Research Article

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(p, q)-Compactness in spaces of holomorphic mappings

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Abstract: Based on the concept of (p, q)-compact operator for $p \in [1, \infty]$ and $q \in [1, p^*]$, we introduce and study the notion of (p, q)-compact holomorphic mapping between Banach spaces. We prove that the space formed by such mappings is a surjective $pq/(p + q)$-Banach bounded-holomorphic ideal that can be generated by composition with the ideal of (p, q)-compact operators. In addition, we study Mujica's linearization of such mappings, its relation with the $(u^*v^* + tv^* + tu^*)/tu^*v^*$-Banach bounded-holomorphic composition ideal of the (t, u, v)-nuclear holomorphic mappings for $t, u, v \in [1, \infty]$, its holomorphic transposition via the injective hull of the ideal of $(p, q^*, 1)$-nuclear operators, the Möbius invariance of (p, q)-compact holomorphic mappings on $\mathbb{D}$, and its full compact factorization through a compact holomorphic mapping, a (p, q)-compact operator, and a compact operator.

Keywords: vector-valued holomorphic function, linearization, factorization theorems, (p, q)-compact operator, (p, q)-compact holomorphic mapping, (t, u, v)-nuclear holomorphic mapping

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1 Introduction

Let E and F be Banach spaces over $\mathbb{K}$. As usual, B_E stands for the closed unit ball of E, E^* for the dual space of E, and $\mathcal{L}(E, F)$ for the Banach space of all bounded linear operators from E into F, with the operator canonical norm. For a set $A \subseteq E$, $\mathrm{lin}(A)$ and $\overline{\mathrm{abco}}(A)$ represent the linear hull and the norm-closed absolutely convex hull of A in E, respectively.

Given $p \in [1, \infty)$, $\ell_p(E)$ denotes the Banach space of all absolutely p-summable sequences (x_n) in E endowed with the norm:

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p},$$

and $c_0(E)$ is the Banach space of all sequences in E converging to zero equipped with the norm:

$$\|(x_n)\|_\infty = \sup\{\|x_n\| : n \in \mathbb{N}\}.$$

In the case of sequences in $\mathbb{K}$, we will just write ℓ_p and c_0.

Let $p \in [1, \infty]$ and let p^* denote the *conjugate index of* p defined by $p^* = p/(p - 1)$ if $p \neq 1$, $p^* = \infty$ if $p = 1$, and $p^* = 1$ if $p = \infty$.

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For each $p \in (1, \infty)$, the *p-convex hull* of a sequence $(x_n) \in \ell_p(E)$ is defined by

$$p\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_{p^*}} \right\}.$$

Moreover, the 1-*convex hull* of a sequence $(x_n) \in \ell_1(E)$ is given by

$$1\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{c_0} \right\},$$

and the ∞-*convex hull* of a sequence $(x_n) \in c_0(E)$ by

$$\infty\text{-conv}(x_n) = \left\{ \sum_{n=1}^{\infty} a_n x_n : (a_n) \in B_{\ell_1} \right\}.$$

According to Grothendieck's criterion of compactness [1], a set $K \subseteq E$ is relatively compact if and only if there exists a sequence $(x_n) \in c_0(E)$ such that $K \subseteq \overline{\text{abco}}(\{x_n : n \in \mathbb{N}\})$. Note that $\overline{\text{abco}}(\{x_n : n \in \mathbb{N}\})$ is compact and coincides with $\infty\text{-conv}(x_n)$.

Motivated by this result, a stronger property of compactness in Banach spaces was introduced by Sinha and Karn [2] in 2002. Namely, given $p \in [1, \infty]$, a set $K \subseteq E$ is said to be *relatively p-compact* if there is a sequence $(x_n) \in \ell_p(E)$ (or $(x_n) \in c_0(E)$ if $p = \infty$) such that $K \subseteq p\text{-conv}(x_n)$. Lassalle and Turco [3] provided a *measure of the p-compactness of K* by defining

$$m_p(K) = \begin{cases} \inf\{\|(x_n)\|_p : (x_n) \in \ell_p(E), K \subseteq p\text{-conv}(x_n)\}, & \text{if } 1 \le p < \infty, \\ \inf\{\|(x_n)\|_\infty : (x_n) \in c_0(E), K \subseteq \infty\text{-conv}(x_n)\}, & \text{if } p = \infty. \end{cases}$$

An operator $T \in \mathcal{L}(E, F)$ is said to be *p-compact* if $T(B_E)$ is a relatively p-compact subset of F. Theorem 4.2 in [2] states that the class of p-compact operators between Banach spaces $\mathcal{K}_p$ is a Banach operator ideal equipped with the norm k_p, where $k_p(T) = m_p(T(B_E))$ for all $T \in \mathcal{K}_p(E, F)$.

The study of holomorphic mappings between Banach spaces with relatively compact range was initiated by Mujica [4] and continued by Sepulcre and the two authors of this work in [5]. The investigation on the p-compactness in the polynomial and holomorphic settings was addressed by Aron et al. [6], Aron et al. [7], and Jiménez-Vargas [8].

Influenced by these concepts, Ain et al. [9] introduced a more general property of compactness in 2012. Precisely, given $p \in [1, \infty]$ and $q \in [1, p^*]$, a set $K \subseteq E$ is said to be *relatively (p,q)-compact* if there exists a sequence $(x_n) \in \ell_p(E)$ ($(x_n) \in c_0(E)$ if $p = \infty$) such that $K \subseteq q^*\text{-conv}(x_n)$. As in the p-compact case, a *measure of the (p,q)-compactness of K* could be defined as

$$m_{(p,q)}(K) = \begin{cases} \inf\{\|(x_n)\|_p : (x_n) \in \ell_p(E), K \subseteq q^*\text{-conv}(x_n)\}, & \text{if } 1 \le p < \infty,\ 1 \le q \le p^*, \\ \inf\{\|(x_n)\|_\infty : (x_n) \in c_0(E), K \subseteq \infty\text{-conv}(x_n)\}, & \text{if } p = \infty,\ q = 1. \end{cases}$$

Note that the $(\infty, 1)$-compactness coincides with the compactness in view of Grothendieck's criterion, while the (p, p^*)-compactness is precisely the p-compactness of Sinha and Karn [2].

In an analogous form, an operator $T \in \mathcal{L}(E, F)$ is said to be (p, q)-*compact* if $T(B_E)$ is a relatively (p, q)-compact subset of F. Proposition 2.1 in [9] shows that the class of (p, q)-compact operators between Banach spaces, denoted $\mathcal{K}_{(p,q)}$, is an operator ideal. Furthermore, given $s = pq/(p + q)$, $[\mathcal{K}_{(p,q)}, k_{(p,q)}]$ is an s-Banach operator ideal endowed with the s-norm $k_{(p,q)}(T) = m_{(p,q)}(T(B_E))$ for all $T \in \mathcal{K}_{(p,q)}(E, F)$, which becomes a Banach operator ideal if and only if $q = p^*$.

Since then, (p, q)-compact sets and (p, q)-compact operators between Banach spaces have been studied by various authors. For example, Ain and Oja [10,11] characterized relatively (p, q)-compact sets and studied (p, q)-null sequences, while Kim [12,13] investigated the injective and surjective hulls and an approximation property of the ideal of (p, q)-compact operators.

Our aim in this study is now to extend the property of (p, q)-compactness to the holomorphic setting as follows.

Let U be an open subset of a complex Banach space E and let F be a complex Banach space, $p \in [1, \infty]$ and $q \in [1, p^*]$. Let $\mathcal{H}(U, F)$ denote the linear space of all holomorphic mappings from U to F. A mapping $f \in \mathcal{H}(U, F)$ is said to be (p, q)-*compact* if $f(U)$ is a relatively (p, q)-compact subset of F. If $\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$ denotes the space of all (p, q)-compact holomorphic mappings from U into F, we define $k^{\mathcal{H}^\infty}_{(p,q)}(f) = m_{(p,q)}(f(U))$ for every $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$.

