $\Phi_{i,n}(t') = \Phi_{i,n}(\mathbf{1}) = \mathbf{1}$  for all  $n \in \mathbb{N}$ , it follows that

$$\lim_{n \rightarrow \infty} \|\omega_{i,n} - \omega\|_{L^\infty} = 0.$$

For any  $m \in \mathcal{M}_{L^\infty}$  and  $n \in \mathbb{N}$ , we have

$$|\widehat{\omega_{i,n}}(m)| - |\widehat{\omega}(m)| \leq \|\widehat{\omega_{i,n}} - \widehat{\omega}\|_{C(\mathcal{M}_{L^\infty})} = \|\omega_{i,n} - \omega\|_{L^\infty}.$$

As  $\widehat{\omega_{i,n}}(\mathcal{M}_{L^\infty}) = \sigma(\omega_{i,n}) \subseteq \mathbb{T}$  for all  $n \in \mathbb{N}$ , we deduce that  $\widehat{\omega}(\mathcal{M}_{L^\infty}) \subseteq \mathbb{T}$  and thus  $\sigma(\omega) \subseteq \mathbb{T}$ .

**Property 5.** *There exists a unital algebra monomorphism  $\Phi$  of  $L^\infty([0, 1])$  such that*

$$T(f)(x) = \lambda f(0) + \int_0^x \omega(t)\Phi(f')(t) dt \quad (f \in \text{Lip}([0, 1], x \in [0, 1]).$$

Define the isometric linear embedding  $S$  of  $L^\infty([0, 1])$  into itself by

$$S(f') = T(f)' \quad (f \in \text{Lip}([0, 1]), \text{ a.e. on } [0, 1]).$$

For each  $f \in \text{Lip}([0, 1])$  there are sequences  $\{\omega_{f,n}\}_{n \in \mathbb{N}}$  in  $L^\infty_{\mathbb{T}}([0, 1])$  and  $\{\Phi_{f,n}\}_{n \in \mathbb{N}}$  in  $\text{Aut}(L^\infty([0, 1]))$  such that

$$\lim_{n \rightarrow \infty} \|\omega_{f,n}\Phi_{f,n}(f') - S(f')\|_{L^\infty} = 0.$$

As each  $\omega_{f,n}\Phi_{f,n} \in \text{Iso}(L^\infty([0, 1]))$  by the work in Theorem 3 of [22], we deduce that  $S$  belongs to  $\text{ref}_{\text{top}}(\text{Iso}(L^\infty([0, 1])))$ . As  $L^\infty([0, 1])$  is a uniform algebra, it follows that

$$S(f') = \omega\Phi(f') \quad (f \in \text{Lip}([0, 1]), \text{ a.e. on } [0, 1])$$

for some  $\omega \in L^\infty_{\mathbb{T}}([0, 1])$  and some unital algebra endomorphism  $\Phi$  of  $L^\infty([0, 1])$  (see [23], Theorem 5). Therefore,

$$T(f)' = \omega\Phi(f') \quad (f \in \text{Lip}([0, 1]), \text{ a.e. on } [0, 1]).$$

By Property 2,  $\Phi$  is injective. Finally, we have

$$T(f)(x) = T(f)(0) + \int_0^x \omega(t)\Phi(f')(t) dt \quad (f \in \text{Lip}([0, 1]), x \in [0, 1])$$

and Property 3 gives

$$T(f)(x) = \lambda f(0) + \int_0^x \omega(t)\Phi(f')(t) dt \quad (f \in \text{Lip}([0, 1]), x \in [0, 1]). \quad \square$$

We now prove that every local isometry of  $\text{Lip}([0, 1])$  is a surjective isometry.

**Theorem 3.** *The isometry group of  $\text{Lip}([0, 1])$  is algebraically reflexive.*

**Proof.** Let  $T \in \text{ref}_{\text{alg}}(\text{Iso}(\text{Lip}([0, 1])))$ . Clearly,  $T \in \text{ref}_{\text{top}}(\text{Iso}(\text{Lip}([0, 1])))$  and so Theorem 2 yields

$$T(f)(x) = \lambda f(0) + \int_0^x \omega(t)\Phi(f')(t) dt \quad (f \in \text{Lip}([0, 1]), x \in [0, 1]),$$

where  $\lambda \in \mathbb{T}$ ,  $\omega \in L^\infty_{\mathbb{T}}([0, 1])$  and  $\Phi$  is a unital algebra monomorphism of  $L^\infty([0, 1])$ . It follows that

$$T(f)' = \omega\Phi(f') \quad (f \in \text{Lip}([0, 1]), \text{ a.e. on } [0, 1]).$$

In view of Theorem 1, we only need to show that  $\Phi$  is surjective.