This article contains a complete study on (p, q)-compact holomorphic mappings. We now describe the content of this article. Let us recall that the space of all bounded holomorphic mappings from U into F, denoted $\mathcal{H}^\infty(U, F)$, is a Banach space endowed with the supremum norm. If $\mathcal{G}^\infty(U)$ is the canonical predual space of $\mathcal{H}^\infty(U) \coloneqq \mathcal{H}^\infty(U, \mathbb{C})$ obtained by Mujica in [4], we prove that a mapping $f \in \mathcal{H}^\infty(U, F)$ is (p, q)-compact if and only if its linearization $T_f : \mathcal{G}^\infty(U) \to F$ is a (p, q)-compact operator. This fact allows us to extend to the holomorphic setting some known results on (p, q)-compact operators. For instance, we prove that every mapping $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$ admits a factorization in the form $f = T \circ g$, where G is a complex Banach space, $g \in \mathcal{H}^\infty(U, G)$, and T is a (p, q)-compact operator from G into F. Furthermore, $k^{\mathcal{H}^\infty}_{(p,q)}(f) = \inf\{k_{(p,q)}(T)\|g\|_\infty\}$, where the infimum is taken over all factorization of f as above.

We also prove that $[\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}, k^{\mathcal{H}^\infty}_{(p,q)}]$ is a surjective s-Banach bounded-holomorphic ideal, where $s = pq/(p + q)$. This fact extends some results stated in [5,8,14] on spaces of holomorphic mappings with relatively compact or relatively p-compact range.

The notion of (t, u, v)-nuclear holomorphic mapping for $t, u, v \in [1, \infty]$ is introduced, and it is proved that the space formed by such mappings is an s-Banach bounded-holomorphic ideal, where $s = 1/t + 1/u^* + 1/v^*$. This allows us to ensure that every $(p, 1, q^*)$-nuclear holomorphic mapping is (p, q)-compact holomorphic. We provide three characterizations for (p, q)-compact holomorphic mappings: (1) as those bounded holomorphic mappings whose Mujica's linearization is a (p, q)-compact operator; (2) as those that can be generated by composition with the ideal of (p, q)-compact operators; and (3) as those for which its holomorphic transpose belongs to the injective hull of the ideal of $(p, q^*, 1)$-nuclear operators.

The Möbius invariance of the class of (p, q)-holomorphic mappings defined on $\mathbb{D}$ (the complex open unit disc) is also addressed. Finally, we establish a general result of factorization for (p, q)-compact operators between Banach spaces, which extends some known results and permits us to characterize the members of $\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$ as those bounded holomorphic mappings that admit a full compact factorization through a compact holomorphic mapping, a (p, q)-compact operator, and a compact operator.

2 Results

We will start by collecting some results on holomorphic mappings due to Mujica [4] that will be applied throughout this article.

Theorem 2.1. [4] *Let U be an open subset of a complex Banach space E. Let $\mathcal{G}^\infty(U)$ denote the norm-closed linear subspace of $\mathcal{H}^\infty(U)^*$ generated by the functionals $\delta(x) \in \mathcal{H}^\infty(U)^*$ with $x \in U$, defined by $\delta(x)(f) = f(x)$ for all $f \in \mathcal{H}^\infty(U)$.*

(*i*) *The mapping $g_U : U \to \mathcal{G}^\infty(U)$, defined by $g_U(x) = \delta(x)$ for all $x \in U$, is holomorphic with $\|\delta(x)\| = 1$ for all $x \in U$.*

(*ii*) *For every complex Banach space F and every mapping $f \in \mathcal{H}^\infty(U, F)$, there exists a unique operator $T_f \in \mathcal{L}(\mathcal{G}^\infty(U), F)$ such that $T_f \circ g_U = f$. Furthermore, $\|T_f\| = \|f\|_\infty$.*

(*iii*) *For every complex Banach space F, the mapping $f \mapsto T_f$ is an isometric isomorphism from $\mathcal{H}^\infty(U, F)$ onto $\mathcal{L}(\mathcal{G}^\infty(U), F)$.*

(*iv*) *$\mathcal{H}^\infty(U)$ is isometrically isomorphic to $\mathcal{G}^\infty(U)^*$, via $J_U : \mathcal{H}^\infty(U) \to \mathcal{G}^\infty(U)^*$ given by $J_U(f)(g_U(x)) = f(x)$ for all $f \in \mathcal{H}^\infty(U)$ and $x \in U$.*

(*v*) *$B_{\mathcal{G}^\infty(U)}$ coincides with $\overline{\mathrm{abco}}(g_U(U))$.*

From now on, unless otherwise, E will denote a complex Banach space, U an open subset of E, and F a complex Banach space. The subspaces of $\mathcal{L}(E,F)$ formed by compact operators, finite-rank bounded operators, and approximable operators from E into F will be denoted by $\mathcal{K}(E,F)$, $\mathcal{F}(E,F)$, and $\overline{\mathcal{F}}(E,F)$, respectively.

In view of the following result, we will only focus on the case $(p,q)\in[1,\infty)\times[1,p^*)$. We must point out that the Banach space of all holomorphic mappings with relatively compact range from U into F, denoted $\mathcal{H}^\infty_{\mathcal{K}}(U,F)$, equipped with the supremum norm $\|\cdot\|_\infty$ was studied in [4,5,14], and that the Banach space of all holomorphic mappings with relatively p-compact range from U into F for $p\in[1,\infty]$, denoted $\mathcal{H}^\infty_{\mathcal{K}_p}(U,F)$, endowed with the norm $k_p^{\mathcal{H}^\infty}$ was dealt in [8].

Proposition 2.2. *The following statements are satisfied*:
(*i*) $\mathcal{H}^\infty_{\mathcal{K}_{(\infty,1)}}(U,F)=\mathcal{H}^\infty_{\mathcal{K}}(U,F)$ *and* $k^{\mathcal{H}^\infty}_{(\infty,1)}(f)=\|f\|_\infty$, *for all* $f\in\mathcal{H}^\infty_{\mathcal{K}_{(\infty,1)}}(U,F)$.
(*ii*) *For* $p\in[1,\infty)$, $\mathcal{H}^\infty_{\mathcal{K}_{(p,p^*)}}(U,F)=\mathcal{H}^\infty_{\mathcal{K}_p}(U,F)$, *and* $k^{\mathcal{H}^\infty}_{(p,p^*)}(f)=k_p^{\mathcal{H}^\infty}(f)$, *for all* $f\in\mathcal{H}^\infty_{\mathcal{K}_{(p,p^*)}}(U,F)$.

Proof. (*i*) Let $f\in\mathcal{H}^\infty_{\mathcal{K}_{(\infty,1)}}(U,F)$ and let $(x_n)\in c_0(F)$ be such that $f(U)\subseteq\infty\text{-conv}(x_n)$. Note that $\infty\text{-conv}(x_n)$ is compact in F, and then, $f(U)$ is relatively compact in F. Thus, $f\in\mathcal{H}^\infty_{\mathcal{K}}(U,F)$ with $\|f\|_\infty\le\|(x_n)\|_\infty$, and taking the infimum over all such sequences (x_n), we have $\|f\|_\infty\le m_{(\infty,1)}(f(U))=k^{\mathcal{H}^\infty}_{(\infty,1)}(f)$.

Conversely, let $f\in\mathcal{H}^\infty_{\mathcal{K}}(U,F)$. Then, $f(U)$ is a relatively compact subset of F and, by the Grothendieck's criterion of compactness, for every $\varepsilon>0$, there is a sequence $(x_n)\in c_0(F)$ such that $f(U)\subseteq\infty\text{-conv}(x_n)$ with $\|(x_n)\|_\infty\le\|f\|_\infty+\varepsilon$. Hence, $f\in\mathcal{H}^\infty_{\mathcal{K}_{(\infty,1)}}(U,F)$ with $k^{\mathcal{H}^\infty}_{(\infty,1)}(f)\le\|f\|_\infty$.

(*ii*) Let $f\in\mathcal{H}^\infty_{\mathcal{K}_{(p,p^*)}}(U,F)$ and let $(x_n)\in\ell_p(F)$ be such that $f(U)\subseteq p\text{-conv}(x_n)$. Hence, $f(U)$ is a relatively p-compact subset of F, and thus, $f\in\mathcal{H}^\infty_{\mathcal{K}_p}(U,F)$.

Conversely, let $f\in\mathcal{H}^\infty_{\mathcal{K}_p}(U,F)$, i.e., $f(U)$ is a relatively p-compact subset of F. Hence, there exists a sequence $(x_n)\in\ell_p(F)$ such that $f(U)\subseteq p\text{-conv}(x_n)$. This allows us to assure that $f\in\mathcal{H}^\infty_{\mathcal{K}_{(p,p^*)}}(U,F)$.

Finally, note that $k_p^{\mathcal{H}^\infty}(f)=m_p(f(U))=m_{(p,p^*)}(f(U))=k^{\mathcal{H}^\infty}_{(p,p^*)}(f)$. □

2.1 Linearization and factorization

We next characterize (p,q)-compact holomorphic mappings $f:U\to F$ in terms of the (p,q)-compactness of its linearization $T_f\in\mathcal{L}(\mathcal{G}^\infty(U),F)$.

Theorem 2.3. *Let* $p\in[1,\infty)$, $q\in[1,p^*)$, *and* $f\in\mathcal{H}^\infty(U,F)$. *The following conditions are equivalent*:
(*i*) $f:U\to F$ *is a* (p,q)*-compact holomorphic mapping.*
(*ii*) $T_f:\mathcal{G}^\infty(U)\to F$ *is a* (p,q)*-compact operator.*

In this case, $k^{\mathcal{H}^\infty}_{(p,q)}(f)=k_{(p,q)}(T_f)$. *Furthermore, the mapping* $f\mapsto T_f$ *is an isometric isomorphism from* $(\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U,F),k^{\mathcal{H}^\infty}_{(p,q)})$ *onto* $(\mathcal{K}_{(p,q)}(\mathcal{G}^\infty(U),F),k_{(p,q)})$.

Proof. Applying Theorem 2.1, we obtain the following inclusions:

$$\begin{aligned}f(U)=T_f(g_U(U))&\subseteq T_f(\overline{\operatorname{abco}}(g_U(U)))=T_f(B_{\mathcal{G}^\infty(U)})\\&\subseteq\overline{\operatorname{abco}}(T_f(g_U(U)))=\overline{\operatorname{abco}}(f(U)).\end{aligned}$$

(*i*) ⇒ (*ii*): Suppose that $f\in\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U,F)$, and thus, $f(U)$ is a relatively (p,q)-compact set in F. In light of [15, p. 1094], $\overline{\operatorname{abco}}(f(U))$ is also relatively (p,q)-compact and $m_{(p,q)}(f(U))=m_{(p,q)}(\overline{\operatorname{abco}}(f(U)))$. Hence, there exists a sequence $(x_n)\in\ell_p(E)$ so that $\overline{\operatorname{abco}}(f(U))\subseteq q^*\text{-conv}(x_n)$. It follows that $T_f(B_{\mathcal{G}^\infty(U)})\subseteq q^*\text{-conv}(x_n)$ by the second inclusion above. Hence, $T_f\in\mathcal{K}_{(p,q)}(\mathcal{G}^\infty(U),F)$ with

$$k_{(p,q)}(T_f)=m_{(p,q)}(T_f(B_{\mathcal{G}^\infty(U)}))\le m_{(p,q)}(\overline{\operatorname{abco}}(f(U)))=m_{(p,q)}(f(U))=k^{\mathcal{H}^\infty}_{(p,q)}(f).$$

(*ii*) ⇒ (*i*): Assume that $T_f \in \mathcal{K}_{(p,q)}(\mathcal{G}^\infty(U), F)$. Then, $T_f(B_{\mathcal{G}^\infty(U)})$ is relatively (p,q)-compact in F. By using the first inclusion above, $f(U)$ is also relatively (p,q)-compact, i.e., $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$, with

$$k^{\mathcal{H}^\infty}_{(p,q)}(f) = m_{(p,q)}(f(U)) \leq m_{(p,q)}(T_f(B_{\mathcal{G}^\infty(U)})) = k_{(p,q)}(T_f).$$

Combining assertion (iii) in Theorem 2.1 and what was proved above, we can ensure the last assertion of the result. □

Let $1 \leq p \leq r \leq \infty$, $1 \leq q \leq p^*$, and $1 \leq s \leq r^*$. Assume that $s \leq q$ with

$$\frac{1}{r} + \frac{1}{s} \leq \frac{1}{p} + \frac{1}{q}.$$

Then, Corollary 3.3 in [9] states that $\mathcal{K}_{(p,q)}(\mathcal{G}^\infty(U), F) \subseteq \mathcal{K}_{(r,s)}(\mathcal{G}^\infty(U), F)$ with $k_{(r,s)}(f) \leq k_{(p,q)}(f)$ for all $f \in \mathcal{K}_{(p,q)}(\mathcal{G}^\infty(U), F)$.

Taking into account this fact, Theorem 2.3 yields the following result on inclusions between $\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}$-spaces.

Corollary 2.4. *Let p, q, r, s be satisfying the same conditions as above. Then, $\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F) \subseteq \mathcal{H}^\infty_{\mathcal{K}_{(r,s)}}(U, F)$ with $k^{\mathcal{H}^\infty}_{(r,s)}(f) \leq k^{\mathcal{H}^\infty}_{(p,q)}(f)$ for all $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$.*

As a consequence, we deduce [8, Corollary 1.10]: if $1 \leq p \leq q \leq \infty$, then $\mathcal{H}^\infty_{\mathcal{K}_p}(U, F) \subseteq \mathcal{H}^\infty_{\mathcal{K}_q}(U, F)$ with $k^{\mathcal{H}^\infty}_q(f) \leq k^{\mathcal{H}^\infty}_p(f)$ for all $f \in \mathcal{H}^\infty_{\mathcal{K}_p}(U, F)$.

The combination of Theorem 2.3 and [14, Theorem 2.4] gives the following result about the factorization of (p,q)-compact holomorphic mappings.

Corollary 2.5. *Let $p \in [1, \infty)$, $q \in [1, p^*)$, and $f \in \mathcal{H}^\infty(U, F)$. The following are equivalent:*
(*i*) *$f: U \to F$ is (p,q)-compact holomorphic.*
(*ii*) *$f = T \circ g$, for some complex Banach space G, $g \in \mathcal{H}^\infty(U, G)$ and $T \in \mathcal{K}_{(p,q)}(G, F)$.*

In this case, we have

$$k^{\mathcal{H}^\infty}_{(p,q)}(f) = \|f\|_{\mathcal{K}_{(p,q)} \circ \mathcal{H}^\infty} := \inf\{k_{(p,q)}(T)\|g\|_\infty\},$$

where this infimum (in fact attained at $T_f \circ g_U$) is taken over all factorizations of f as above. Furthermore, the correspondence $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F), \|\cdot\|_{\mathcal{K}_{(p,q)} \circ \mathcal{H}^\infty})$ onto $(\mathcal{K}_{(p,q)}(\mathcal{G}^\infty(U), F), k_{(p,q)})$.

Let us recall (see [4, p. 72]) that a mapping $f \in \mathcal{H}^\infty(U, F)$ has finite-rank if $\mathrm{lin}(f(U))$ is a finite dimensional subspace of F. The set of all finite-rank bounded holomorphic mappings from U into F is denoted by $\mathcal{H}^\infty_{\mathcal{F}}(U, F)$. In view of [5, Theorem 2.1] and Theorem 2.3, we can assure that $\mathcal{H}^\infty_{\mathcal{F}}(U, F)$ is a linear subspace of $\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$. We next introduce a greater class of holomorphic mappings.

Definition 2.6. Let $p \in [1, \infty)$ and $q \in [1, p^*)$. A mapping $f \in \mathcal{H}^\infty(U, F)$ is said to be (p,q)-*approximable* if there exists a sequence (f_n) in $\mathcal{H}^\infty_{\mathcal{F}}(U, F)$ such that $k^{\mathcal{H}^\infty}_{(p,q)}(f_n - f) \overset{n\to\infty}{\to} 0$. We denote by $\mathcal{H}^\infty_{\overline{\mathcal{F}}_{(p,q)}}(U, F)$ the space of all (p,q)-approximable bounded holomorphic maps from U to F.