First, we prove that  $\Phi$  is weak\* continuous. By hypothesis, for each  $f \in \text{Lip}([0, 1])$ , we have

$$T(f)(x) = \lambda_f f(0) + \int_0^x \omega_f(t) \Phi_f(f')(t) dt \quad (x \in [0, 1]),$$

where  $\lambda_f \in \mathbb{T}$ ,  $\omega_f \in L^\infty_{\mathbb{T}}([0, 1])$  and  $\Phi_f \in \text{Aut}(L^\infty([0, 1]))$ . Therefore,

$$T(f)' = \omega \Phi_f(f') \quad (\text{a.e. on } [0, 1]).$$

Consider the map  $S$  of  $L^\infty([0, 1])$  into itself defined in Property 5. Hence we may write  $S(f') = \omega_f \Phi_f(f')$  a.e. on  $[0, 1]$ . As  $\omega_f \Phi_f \in \text{Iso}(L^\infty([0, 1]))$  by Theorem 3 in [22], it follows that  $S \in \text{ref}_{\text{alg}}(\text{Iso}(L^\infty([0, 1])))$ . This implies that  $S \in \text{Iso}(L^\infty([0, 1]))$  by Corollary 1 in [18]. Again, by applying Theorem 3 in [22], gives  $S(f') = \tau \Psi(f')$  for all  $f \in \text{Lip}([0, 1])$ , where  $\tau \in L^\infty_{\mathbb{T}}([0, 1])$  and  $\Psi \in \text{Aut}(L^\infty([0, 1]))$ . Finally, we have

$$\Phi(f') = \bar{\omega} T(f)' = \bar{\omega} S(f') = \bar{\omega} \tau \Psi(f') \quad (f \in \text{Lip}([0, 1]), \text{ a.e. on } [0, 1]),$$

and as  $\Psi$  is weak\* continuous, so is also  $\Phi$ .

Consider now the function  $h(x) = x^2/2$  for all  $x \in [0, 1]$ . By hypothesis, Theorem 1 assures the existence of some  $\lambda_h \in \mathbb{T}$ ,  $\omega_h \in L^\infty_{\mathbb{T}}([0, 1])$  and  $\Phi_h \in \text{Aut}(L^\infty([0, 1]))$  for which

$$T(h)(x) = \lambda_h h(0) + \int_0^x \omega_h(t) \Phi_h(h')(t) dt \quad (x \in [0, 1]).$$

Therefore, we have

$$\int_0^x \omega(t) \Phi(h')(t) dt = \int_0^x \omega_h(t) \Phi_h(h')(t) dt \quad (x \in [0, 1]).$$

As  $h' = \iota$ , it follows that  $\omega \Phi(\iota) = \omega_h \Phi_h(\iota)$  a.e. on  $[0, 1]$ . Since  $\Phi$  and  $\Phi_h$  are weak\* continuous algebra homomorphisms of  $L^\infty([0, 1])$ , Theorem 1 in [18] guarantees that there exist measurable functions  $\phi, \phi_h: [0, 1] \rightarrow [0, 1]$  such that  $\Phi(f) = f \circ \phi$  and  $\Phi_h(f) = f \circ \phi_h$  for all  $f \in L^\infty([0, 1])$ . As  $\sigma(\omega), \sigma(\omega_h) \subseteq \mathbb{T}$ , from  $\omega \phi = \omega_h \phi_h$  a.e. on  $[0, 1]$ , we infer that  $\phi = \phi_h$  a.e. on  $[0, 1]$  and thus  $\Phi = \Phi_h$ . Therefore,  $\Phi$  is surjective as required. This completes the proof.  $\square$

Another application of Theorem 2 allows us to state the algebraic reflexivity for other distinguished subsets of linear maps on  $\text{Lip}([0, 1])$  as, for example, isometric reflections and generalized bi-circular projections.

**Definition 1.** Let  $E$  be a Banach space. An isometric reflection of  $E$  is a linear isometry  $T: E \rightarrow E$  such that  $T^2 = \text{Id}$ , where  $\text{Id}$  denotes the identity operator of  $E$ . We denote by  $\text{Iso}^2(E)$  the set of all isometric reflections of  $E$ .

The next result provides a characterization of isometric reflections on  $\text{Lip}([0, 1])$ .

**Proposition 1.** A map  $T: \text{Lip}([0, 1]) \rightarrow \text{Lip}([0, 1])$  is an isometric reflection if and only if there exist a constant  $\lambda \in \{\pm 1\}$ , a function  $\omega \in L^\infty_{\mathbb{T}}([0, 1])$  and a map  $\Phi \in \text{Aut}(L^\infty([0, 1]))$  with  $\Phi(\omega) = \bar{\omega}$  almost everywhere on  $[0, 1]$  and  $\Phi^2 = \text{Id}$  such that

$$T(f)(x) = \lambda f(0) + \int_0^x \omega(t) \Phi(f')(t) dt$$

for all  $x \in [0, 1]$  and  $f \in \text{Lip}([0, 1])$ .



**Proof.** Let  $T \in \text{Iso}^2(\text{Lip}([0, 1]))$ . We have

$$T(f)(x) = \lambda f(0) + \int_0^x \omega(t)\Phi(f')(t) dt \quad (f \in \text{Lip}([0, 1]), x \in [0, 1]),$$

with  $\lambda, \omega$  and  $\Phi$  being as in Theorem 1. As  $T^2 = \text{Id}$ , we obtain the equation:

$$\begin{aligned} f(x) &= T(T(f))(x) \\ &= \lambda T(f)(0) + \int_0^x \omega(t)\Phi(T(f)')(t) dt \\ &= \lambda^2 f(0) + \int_0^x \omega(t)\Phi(\omega\Phi(f'))(t) dt \\ &= \lambda^2 f(0) + \int_0^x \omega(t)\Phi(\omega)(t)\Phi^2(f')(t) dt \quad (f \in \text{Lip}([0, 1]), x \in [0, 1]). \end{aligned}$$

Taking  $f = 1$  in the equation, we deduce that  $\lambda \in \{\pm 1\}$ . Substituting  $f = \iota$ , we obtain

$$x = \int_0^x \omega(t)\Phi(\omega)(t) dt \quad (x \in [0, 1]),$$

Therefore,  $\omega\Phi(\omega) = 1$  a.e. on  $[0, 1]$  and so  $\Phi(\omega) = \bar{\omega}$  a.e. on  $[0, 1]$ . Then, the equation can be written as

$$f(x) = f(0) + \int_0^x \Phi^2(f')(t) dt \quad (f \in \text{Lip}([0, 1]), x \in [0, 1]),$$

Thus,

$$\int_0^x f'(t) dt = f(x) - f(0) = \int_0^x \Phi^2(f')(t) dt \quad (f \in \text{Lip}([0, 1]), x \in [0, 1]),$$

and this implies that, for each  $f \in \text{Lip}([0, 1])$ ,  $\Phi^2(f') = f'$  a.e. on  $[0, 1]$ . Therefore  $\Phi^2 = \text{Id}$ , as required.

Conversely, assume that  $T$  has the form as in the statement:

$$T(f)(x) = \lambda f(0) + \int_0^x \omega(t)\Phi(f')(t) dt \quad (f \in \text{Lip}([0, 1]), x \in [0, 1]).$$