Corollary 2.7. *Let $p \in [1, \infty)$ and $q \in [1, p^*)$. Every (p,q)-approximable holomorphic mapping is (p,q)-compact holomorphic.*

Proof. Let $f \in \mathcal{H}^\infty_{\overline{\mathcal{F}}_{(p,q)}}(U, F)$. Then, there is a sequence (f_n) in $\mathcal{H}^\infty_{\mathcal{F}}(U, F)$ such that $k^{\mathcal{H}^\infty}_{(p,q)}(f_n - f) \to 0$ as $n \to \infty$. Since $T_{f_n} \in \mathcal{F}(\mathcal{G}^\infty(U), F)$ by [5, Theorem 2.1], $\mathcal{F}(\mathcal{G}^\infty(U), F) \subseteq \mathcal{K}_{(p,q)}(\mathcal{G}^\infty(U), F)$ by [9, Proposition 2.1] and $k_{(p,q)}(T_{f_n} - T_f) = k_{(p,q)}(T_{f_n - f}) = k^{\mathcal{H}^\infty}_{(p,q)}(f_n - f)$ for all $n \in \mathbb{N}$ by Theorems 2.1 and 2.3, we obtain that $T_f \in \mathcal{K}_{(p,q)}(\mathcal{G}^\infty(U), F)$ because $[\mathcal{K}_{(p,q)}, k_{(p,q)}]$ is an s-Banach operator ideal [9, Proposition 2.1 and p. 151]. We conclude that $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$ by Theorem 2.3. □

2.2 s-Banach ideal property

By [14, Definition 2.1], a *bounded-holomorphic ideal* is a subclass $\mathcal{I}^{\mathcal{H}^\infty}$ of the class $\mathcal{H}^\infty$ of all bounded holomorphic mappings such that for any complex Banach space E, any open subset U of E and any complex Banach space F, the components

$$\mathcal{I}^{\mathcal{H}^\infty}(U,F) \coloneqq \mathcal{I}^{\mathcal{H}^\infty} \cap \mathcal{H}^\infty(U,F)$$

satisfy the following three properties:

(I1) $\mathcal{I}^{\mathcal{H}^\infty}(U,F)$ is a linear subspace of $\mathcal{H}^\infty(U,F)$.

(I2) For any $g \in \mathcal{H}^\infty(U)$ and $y \in F$, the map $g \cdot y : x \mapsto g(x)y$ from U into F is in $\mathcal{I}^{\mathcal{H}^\infty}(U,F)$.

(I3) *The ideal property*: If H, G are complex Banach spaces, V is an open subset of H, $h \in \mathcal{H}(V,U)$, $f \in \mathcal{I}^{\mathcal{H}^\infty}(U,F)$, and $S \in \mathcal{L}(F,G)$, then $S \circ f \circ h \in \mathcal{I}^{\mathcal{H}^\infty}(V,G)$.

Given $s \in (0,1]$, let us recall that an *s-norm* on a linear space X over $\mathbb{K}$ is a function $f : X \to \mathbb{R}$ satisfying that $x = 0$ whenever $f(x) = 0$, $f(\lambda x) = |\lambda| f(x)$ for all $\lambda \in \mathbb{K}$ and $x \in X$, and $f(x+y)^s \leq f(x)^s + f(y)^s$ for all $x, y \in X$. We say that (X, f) is an *s-normed space*, and it is said that (X, f) is an *s-Banach space* if every Cauchy sequence in (X, f) converges in (X, f).

Inspired by the notion of s-Banach operator ideal introduced by Pietsch [16, 6.2.2], we present the holomorphic analogue that extends the concept of (Banach) normed bounded-holomorphic ideal stated in [14, Definition 2.1]. A 1-Banach bounded-holomorphic ideal is simply a Banach bounded-holomorphic ideal.

Definition 2.8. Let $s \in (0,1]$. A bounded-holomorphic ideal $\mathcal{I}^{\mathcal{H}^\infty}$ is said to be s-normed (s-Banach) if there exists a function $\|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty}} : \mathcal{I}^{\mathcal{H}^\infty} \to \mathbb{R}_0^+$ such that for every complex Banach space E, every open subset U of E, and every complex Banach space F, the following three conditions are satisfied:

(N1) $(\mathcal{I}^{\mathcal{H}^\infty}(U,F), \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty}})$ is an s-normed (s-Banach) space with $\|f\|_\infty \leq \|f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$ for all $f \in \mathcal{I}^{\mathcal{H}^\infty}(U,F)$.

(N2) $\|g \cdot y\|_{\mathcal{I}^{\mathcal{H}^\infty}} = \|g\|_\infty \|y\|$ for all $g \in \mathcal{H}^\infty(U)$ and $y \in F$.

(N3) If H, G are complex Banach spaces, V is an open subset of H, $h \in \mathcal{H}(V,U)$, $f \in \mathcal{I}^{\mathcal{H}^\infty}(U,F)$, and $S \in \mathcal{L}(F,G)$, then $\|S \circ f \circ h\|_{\mathcal{I}^{\mathcal{H}^\infty}} \leq \|S\| \|f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$.

An s-normed bounded-holomorphic ideal $[\mathcal{I}^{\mathcal{H}^\infty}, \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty}}]$ is said to be:

(R) regular if for any $f \in \mathcal{H}^\infty(U,F)$, we have that $f \in \mathcal{I}^{\mathcal{H}^\infty}(U,F)$ with $\|f\|_{\mathcal{I}^{\mathcal{H}^\infty}} = \|\kappa_F \circ f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$ whenever $\kappa_F \circ f \in \mathcal{I}^{\mathcal{H}^\infty}(U,F^{**})$, where κ_F denotes the canonical isometric linear embedding of F into F^{**}.

(S) surjective if for any mapping $f \in \mathcal{H}^\infty(U,F)$, any open subset V of a complex Banach space G, and any surjective mapping $\pi \in \mathcal{H}(V,U)$, we have that $f \in \mathcal{I}^{\mathcal{H}^\infty}(U,F)$ with $\|f\|_{\mathcal{I}^{\mathcal{H}^\infty}} = \|f \circ \pi\|_{\mathcal{I}^{\mathcal{H}^\infty}}$ whenever $f \circ \pi \in \mathcal{I}^{\mathcal{H}^\infty}(V,F)$.

We are now in a position to study the structure of $\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}$ as an s-Banach bounded holomorphic ideal.

Theorem 2.9. *Let $p \in [1,\infty)$, $q \in [1,p^*)$, and $s = pq/(p+q)$. Then, $[\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}, k^{\mathcal{H}^\infty}_{(p,q)}]$ is a surjective s-Banach bounded-holomorphic ideal. Furthermore, $[\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U,F), k^{\mathcal{H}^\infty}_{(p,q)}]$ is regular for the class of reflexive Banach spaces F.*

Proof. Note that $[\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}, k^{\mathcal{H}^\infty}_{(p,q)}]$ is a bounded-holomorphic ideal by Corollary 2.5 and [14, Corollary 2.5]. By [9, p. 151], $[\mathcal{K}_{(p,q)}, k_{(p,q)}]$ is an s-Banach space, and therefore, we deduce that $[\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}, k^{\mathcal{H}^\infty}_{(p,q)}]$ is also so by using the fact that both spaces are isometrically isomorphic by Corollary 2.5.

(S) Let $f \in \mathcal{H}^\infty(U,F)$ and assume that $f \circ \pi \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(V,F)$, where V is an open subset of a complex Banach space G and $\pi \in \mathcal{H}(V,U)$ is surjective. Since $f(U) = (f \circ \pi)(V)$, it follows immediately that $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U,F)$ with $k^{\mathcal{H}^\infty}_{(p,q)}(f) = k^{\mathcal{H}^\infty}_{(p,q)}(f \circ \pi)$. Hence, $[\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}, k^{\mathcal{H}^\infty}_{(p,q)}]$ is surjective.

(R) Assume that F is reflexive, and thus, $\ell_p(F^{**}) = \kappa_F(\ell_p(F))$. Let $f \in \mathcal{H}^\infty(U, F)$ and suppose that $\kappa_F \circ f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F^{**})$. Consider a sequence $(x_n) \in \ell_p(F)$ such that $(\kappa_F \circ f)(U) \subseteq q^*\text{-conv}(\kappa_F(x_n))$. Thus, $\kappa_F(f(U)) \subseteq \kappa_F(q^*\text{-conv}(x_n))$, and due to the injectivity of κ_F, we have $f(U) \subseteq q^*\text{-conv}(x_n)$. Hence, $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$ with $k^{\mathcal{H}^\infty}_{(p,q)}(f) \le \|(x_n)\|_p = \|(\kappa_F(x_n))\|_p$ and taking the infimum over all such sequences $(\kappa_F(x_n))$, we obtain $k^{\mathcal{H}^\infty}_{(p,q)}(f) \le k^{\mathcal{H}^\infty}_{(p,q)}(\kappa_F \circ f)$. The converse inequality follows since $[\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}, k^{\mathcal{H}^\infty}_{(p,q)}]$ satisfies the condition (N3) in Definition 2.8. □

In view of Proposition 2.2, Theorem 2.9 extends Proposition 3.2 in [14] and Theorem 2.5 in [8] on the structure as a Banach bounded-holomorphic ideal of $[\mathcal{H}^\infty_{\mathcal{K}}, \|\cdot\|_\infty]$ and $[\mathcal{H}^\infty_{\mathcal{K}_p}, k_p^{\mathcal{H}^\infty}]$, respectively.