For each  $f \in \text{Lip}([0, 1])$  and all  $x \in [0, 1]$ , an easy calculation gives

$$T^2(f)(x) = \lambda^2 f(0) + \int_0^x \omega(t)\Phi(\omega)(t)\Phi^2(f')(t) dt,$$

and therefore

$$T^2(f)(x) = f(0) + \int_0^x f'(t) dt = f(x). \quad \square$$

**Corollary 2.** *The group of isometric reflections of  $\text{Lip}([0, 1])$  is algebraically reflexive.*

**Proof.** Let  $T \in \text{ref}_{\text{alg}}(\text{Iso}^2(\text{Lip}([0, 1])))$ . For each  $f \in \text{Lip}([0, 1])$ , we can take some  $T_f \in \text{Iso}^2(\text{Lip}[0, 1])$  such that  $T_f(f) = T(f)$ . Hence  $T \in \text{ref}_{\text{alg}}(\text{Iso}(\text{Lip}([0, 1])))$  and then  $T \in \text{Iso}(\text{Lip}([0, 1]))$  by Theorem 3. On a hand, by Theorem 1, there exist  $\lambda \in \mathbb{T}, \omega \in L^\infty_{\mathbb{T}}([0, 1])$  and  $\Phi \in \text{Aut}(L^\infty([0, 1]))$  such that

$$T(f)(x) = \lambda f(0) + \int_0^x \omega(t)\Phi(f')(t) dt \quad (x \in [0, 1]).$$

On the other hand, Proposition 1 guarantees the existence of  $\lambda_f \in \{\pm 1\}$ ,  $\omega_f \in L^\infty_{\mathbb{T}}([0, 1])$  and  $\Phi_f \in \text{Aut}(L^\infty([0, 1]))$  with  $\Phi_f^2 = \text{Id}$  and  $\Phi_f(\omega_f) = \overline{\omega_f}$  a.e. on  $[0, 1]$  for which

$$T_f(f)(x) = \lambda_f f(0) + \int_0^x \omega_f(t) \Phi_f(f')(t) dt \quad (x \in [0, 1]).$$

As  $T(f) = T_f(f)$ , we have the formula

$$\lambda f(0) + \int_0^x \omega(t) \Phi(f')(t) dt = \lambda_f f(0) + \int_0^x \omega_f(t) \Phi_f(f')(t) dt \quad (x \in [0, 1]).$$

First, applying the formula for  $f = 1$ , one obtains  $\lambda = \lambda_1 \in \{\pm 1\}$ . Second, putting  $f = h$ , where  $h(x) = x^2/2$  if  $x \in [0, 1]$ , we get  $\omega \Phi(\iota) = \omega_h \Phi_h(\iota)$  a.e. on  $[0, 1]$ . A reasoning similar to that of the proof of Theorem 3 yields  $\Phi = \Phi_h$ . Now,  $\omega \Phi(\iota) = \omega_h \Phi_h(\iota)$  a.e. on  $[0, 1]$  gives  $\omega = \omega_h$  a.e. on  $[0, 1]$  because  $\Phi(\iota)(x) \neq 0$  for all  $x \in [0, 1]$  (otherwise, if  $\Phi(\iota)(x_0) = 0$  for some  $x_0 \in [0, 1]$ , we have  $\Phi(p)(x_0) = 0$  if  $p$  is a polynomial function, and since polynomials are weak\* dense in  $L^\infty([0, 1])$  and  $\Phi$  is weak\* continuous, it is clear that  $\Phi(f)(x_0) = 0$  for all  $f \in L^\infty([0, 1])$ , which contradicts, for example, that  $\Phi$  is unital). Therefore,  $\Phi^2 = \Phi_h^2 = \text{Id}$  and  $\Phi(\omega) = \Phi_h(\omega_h) = \overline{\omega_h} = \overline{\omega}$  a.e. on  $[0, 1]$ . This shows that  $T \in \text{Iso}^2(\text{Lip}([0, 1]))$ .  $\square$

**Definition 2.** Let  $E$  be a Banach space. A projection of  $E$  is a map  $P: E \rightarrow E$  such that  $P^2 = P$ . A generalized bi-circular projection of  $E$  is a linear projection  $P: E \rightarrow E$  such that  $P + \tau(\text{Id} - P)$  is a linear surjective isometry for some  $\tau \in \mathbb{T} \setminus \{1\}$ . We denote by  $\text{GBP}(E)$  the set of all generalized bi-circular projections of  $E$ .

The next theorem describes this kind of projections on  $\text{Lip}([0, 1])$ .