2.3 Relation with (t, u, v)-nuclear holomorphic mappings

For $p \in [1, \infty]$, $\ell_p^{\text{weak}}(E)$ denotes the Banach space of all weakly p-summable sequences in E, endowed with the norm:

$$\|(x_n)\|_p^{\text{weak}} = \sup_{x^* \in B_{E^*}} \|(x^*(x_n))\|_p.$$

By [16, Definition 18.1.1], an operator $T \in \mathcal{L}(E, F)$ is said to be (t, u, v)-*nuclear* (with $t, u, v \in [1, \infty]$ and $1 + 1/t \ge 1/u + 1/v$) if

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^* \cdot y_n$$

in the norm topology of $\mathcal{L}(E, F)$, where $(\lambda_n) \in \ell_t$, $(x_n^*) \in \ell_{v^*}^{\text{weak}}(E^*)$ and $(y_n) \in \ell_{u^*}^{\text{weak}}(F)$. In the case $t = \infty$, we put $(\lambda_n) \in c_0$. We will say that $\sum_{n\ge1} \lambda_n x_n^* \cdot y_n$ is a (t, u, v)-*nuclear representation of* T. We denote by $\mathcal{N}_{(t,u,v)}$ the space of all (t, u, v)-nuclear operators, and if we set $1/s = 1/t + 1/u^* + 1/v^*$, then it becomes an s-Banach operator ideal under the norm:

$$\nu_{(t,u,v)}(T) := \inf\{\|(\lambda_n)\|_t \|(x_n^*)\|_{v^*}^{\text{weak}} \|(y_n)\|_{u^*}^{\text{weak}}\},$$

where the infimum is taken over all (t, u, v)-nuclear representations of T (see [16, Theorem 18.1.2]).

In [16, Theorem 18.1.3], Pietsch proved that an operator $T \in \mathcal{L}(E, F)$ is (t, u, v)-nuclear if and only if there exists a commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\quad T \quad} & F \\ {\scriptstyle S}\downarrow & & \uparrow{\scriptstyle R} \\ \ell_{v^*} & \xrightarrow{\quad M_\lambda \quad} & \ell_u \end{array}$$

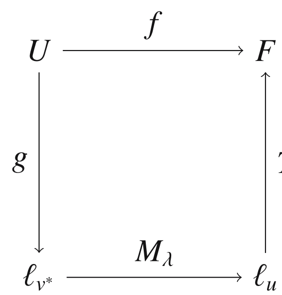
where $S \in \mathcal{L}(E, \ell_{v^*})$, $R \in \mathcal{L}(\ell_u, F)$, and M_λ is the diagonal operator from ℓ_{v^*} into ℓ_u given by $M_\lambda((x_n)) = (\lambda_n x_n)$ for all $(x_n) \in \ell_{v^*}$, where $\lambda = (\lambda_n) \in \ell_t$ if $1 \le t < \infty$ and $\lambda = (\lambda_n) \in c_0$ if $t = \infty$. In this case,

$$\nu_{(t,u,v)}(T) = \inf\{\|R\| \|(\lambda_n)\|_t \|S\|\},$$

and the infimum is taken over all possible factorizations of T.

This motivates the introduction of a new class of holomorphic mappings.

Definition 2.10. Let $t, u, v \in [1, \infty]$ such that $1 + 1/t \ge 1/u + 1/v$. A mapping $f : U \to F$ is said to be (t, u, v)-nuclear holomorphic if there exist an operator $T \in \mathcal{L}(\ell_u, F)$, a diagonal operator $M_\lambda \in \mathcal{L}(\ell_{v^*}, \ell_u)$ induced by a sequence $\lambda = (\lambda_n) \in \ell_t$ ($\lambda \in c_0$ if $t = \infty$), and a mapping $g \in \mathcal{H}^\infty(U, \ell_{v^*})$ such that $f = T \circ M_\lambda \circ g$, i.e., the following diagram commutes



The triple (T, M_λ, g) is called a (t, u, v)-nuclear holomorphic factorization of f. We set

$$\nu^{\mathcal{H}^\infty}_{(t,u,v)}(f) = \inf\{\|T\|\|M_\lambda\|\|g\|_\infty\},$$

where the infimum is taken over all such factorizations of f. We will denote by $\mathcal{H}^\infty_{\mathcal{N}_{(t,u,v)}}(U, F)$ the set of all (t, u, v)-nuclear holomorphic mappings from U into F.

Our next goal is to show that $[\mathcal{H}^\infty_{\mathcal{N}_{(t,u,v)}}, \nu^{\mathcal{H}^\infty}_{(t,u,v)}]$ is an s-Banach bounded-holomorphic ideal. For it, we first study its linearization.

Theorem 2.11. *Let $t, u, v \in [1, \infty]$ such that $1 + 1/t \geq 1/u + 1/v$, and let $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent*:
(*i*) *$f : U \to F$ is a (t, u, v)-nuclear holomorphic mapping.*
(*ii*) *$T_f : \mathcal{G}^\infty(U) \to F$ is a (t, u, v)-nuclear operator.*

In this case, $\nu^{\mathcal{H}^\infty}_{(t,u,v)}(f) = \nu_{(t,u,v)}(T_f)$. Furthermore, the mapping $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{H}^\infty_{\mathcal{N}_{(t,u,v)}}(U, F), \nu^{\mathcal{H}^\infty}_{(t,u,v)})$ onto $(\mathcal{N}_{(t,u,v)}(\mathcal{G}^\infty(U), F), \nu_{(t,u,v)})$.

Proof. $(i) \Rightarrow (ii)$: Assume that $f \in \mathcal{H}^\infty_{\mathcal{N}_{(t,u,v)}}(U, F)$. Then, $f = T \circ M_\lambda \circ g$, where $T \in \mathcal{L}(\ell_u, F)$, $(\lambda_n) \in \ell_t$ (or c_0 if $t = \infty$) and $g \in \mathcal{H}^\infty(U, \ell_{v^*})$. Theorem 2.1 gives

$$T_f \circ g_U = f = T \circ M_\lambda \circ g = T \circ M_\lambda \circ T_g \circ g_U,$$

where $T_g \in \mathcal{L}(\mathcal{G}^\infty(U), \ell_{v^*})$. Hence, $T_f = T \circ M_\lambda \circ T_g$. By [16, 18.1.3], it follows that $T_f \in \mathcal{N}_{(t,u,v)}(\mathcal{G}^\infty(U), F)$ with

$$\nu_{(t,u,v)}(T_f) \leq \|T\|\|M_\lambda\|\|T_g\| = \|T\|\|M_\lambda\|\|g\|_\infty.$$

Taking the infimum over all such factorizations of f, we have $\nu_{(t,u,v)}(T_f) \leq \nu^{\mathcal{H}^\infty}_{(t,u,v)}(f)$.

$(ii) \Rightarrow (i)$: Suppose that $T_f \in \mathcal{N}_{(t,u,v)}(\mathcal{G}^\infty(U), F)$ and let $\varepsilon > 0$. Again, by [16, 18.1.3], there exist $(\lambda_n) \in \ell_t$ (or c_0 if $t = \infty$), $R \in \mathcal{L}(\mathcal{G}^\infty(U), \ell_{v^*})$, and $S \in \mathcal{L}(\ell_u, F)$ such that $T_f = S \circ M_\lambda \circ R$ and

$$\|S\|\|M_\lambda\|\|R\| \leq (1 + \varepsilon)\nu_{(t,u,v)}(T_f).$$

Then, $f = T_f \circ g_U = S \circ M_\lambda \circ R \circ g_U$. Consider the mapping $g : U \to \ell_{v^*}$ defined by

$$g(x) = (R \circ g_U)(x) \quad (x \in U).$$

It is clear that $g \in \mathcal{H}(U, \ell_{v^*})$ and

$$\|g(x)\| = \|(R \circ g_U)(x)\| \leq \|R\|\|\delta(x)\| = \|R\|,$$

for all $x \in U$. Thus, $g \in \mathcal{H}^\infty(U, \ell_{v^*})$ and $f = S \circ M_\lambda \circ g$, i.e., $f \in \mathcal{H}^\infty_{\mathcal{N}_{(t,u,v)}}(U, F)$ with

$$\nu^{\mathcal{H}^\infty}_{(t,u,v)}(f) \leq \|S\|\|M_\lambda\|\|g\|_\infty \leq \|S\|\|M_\lambda\|\|R\| \leq (1 + \varepsilon)\nu_{(t,u,v)}(T_f).$$

Just letting $\varepsilon \to 0$, we obtain $\nu^{\mathcal{H}^\infty}_{(t,u,v)}(f) \leq \nu_{(t,u,v)}(T_f)$.

The last assertion can be shown using assertion (iii) in Theorem 2.1. □

Applying Theorem 2.11 and [14, Theorem 2.4], we obtain the following result on factorization of (t, u, v)-nuclear holomorphic mappings.

Corollary 2.12. *Let $t, u, v \in [1, \infty]$ with $1 + 1/t \geq 1/u + 1/v$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent:*
(i) *$f : U \to F$ is (t, u, v)-nuclear holomorphic.*
(ii) *$f = T \circ g$, for some complex Banach space G, $g \in \mathcal{H}^\infty(U, G)$ and $T \in \mathcal{N}_{(t,u,v)}(G, F)$.*

In this case, we have

$$\nu^{\mathcal{H}^\infty}_{(t,u,v)}(f) = \|f\|_{\mathcal{N}_{(t,u,v)} \circ \mathcal{H}^\infty} := \inf\{\nu_{(t,u,v)}(T)\|g\|_\infty\},$$

where the infimum is taken over all factorizations of f as in (ii), and this infimum is attained in $T_f \circ g_U$. Furthermore, the mapping $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{H}^\infty_{\mathcal{N}_{(t,u,v)}}(U, F), \|\cdot\|_{\mathcal{N}_{(t,u,v)} \circ \mathcal{H}^\infty})$ onto $(\mathcal{N}_{(t,u,v)}(\mathcal{G}^\infty(U), F), \nu_{(t,u,v)})$.

Now, using [14, Corollary 2.5], the fact that $[\mathcal{N}_{(t,u,v)}, \nu_{(t,u,v)}]$ is an s-Banach space (see [16, Theorem 18.1.2]) and Corollary 2.12, we arrive at the announced fact.

Corollary 2.13. *Let $t, u, v \in [1, \infty]$ and $1/s = 1/t + 1/u^* + 1/v^*$. Then, $[\mathcal{H}^\infty_{\mathcal{N}_{(t,u,v)}}, \nu^{\mathcal{H}^\infty}_{(t,u,v)}]$ is an s-Banach bounded-holomorphic ideal.*

The following relationship is easily obtained.

Corollary 2.14. *Let $p \in [1, \infty)$ and $q \in [1, p^*)$. Then, $\mathcal{H}^\infty_{\mathcal{N}_{(p,1,q^*)}}(U, F) \subseteq \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$ and $k^{\mathcal{H}^\infty}_{(p,q)}(f) \leq \nu^{\mathcal{H}^\infty}_{(p,1,q^*)}(f)$ for all $f \in \mathcal{H}^\infty_{\mathcal{N}_{(p,1,q^*)}}(U, F)$.*

Proof. By Theorem 2.11, $T_f \in \mathcal{N}_{(p,1,q^*)}(\mathcal{G}^\infty(U), F)$ with $\nu_{(p,1,q^*)}(T_f) = \nu^{\mathcal{H}^\infty}_{(p,1,q^*)}(f)$. By [9, Theorems 3.2 and 3.4], $T_f \in \mathcal{K}_{(p,q)}(\mathcal{G}^\infty(U), F)$ and $k_{(p,q)}(T_f) \leq \nu_{(p,1,q^*)}(T_f)$. Hence, $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$ with

$$k^{\mathcal{H}^\infty}_{(p,q)}(f) = k_{(p,q)}(T_f) \leq \nu_{(p,1,q^*)}(T_f) = \nu^{\mathcal{H}^\infty}_{(p,1,q^*)}(f),$$

by Theorem 2.3. □

Let us recall (see [8, Definition 1.18]) that the *surjective hull of a bounded-holomorphic ideal $\mathcal{I}^{\mathcal{H}^\infty}$* is the smallest surjective ideal, which contains $\mathcal{I}^{\mathcal{H}^\infty}$, and it is denoted by $(\mathcal{I}^{\mathcal{H}^\infty})^{\text{sur}}$.

We have proven that $\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}$ is a surjective s-Banach bounded-holomorphic ideal, which contains $\mathcal{H}^\infty_{\mathcal{N}_{(p,1,q^*)}}$, and therefore, $(\mathcal{H}^\infty_{\mathcal{N}_{(p,1,q^*)}})^{\text{sur}} \subseteq \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}$, but we do not know whether both sets are equal as it occurs in [9, Theorem 3.2], in the linear context.

2.4 Transposition

Let us recall that the *transpose of a mapping $f \in \mathcal{H}^\infty(U, F)$* is the operator $f^t \in \mathcal{L}(F^*, \mathcal{H}^\infty(U))$ defined by

$$f^t(y^*) = y^* \circ f \quad (y^* \in F^*).$$

Moreover, $\|f^t\| = \|f\|_\infty$ and $f^t = J_U^{-1} \circ (T_f)^*$ (see [5, Proposition 1.6]).

It is well known by [9, Theorem 4.2] that the ideal $\mathcal{K}_{(p,q)}$ is related by duality with the injective hull of the ideal of $(p, q^*, 1)$-nuclear operators. Let us recall that if $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is an s-Banach operator ideal, the components of the s-Banach operator ideal $\mathcal{A}^{\text{inj}}$, the injective hull of $\mathcal{A}$, are defined as

$$\mathcal{A}^{\text{inj}}(E, F) = \{T \in \mathcal{L}(E, F) : \iota_F \circ T \in \mathcal{A}(E, \ell_\infty(B_{F^*}))\},$$

with $\|T\|_{\mathcal{A}^{\mathrm{inj}}} = \|\iota_F \circ T\|_{\mathcal{A}}$ for $T \in \mathcal{A}^{\mathrm{inj}}(E, F)$, where $\iota_F : F \to \ell_\infty(B_{F^*})$ is the map defined by $y \mapsto (y^*(y))_{y^* \in B_{F^*}}$.

Thus, we can characterize (p, q)-compact holomorphic mappings via their transposes.

Theorem 2.15. *Let $p \in [1, \infty)$ and $q \in [1, p^*)$. If $f \in \mathcal{H}^\infty(U, F)$, then $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$ if and only if $f^t \in \mathcal{N}^{\mathrm{inj}}_{(p,q^*,1)}(\mathcal{G}^\infty(U), F)$. In this case, $k^{\mathcal{H}^\infty}_{(p,q)}(f) = \|f^t\|_{\mathcal{N}^{\mathrm{inj}}_{(p,q^*,1)}}$.*

Proof. Applying Theorem 2.3, [9, Theorem 4.2], and [16, 8.4.2], we have

$$\begin{aligned} f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F) &\Leftrightarrow T_f \in \mathcal{K}_{(p,q)}(\mathcal{G}^\infty(U), F) \\ &\Leftrightarrow (T_f)^* \in \mathcal{N}^{\mathrm{inj}}_{(p,q^*,1)}(F^*, \mathcal{G}^\infty(U)^*) \\ &\Leftrightarrow f^t = J_U^{-1} \circ (T_f)^* \in \mathcal{N}^{\mathrm{inj}}_{(p,q^*,1)}(F^*, \mathcal{H}^\infty(U)). \end{aligned}$$

Furthermore, $k^{\mathcal{H}^\infty}_{(p,q)}(f) = k_{(p,q)}(T_f) = \|(T_f)^*\|_{\mathcal{N}^{\mathrm{inj}}_{(p,q^*,1)}} = \|f^t\|_{\mathcal{N}^{\mathrm{inj}}_{(p,q^*,1)}}$. □

2.5 Möbius invariance

The *Möbius group of* $\mathbb{D}$, denoted by $\mathrm{Aut}(\mathbb{D})$, consists of all one-to-one holomorphic functions ϕ that map $\mathbb{D}$ onto itself. Each $\phi \in \mathrm{Aut}(\mathbb{D})$ has the form $\phi = \lambda\phi_a$, with $\lambda \in \mathbb{T}$ and $a \in \mathbb{D}$, where

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z} \quad (z \in \mathbb{D}).$$

Given a complex Banach space F, let us recall (see [17]) that a linear space $\mathcal{A}(\mathbb{D}, F)$ of holomorphic mappings from $\mathbb{D}$ into F, endowed with a seminorm $p_{\mathcal{A}}$, is said to be *Möbius-invariant* if for all $\phi \in \mathrm{Aut}(\mathbb{D})$ and $f \in \mathcal{A}(\mathbb{D}, F)$, we have that $f \circ \phi \in \mathcal{A}(\mathbb{D}, F)$ with $p_{\mathcal{A}}(f \circ \phi) = p_{\mathcal{A}}(f)$.