**Proposition 2.** A map  $P: \text{Lip}([0, 1]) \rightarrow \text{Lip}([0, 1])$  is a generalized bi-circular projection if and only if there exist a number  $\lambda \in \{-1, 1\}$ , a function  $\omega \in L^\infty_{\mathbb{T}}([0, 1])$  and a map  $\Phi \in \text{Aut}(L^\infty([0, 1]))$  with  $\Phi(\omega) = \overline{\omega}$  almost everywhere on  $[0, 1]$  and  $\Phi^2 = \text{Id}$  such that

$$P(f)(x) = \frac{1}{2} \left[ f(x) + \lambda f(0) + \int_0^x \omega(t) \Phi(f')(t) dt \right] \quad (f \in \text{Lip}([0, 1]), x \in [0, 1]).$$

**Proof.** We use some arguments of the proof of [15] Theorem 3.2. If  $P \in \text{GBP}(\text{Lip}([0, 1]))$ , then  $T = P + \tau(\text{Id} - P) \in \text{Iso}(\text{Lip}([0, 1]))$  for some  $\tau \in \mathbb{T} \setminus \{1\}$ , and we obtain the equality (1):

$$P(f) = (1 - \tau)^{-1} [T(f) - \tau f] \quad (f \in \text{Lip}([0, 1])).$$

As  $P^2 = P$ , an easy calculation yields the equation

$$T^2(f) - (\tau + 1)T(f) + \tau f = 0 \quad (f \in \text{Lip}([0, 1])).$$

By Theorem 1, there are  $\lambda \in \mathbb{T}$ ,  $\omega \in L^\infty_{\mathbb{T}}([0, 1])$  and  $\Phi \in \text{Aut}(L^\infty([0, 1]))$  such that

$$T(f)(x) = \lambda f(0) + \int_0^x \omega(t) \Phi(f')(t) dt \quad (f \in \text{Lip}([0, 1]), x \in [0, 1]).$$

Using this expression of  $T$ , the above-cited equation becomes Equation (2):

$$\begin{aligned} &\lambda^2 f(0) + \int_0^x \omega(t) \Phi(\omega)(t) \Phi^2(f')(t) dt \\ &\quad - (\tau + 1) \lambda f(0) \\ &\quad - (\tau + 1) \int_0^x \omega(t) \Phi(f')(t) dt + \tau f(x) = 0 \end{aligned}$$

for all  $f \in \text{Lip}([0, 1])$  and  $x \in [0, 1]$ . Differentiating respect of  $x$ , for almost every point in  $[0, 1]$ , we obtain the Equation (3):

$$\omega\Phi(\omega)\Phi^2(g) - (\tau + 1)\omega\Phi(g) + \tau g = 0,$$

where  $g = f'$  with  $f \in \text{Lip}([0, 1])$ . Taking  $g = \iota$  and  $g = \iota^2$  in (3), we get

$$\omega\Phi(\omega)\Phi^2(\iota) - (\tau + 1)\omega\Phi(\iota) + \tau\iota = 0, \omega\Phi(\omega)\Phi^2(\iota^2) - (\tau + 1)\omega\Phi(\iota^2) + \tau\iota^2 = 0.$$

Subtracting the second equation from the first one multiplied by  $\iota$ , we have

$$\omega\Phi(\omega) \left[ \Phi^2(\iota^2) - \iota\Phi^2(\iota) \right] - (\tau + 1)\omega \left[ \Phi^2(\iota) - \iota\Phi(\iota) \right] = 0,$$

or, equivalently, the Equation (4):

$$\Phi(\omega)\Phi^2(\iota) \left[ \Phi^2(\iota) - \iota \right] - (\tau + 1)\Phi(\iota) \left[ \Phi(\iota) - \iota \right] = 0.$$

We distinguish three cases:

Case 1. If  $\Phi \neq \text{Id}$  (that is,  $\Phi(i) \neq i$ ) but  $\Phi^2 = \text{Id}$  (which implies  $\Phi(i) \neq 0$ ), we have  $\tau = -1$  by (4). Now, we deduce that  $\lambda^2 = 1$  from (2), and that  $\Phi(\omega) = \bar{\omega}$  a.e. on  $[0, 1]$  from (3) taking  $g = 1$ . Finally, from (1) we infer that

$$\begin{aligned} P(f)(x) &= \frac{1}{2} [T(f)(x) + f(x)] \\ &= \frac{1}{2} \left[ f(x) + \lambda f(0) + \int_0^x \omega(t)\Phi(f')(t) dt \right] \quad (f \in \text{Lip}([0, 1]), x \in [0, 1]), \end{aligned}$$

as stated in the theorem.

Case 2. If  $\Phi = \text{Id}$ , then the Equations (2) and (3) yield  $\lambda^2 - (\tau + 1)\lambda + \tau = 0$  and  $\omega^2 - (\tau + 1)\omega + \tau = 0$  a.e. on  $[0, 1]$ , respectively. Therefore,  $\lambda \in \{\tau, 1\}$  and  $\omega \in \{\tau\mathbf{1}, \mathbf{1}\}$  a.e. on  $[0, 1]$ .

First, if  $\lambda = 1$  and  $\omega = \mathbf{1}$  a.e. on  $[0, 1]$ , or  $\lambda = \tau$  and  $\omega = \tau\mathbf{1}$  a.e. on  $[0, 1]$ , an easy calculation shows that  $P = 0$ , which can be expressed in the form as in the statement taking  $\lambda = 1, \omega = \mathbf{1}$  a.e. on  $[0, 1]$  and  $\Phi = \text{Id}$ .

Second, if  $\lambda = 1$  and  $\omega = \tau\mathbf{1}$  a.e. on  $[0, 1]$ , we obtain  $P(f) = f(0)\mathbf{1}$  for all  $f \in \text{Lip}([0, 1])$ , which has the required form taking now  $\lambda = 1, \omega = -\mathbf{1}$  a.e. on  $[0, 1]$  and  $\Phi = \text{Id}$ .

Finally, if  $\lambda = \tau$  and  $\omega = \mathbf{1}$  a.e. on  $[0, 1]$ , one gets  $P(f) = f - f(0)\mathbf{1}$  for all  $f \in \text{Lip}([0, 1])$ , which takes the desired form for  $\lambda = -1, \omega = \mathbf{1}$  a.e. on  $[0, 1]$  and  $\Phi = \text{Id}$ .

Case 3. Assume that  $\Phi \neq \text{Id}$  and  $\Phi^2 \neq \text{Id}$ . Let us recall that  $\Phi(f) = f \circ \phi$  for all  $f \in L^\infty([0, 1])$ , where  $\phi: [0, 1] \rightarrow [0, 1]$  is a measurable function. Thus, there exists  $x_0 \in [0, 1]$  such that  $\phi(x_0) \neq x_0$  and  $\phi^2(x_0) \neq x_0$ . We can take a polynomial  $p \in L^\infty([0, 1])$  such that  $p(\phi(x_0)) = p(\phi^2(x_0)) = 0$  and  $p(x_0) = 1$ . Substituting  $g$  by  $p$  in the Equation (3) and evaluating at  $x_0$ , we obtain  $\tau = 0$ , a contradiction.

Conversely, if  $P$  has the form as in the statement of the theorem, that is, as the mean of the identity operator and an isometric reflection of  $\text{Lip}([0, 1])$ , we obtain that  $P \in \text{GBP}(\text{Lip}([0, 1]))$  with  $\tau = -1$ .  $\square$

**Corollary 3.** *The set of generalized bi-circular projections of  $\text{Lip}([0, 1])$  is algebraically reflexive.*

**Proof.** Let  $P \in \text{ref}_{\text{alg}}(\text{GBP}(\text{Lip}([0, 1])))$ . By Proposition 2, for each  $f \in \text{Lip}([0, 1])$ , there exists a  $T \in \text{Iso}^2(\text{Lip}([0, 1]))$  such that  $P(f) = (1/2)[f + T(f)]$ . Therefore, for each

$f \in \text{Lip}([0, 1])$ , we have  $2P(f) - f = T(f)$  and so  $2P - \text{Id} \in \text{ref}_{\text{alg}}(\text{Iso}^2(\text{Lip}([0, 1])))$ . Thus,  $2P - \text{Id} \in \text{Iso}^2(\text{Lip}([0, 1]))$  by Corollary 2, and therefore  $P \in \text{GBP}(\text{Lip}([0, 1]))$ .  $\square$

#### 4. Discussion

Given a Banach space  $E$ , an operator  $T \in \mathcal{B}(E)$  is a local isometry of  $E$  whenever for every  $e \in E$ , there exists a  $T_e \in \text{Iso}(E)$ , possibly depending on  $e$ , such that  $T_e(e) = T(e)$ , and  $T$  is an approximate local isometry of  $E$  if for every  $e \in E$ , there is a sequence  $\{T_{e,n}\}_{n \in \mathbb{N}}$  in  $\text{Iso}(E)$  such that  $\lim_{n \rightarrow \infty} T_{e,n}(e) = T(e)$ .