We obtain the following result closely related to the invariance by Möbius transformations of $\mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(\mathbb{D}, F)$.

Proposition 2.16. *Let $p \in [1, \infty)$ and $q \in [1, p^*)$.* If *$f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(\mathbb{D}, F)$ and $\phi \in \mathrm{Aut}(\mathbb{D})$, then $f \circ \phi \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(\mathbb{D}, F)$ with $k^{\mathcal{H}^\infty}_{(p,q)}(f \circ \phi) = k^{\mathcal{H}^\infty}_{(p,q)}(f)$.*

Proof. Let $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(\mathbb{D}, F)$ and $\phi \in \mathrm{Aut}(\mathbb{D})$. Note that there exists a sequence $(x_n) \in \ell_p(F)$ such that $f(\mathbb{D}) \subseteq q^*\text{-conv}(x_n)$. Thus, due to ϕ maps $\mathbb{D}$ onto itself, we have that $(f \circ \phi)(\mathbb{D}) \subseteq q^*\text{-conv}(x_n)$. Hence, $f \circ \phi \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(\mathbb{D}, F)$ with $k^{\mathcal{H}^\infty}_{(p,q)}(f \circ \phi) \le k^{\mathcal{H}^\infty}_{(p,q)}(f)$. Since $\phi^{-1} \in \mathrm{Aut}(\mathbb{D})$, the previous proof yields the converse inequality $k^{\mathcal{H}^\infty}_{(p,q)}(f) \le k^{\mathcal{H}^\infty}_{(p,q)}(f \circ \phi)$. □

2.6 Full compact factorization

With the aim to present such a factorization theorem for (p, q)-compact holomorphic mappings, we will prove previously that an operator $T \in \mathcal{L}(E, F)$ is (p, q)-compact if and only if it factors through two compact operators and a (p, q)-compact operator. This fact improves [18, Theorem 3.1] and [19, Proposition 2.9] but its proof is based on them. For it, we will first prove that a (p, q)-compact operator factors through a quotient space of ℓ_1.

Theorem 2.17. *Let E and F be Banach spaces, $T \in \mathcal{L}(E, F)$, $p \in [1, \infty)$, and $q \in [1, p^*)$. The following assertions are equivalent*:

(*i*) *T is (p, q)-compact.*

(*ii*) *There exist a sequence $y \in \ell_p(F)$, operators $T_y \in \mathcal{K}_{(p,q)}(\ell_q, F)$ and $R_0 \in \mathcal{L}(E, \ell_q/\ker(T_y))$, a closed subspace $M \subseteq \ell_1$, and operators $T_0 \in \mathcal{K}_{(p,q)}(\ell_q/\ker(T_y), \ell_1/M)$ and $S \in \mathcal{K}(\ell_1/M, F)$ such that $T = S \circ T_0 \circ R_0$.*

In this case, $k_{(p,q)}(T) = \inf\{\|S\| k_{(p,q)}(T_0) \|R_0\|\}$, where the infimum is taken over all factorizations of T as in (*ii*).

Proof. (*i*) ⇒ (*ii*): Let $T \in \mathcal{K}_{(p,q)}(E, F)$. Then, given $\varepsilon > 0$, there is a sequence $y = (y_n) \in \ell_p(F)$ with $\|(y_n)\|_p \le k_{(p,q)}(T) + \varepsilon$ such that

$$T(B_E) \subseteq q^*\text{-conv}(y_n) := \left\{ \sum_{n=1}^{\infty} \alpha_n y_n : (\alpha_n) \in B_{\ell_q} \right\}.$$

Define the bounded linear operators $T_y : \ell_q \to F$ and $\widehat{T_y} : \ell_q / \ker(T_y) \to F$ by

$$T_y(\alpha) = \sum_{n=1}^{\infty} \alpha_n y_n \quad \text{and} \quad \widehat{T_y}([\alpha]) = T_y(\alpha),$$

for all $\alpha = (\alpha_n) \in \ell_q$, respectively. Clearly, $T_y \in \mathcal{K}_{(p,q)}(\ell_q, F)$, and thus, $\widehat{T_y} \in \mathcal{K}_{(p,q)}(\ell_q / \ker(T_y), F)$ with $k_{(p,q)}(\widehat{T_y}) \le \|(y_n)\|_p$.

For each $x \in E$, there exists a sequence $\beta = (\beta_n) \in \ell_q$ such that $T(x) = \sum_{n=1}^{\infty} \beta_n y_n$. Hence, $R_0(x) = [\beta]$ defines a bounded linear operator from E into $\ell_q / \ker(T_y)$ with $\|R_0\| \le 1$. Clearly, $T = \widehat{T_y} \circ R_0$.

We can assume that $y_n \neq 0$ for all $n \in \mathbb{N}$. Let (β_n) be a sequence in B_{c_0} with $\beta_n > 0$ for all $n \in \mathbb{N}$ such that $(y_n / \beta_n) \in \ell_p(F)$ with $\|(y_n / \beta_n)\|_p \le \|(y_n)\|_p + \varepsilon$. Taking $\lambda = (\lambda_n) = (\|y_n\| / \beta_n)$ and $z = (z_n) = (y_n / \lambda_n)$, then $\lambda \in \ell_p$, $z \in B_{c_0(F)}$, and

$$T(B_E) \subseteq \left\{ \sum_{n=1}^{\infty} \alpha_n \lambda_n z_n : (\alpha_n) \in B_{\ell_q} \right\}.$$

Define the bounded linear operators $S_0 : \ell_1 \to F$ and $S : \ell_1 / \ker(S_0) \to F$ by

$$S_0(\gamma) = \sum_{n=1}^{\infty} \gamma_n z_n \quad \text{and} \quad S([\gamma]) = S_0(\gamma),$$

for all $\gamma = (\gamma_n) \in \ell_1$, respectively. Clearly, $S \in \mathcal{K}(\ell_1 / \ker(S_0), F)$ with $\|S\| \le \|(z_n)\|_\infty \le 1$. Now, we can define the bounded linear operator $T_0 : \ell_q / \ker(T_y) \to \ell_1 / \ker(S_0)$ by

$$T_0([\alpha]) = [(\lambda_n \alpha_n)],$$

for all $\alpha = (\alpha_n) \in \ell_q$. Since $(\lambda_n e_n) \in \ell_p(\ell_1)$, where $\{e_n : n \in \mathbb{N}\}$ is the canonical basis of ℓ_1; note that $T_0 \in \mathcal{K}_{(p,q)}(\ell_q / \ker(T_y), \ell_1 / \ker(S_0))$ with $k_{(p,q)}(T_0) \le \|(\lambda_n e_n)\|_p = \|(\lambda_n)\|_p$. Plainly, $\widehat{T_y} = S \circ T_0$, and thus, $T = S \circ T_0 \circ R_0$ with $\|S\| k_{(p,q)}(T_0) \|R_0\| \le k_{(p,q)}(T) + 2\varepsilon$. Since ε is arbitrary, it follows that $\inf\{\|S\| k_{(p,q)}(T_0) \|R_0\|\} \le k_{(p,q)}(T)$.

(*ii*) ⇒ (*i*): Assume that $T = S \circ T_0 \circ R_0$ with S, T_0, R_0 being as in the statement. By the ideal property of $\mathcal{K}_{(p,q)}$, we have that $T \in \mathcal{K}_{(p,q)}(E, F)$ with $k_{(p,q)}(T) \le \|S\| k_{(p,q)}(T_0) \|R_0\|$, and taking the infimum over all such representations of T, we obtain that $k_{(p,q)}(T) \le \inf\{\|S\| k_{(p,q)}(T_0) \|R_0\|\}$. □

Although a great part of the demonstration of the following result is similar to that of Theorem 2.17, we provide its complete proof for the sake of completeness.

Theorem 2.18. *Let E and F be the Banach spaces,* $T \in \mathcal{L}(E, F)$, $p \in [1, \infty)$, *and* $q \in [1, p^*)$. *The following conditions are equivalent*:

(*i*) *T is (p, q)-compact.*

(*ii*) *There exist Banach spaces H and G, an operator* $T_0 \in \mathcal{K}_{(p,q)}(H, G)$, *and operators* $S \in \mathcal{K}(G, F)$ *and* $R \in \mathcal{K}(E, H)$ *such that* $T = S \circ T_0 \circ R$.

In this case, $k_{(p,q)}(T) = \inf\{\|S\| k_{(p,q)}(T_0) \|R\|\}$, *where the infimum is taken over all factorizations of T as in* (*ii*).

Proof. (*i*) ⇒ (*ii*): Let $T \in \mathcal{K}_{(p,q)}(E, F)$. Given $\varepsilon > 0$, we can take $y = (y_n) \in \ell_p(F)$ with $\|(y_n)\|_p \le k_{(p,q)}(T) + \varepsilon$ so that

$$T(B_E) \subseteq q^*\text{-conv}(y_n) := \left\{ \sum_{n=1}^{\infty} \alpha_n y_n : (\alpha_n) \in B_{\ell_q} \right\}.$$

Assuming $y_n \neq 0$ for all $n \in \mathbb{N}$, let (β_n) be a sequence in B_{c_0} with $\beta_n > 0$ for all $n \in \mathbb{N}$ such that $(y_n/\beta_n) \in \ell_p(F)$ with $\|(y_n/\beta_n)\|_p \leq \|(y_n)\|_p + \varepsilon$. If $\lambda = (\lambda_n) = (\|y_n\|/\beta_n)$ and $z = (z_n) = (y_n/\lambda_n)$, then $\lambda \in \ell_p$, $z \in B_{c_0(F)}$, and

$$T(B_E) \subseteq \left\{\sum_{n=1}^{\infty} a_n \lambda_n z_n : (a_n) \in B_{\ell_q}\right\} \subseteq \left\{\sum_{n=1}^{\infty} a_n z_n : (a_n) \in L\right\},$$

where $L := \left\{(a_n) \in B_{\ell_q} : \sum_{n=1}^{\infty}(|a_n|^q/\lambda_n^q) \leq 1\right\}$ is a compact set in B_{ℓ_q}.

Define the bounded linear operators $T_z : \ell_q \to F$ and $\widehat{T_z} : \ell_q/\ker(T_z) \to F$ by

$$T_z(a) = \sum_{n=1}^{\infty} a_n z_n \quad \text{and} \quad \widehat{T_z}([a]) = T_z(a),$$

for all $a = (a_n) \in \ell_q$, respectively. Clearly, $\widehat{T_z} \in \mathcal{K}_{(p,q)}(\ell_q/\ker(T_z), F)$ with $k_{(p,q)}(\widehat{T_z}) \leq \|(z_n)\|_p$.

For each $x \in B_E$, there exists a sequence $a = (a_n) \in L$ such that $T(x) = \sum_{n=1}^{\infty} a_n z_n$. Hence, $Q_z(x) = [a]$ defines a bounded linear operator from E into $\ell_q/\ker(T_z)$ with $\|Q_z\| \leq 1$. Since $Q_z(B_E) \subseteq \pi(L)$, where $\pi : \ell_q \to \ell_q/\ker(T_z)$ is the canonical projection, it follows that $Q_z \in \mathcal{K}(E, \ell_q/\ker(T_z))$. Clearly, $T = \widehat{T_z} \circ Q_z$, and the ideal property of $\mathcal{K}_{(p,q)}$ yields

$$k_{(p,q)}(T) \leq k_{(p,q)}(\widehat{T_z})\|Q_z\| \leq k_{(p,q)}(\widehat{T_z}) \leq \|(z_n)\|_p \leq \|(y_n)\|_p + \varepsilon \leq k_{(p,q)}(T) + 2\varepsilon.$$

Now, applying Theorem 2.17 to $\widehat{T_z}$, we can find a $y \in \ell_p(F)$, $T_{z,y} \in \mathcal{K}_{(p,q)}(\ell_q, F)$, and $R_0 \in \mathcal{L}(\ell_q/\ker(T_z), \ell_q/\ker(T_{z,y}))$, a closed subspace $M \subseteq \ell_1$, and $T_0 \in \mathcal{K}_{(p,q)}(\ell_q/\ker(T_{z,y}), \ell_1/M)$ and $S \in \mathcal{K}(\ell_1/M, F)$ such that $\widehat{T_z} = S \circ T_0 \circ R_0$ with $\|S\|k_{(p,q)}(T_0)\|R_0\| \leq k_{(p,q)}(\widehat{T_z}) + 2\varepsilon$. Hence, $T = S \circ T_0 \circ R$, where $R := R_0 \circ Q_z \in \mathcal{K}(E, \ell_q/\ker(T_{z,y}))$, with $\|S\|k_{(p,q)}(T_0)\|R\| \leq k_{(p,q)}(T) + 4\varepsilon$. Therefore, $\inf\{\|\widehat{S_z}\|k_{(p,q)}(T_0)\|R\|\} \leq k_{(p,q)}(T)$.

$(ii) \Rightarrow (i)$: Assume that $T = S \circ T_0 \circ R$ with S, T_0, R being as in the statement. By the ideal property of $\mathcal{K}_{(p,q)}$, we have that $T \in \mathcal{K}_{(p,q)}(E, F)$ with $k_{(p,q)}(T) \leq \|S\|k_{(p,q)}(T_0)\|R\|$, and taking the infimum over all such representations of T, we obtain $k_{(p,q)}(T) \leq \inf\{\|S\|k_{(p,q)}(T_0)\|R\|\}$. □

We are now able to provide a full compact factorization for (p, q)-compact holomorphic mappings.

Corollary 2.19. *Let $p \in [1, \infty)$, $q \in [1, p^*)$, and let $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent*:
(*i*) *$f : U \to F$ is (p, q)-compact holomorphic.*
(*ii*) *There exist complex Banach spaces H and G, an operator $T_0 \in \mathcal{K}_{(p,q)}(H, G)$, a mapping $g \in \mathcal{H}^\infty_{\mathcal{K}}(U, H)$, and an operator $S \in \mathcal{K}(G, F)$ such that $f = S \circ T_0 \circ g$.*

In this case, $k^{\mathcal{H}^\infty}_{(p,q)}(f) = \inf\{\|S\|k_{(p,q)}(T_0)\|g\|_\infty\}$, where the infimum is extended over all factorizations of f as in (ii).

Proof. $(i) \Rightarrow (ii)$: Suppose that $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$. By Theorem 2.3, $T_f \in \mathcal{K}_{(p,q)}(\mathcal{G}^\infty(U), F)$ with $k_{(p,q)}(T_f) = k^{\mathcal{H}^\infty}_{(p,q)}(f)$. Applying Theorem 2.18, for each $\varepsilon > 0$, there exist complex Banach spaces H and G, an operator $T_0 \in \mathcal{K}_{(p,q)}(H, G)$, and operators $S \in \mathcal{K}(G, F)$ and $R \in \mathcal{K}(\mathcal{G}^\infty(U), H)$ such that $T_f = S \circ T_0 \circ R$ with $\|S\|k_{(p,q)}(T_0)\|R\| \leq k_{(p,q)}(T_f) + \varepsilon$. Note that $R = T_g$ with $\|g\|_\infty = \|R\|$ for some $g \in \mathcal{H}^\infty_{\mathcal{K}}(U, H)$ by [5, Corollary 2.11]. Hence, $f = T_f \circ g_U = S \circ T_0 \circ T_g \circ g_U = S \circ T_0 \circ g$, with

$$\|S\|k_{(p,q)}(T_0)\|g\|_\infty = \|S\|k_{(p,q)}(T_0)\|R\| \leq k_{(p,q)}(T_f) + \varepsilon = k^{\mathcal{H}^\infty}_{(p,q)}(f) + \varepsilon.$$

Just letting $\varepsilon \to 0$, we have $\|S\|k_{(p,q)}(T_0)\|g\|_\infty \leq k^{\mathcal{H}^\infty}_{(p,q)}(f)$.

$(ii) \Rightarrow (i)$: Suppose that $f = S \circ T_0 \circ g$ is a factorization as in (ii). By the ideal property of $\mathcal{K}_{(p,q)}$, we have that $S \circ T_0 \in \mathcal{K}_{(p,q)}(H, F)$ and then, by Corollary 2.5, $f \in \mathcal{H}^\infty_{\mathcal{K}_{(p,q)}}(U, F)$ with

$$k^{\mathcal{H}^\infty}_{(p,q)}(f) \leq k_{(p,q)}(S \circ T_0)\|g\|_\infty \leq \|S\|k_{(p,q)}(T_0)\|g\|_\infty.$$

Taking the infimum over all representations of f, we have $k^{\mathcal{H}^\infty}_{(p,q)}(f) \leq \inf\{\|S\|k_{(p,q)}(T_0)\|g\|_\infty\}$. □

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