One of the main questions addressed in the studies on local isometries is for which Banach spaces  $E$ , every local isometry of  $E$  is a surjective isometry or, equivalently, which Banach spaces  $E$  have an algebraically reflexive isometry group. The topological variant of this question can also be considered, that is, when every approximate local isometry of  $E$  is a surjective isometry or, with other words, when  $\text{Iso}(E)$  is topologically reflexive.

We are interested here in the problems of algebraic and topological reflexivity for the sets of surjective linear isometries, isometric reflections and generalized bi-circular projections on the Banach spaces  $(\text{Lip}([0, 1]), \|\cdot\|_\sigma)$  and  $(\text{Lip}([0, 1]), \|\cdot\|_m)$ .

To address this type of problem, it is necessary to have convenient descriptions of the isometries on the involved spaces. However, Koshimizu [1,2] showed that the surjective linear isometries of such spaces do not have a canonical representation as a weighted composition operator but instead they admit a representation as integral operators.

Although this fact added some initial difficulty to the problem, we have been able to apply a spherical variant of the Gleason–Kahane–Żelazko theorem [11] to establish our results.

Our main theorem states that every approximate local isometry of  $\text{Lip}([0, 1])$  can be represented as a sum of an elementary weighted composition operator and an integral operator. Applying this description, we obtain some important consequences: the groups of surjective linear isometries and isometric reflections and the set of generalized bi-circular projections of  $\text{Lip}([0, 1])$  are algebraically reflexive. In the process, we give complete descriptions of such reflections and projections.

Besides, the advantage of considering approximate local maps rather than local maps is that they are more general and they allow us to state these results more easily.

The problems studied in this paper are closely related to the research on 2-local isometries between Banach spaces, which was raised by Molnár [24]. Let us recall that given a Banach space  $E$ , a set  $\mathcal{S} \subseteq \mathcal{B}(E)$  is called 2-algebraically reflexive if  $2\text{-ref}_{\text{alg}}(\mathcal{S}) = \mathcal{S}$ , where

$$2\text{-ref}_{\text{alg}}(\mathcal{S}) = \left\{ \Delta \in E^E : \forall e, u \in E, \exists S_{e,u} \in \mathcal{S} \mid S_{e,u}(e) = \Delta(e), S_{e,u}(u) = \Delta(u) \right\}.$$

In the case  $\mathcal{S} = \text{Iso}(E)$ , the members of  $2\text{-ref}_{\text{alg}}(\mathcal{S})$  are called 2-local isometries. A complete information on 2-local maps can also be found in [7].

The study of 2-local isometries on Lipschitz spaces with the  $\Sigma$ -norm or the  $M$ -norm was considered in [11,13,25]. The 2-locality problem for surjective isometries on  $\text{Lip}([0, 1])$ , without assuming linearity, has been dealt in [26].

It would be interesting to study the 2-algebraic reflexivity of the sets of surjective linear isometries, isometric reflections and generalized bi-circular projections on  $\text{Lip}([0, 1])$ , equipped with the  $\sigma$ -norm or the  $m$ -norm.

We believe that results of this type have strong potential for further applications, fitting into a quickly growing area of international research.

**Author Contributions:** Investigation, A.J.-V. and M.I.R. All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was partially supported by Junta de Andalucía grant FQM194.

**Conflicts of Interest:** The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

## References

1. Koshimizu, H. Linear isometries on spaces of continuously differentiable and Lipschitz continuous functions. *Nihonkai Math. J.* **2011**, *22*, 73–90.
2. Koshimizu, H. Finite codimensional linear isometries on spaces of differentiable and Lipschitz functions. *Cent. Eur. J. Math.* **2011**, *9*, 139–146.
3. Hatori, O. Hermitian Operators and Isometries on Banach Algebras of Continuous Maps with Values in Unital Commutative  $C^*$ -Algebras. *J. Funct. Spaces* **2018**, *2018*, 8085304.
4. Jarosz, K.; Pathak, V. Isometries between function spaces. *Trans. Am. Math. Soc.* **1988**, *305*, 193–206.
5. Rao, N.V.; Roy, A.K. Linear isometries of some function spaces. *Pac. J. Math.* **1971**, *38*, 177–192.
6. Roy, A.K. Extreme points and linear isometries of the Banach space of Lipschitz functions. *Canad. J. Math.* **1968**, *20*, 1150–1164.
7. Molnár, L. *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces*; Lecture Notes in Mathematics, 1895; Springer: Berlin, Germany, 2007.
8. Botelho, F.; Jamison, J. Algebraic and topological reflexivity of spaces of Lipschitz functions. *Rev. Roum. Math. Pures Appl.* **2011**, *56*, 105–114.
9. Botelho, F.; Jamison, J. Algebraic reflexivity of sets of bounded operators on vector valued Lipschitz functions. *Linear Algebra Appl.* **2010**, *432*, 3337–3342.
10. Jiménez-Vargas, A.; Morales-Campoy, A.; Villegas-Vallecillos, M. Algebraic reflexivity of the isometry group of some spaces of Lipschitz functions. *J. Math. Anal. Appl.* **2010**, *366*, 195–201.
11. Li, L.; Peralta, A.M.; Wang, L.; Wang, Y.-S. Weak-2-local isometries on uniform algebras and Lipschitz algebras. *Publ. Mater.* **2019**, *63*, 241–264.
12. Oi, S. Algebraic reflexivity of isometry groups of algebras of Lipschitz maps. *Linear Algebra Appl.* **2019**, *566*, 167–182.
13. Oi, S. A spherical version of the Kowalski–Słodkowski theorem and its applications. *J. Aust. Math. Soc.* **2020**, 1–26, doi:10.1017/S1446788720000452.
14. Fösner, M.; Ilisević, D.; Li, C. G-invariant norms and bicircular projections. *Linear Algebra Appl.* **2007**, *420*, 596–608.
15. Botelho, F.; Jamison, J. Generalized bi-circular projections on spaces of analytic functions. *Acta Sci. Math.* **2009**, *75*, 527–546.
16. Weaver, N. *Lipschitz Algebras*, 2nd ed.; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA, 2018.
17. Kaniuth, E. *A Course in Commutative Banach Algebras*; Graduate Texts in Mathematics, 246; Springer: New York, NY, USA, 2009.
18. Cabello Sánchez, F. The group of automorphisms of  $L_\infty$  is algebraically reflexive. *Stud. Math.* **2004**, *161*, 19–32.
19. Batty, C.J.K.; Molnár, L. On topological reflexivity of the groups of  $*$ -automorphisms and surjective isometries of  $B(H)$ . *Arch. Math.* **1996**, *67*, 415–421.
20. Font, J.J. On weighted composition operators between spaces of measurable functions. *Quaest. Math.* **1999**, *22*, 143–148.
21. Dales, H.G. *Banach Algebras and Automatic Continuity*; London Mathematical Society Monographs. New Series; Clarendon Press: Oxford, UK, 2000; Volume 24.
22. de Leeuw, K.; Rudin, W.; Wermer, J. The isometries of some function spaces. *Proc. Am. Math. Soc.* **1960**, *11*, 694–698.
23. Sánchez, F.C.; Molnár, L. Reflexivity of the isometry group of some classical spaces. *Rev. Mat. Iberoam.* **2002**, *18*, 409–430.
24. Molnár, L. Some characterizations of the automorphisms of  $B(H)$  and  $C(X)$ . *Proc. Am. Math. Soc.* **2002**, *130*, 111–120.
25. Jiménez-Vargas, A.; Villegas-Vallecillos, M. 2-local isometries on spaces of Lipschitz functions. *Canad. Math. Bull.* **2011**, *54*, 680–692.
26. Hatori, O.; Oi, S. 2-local isometries on functions spaces. In *Recent Trends in Operator Theory and Applications*; Contemp. Math., 737; American Mathematical Society: Providence, RI, USA, 2019; pp. 89–106.