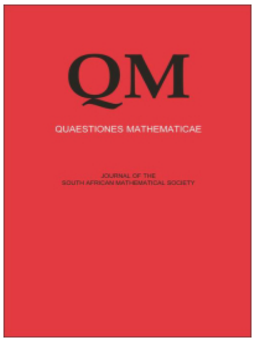

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APPROXIMATE LOCAL ISOMETRIES OF DERIVATIVE HARDY SPACES

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Abstract. For any $1 \leq p \leq \infty$, let $S^p(\mathbb{D})$ be the space of holomorphic functions f on $\mathbb{D}$ such that f' belongs to the Hardy space $H^p(\mathbb{D})$, with the norm $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_p$. We prove that every approximate local isometry of $S^p(\mathbb{D})$ is a surjective isometry and that every approximate 2-local isometry of $S^p(\mathbb{D})$ is a surjective linear isometry. As a consequence, we deduce that the sets of isometric reflections and generalized bi-circular projections on $S^p(\mathbb{D})$ are also topologically reflexive and 2-topologically reflexive.

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Key words: Algebraic reflexivity, topological reflexivity, isometry group, isometric reflection.

1. Introduction. Let $\mathbb{D}$ be the open unit disc in the complex plane and let $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$. The Hardy space $H^\infty(\mathbb{D})$ consists of all bounded functions in $H(\mathbb{D})$. For $1 \leq p < \infty$, the Hardy space $H^p(\mathbb{D})$ is the space of all holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_p = \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \, d\theta \right)^{\frac{1}{p}} < \infty.$$

For $1 \leq p \leq \infty$, let $S^p(\mathbb{D})$ be the derivative Hardy space of holomorphic functions f on $\mathbb{D}$ such that $f' \in H^p(\mathbb{D})$, with the norm

$$\|f\|_\Sigma = \|f\|_\infty + \|f'\|_p \, .$$

In the literature, $S^p(\mathbb{D})$ spaces or some of their variants have been studied under other names such as Novinger-Oberlin spaces, holomorphic Sobolev spaces or Hardy–Sobolev spaces.

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A natural problem is to characterize the surjective linear isometries of these spaces. de Leeuw, Rudin and Wermer [12] did so for $H^\infty(\mathbb{D})$ and $H^1(\mathbb{D})$ (see also [18] by Nagasawa for $H^\infty(\mathbb{D})$). Forelli [7] extended their results for linear isometries (not necessarily surjective) of the spaces $H^p(\mathbb{D})$ with $1 \leq p < \infty$ and $p \neq 2$. For these numbers p, Novinger and Oberlin [19] proved that every linear isometry of $S^p(\mathbb{D})$ has the form of a weighted composition operator induced by a conformal automorphism of $\mathbb{D}$ (bijective holomorphic map of $\mathbb{D}$ onto itself). Furthermore, when $p > 1$ this conformal map must actually be a rotation of $\mathbb{D}$. This connects with the fact stated by Roan [21] that the rotations are the unique conformal maps of $\mathbb{D}$ whose induced composition operators are isometries on $S^p(\mathbb{D})$. The second author and Niwa [16] obtained the same description of Novinger and Oberlin for onto linear isometry of $S^\infty(\mathbb{D})$. From [16, Remark 4.1], $S^\infty(\mathbb{D})$ coincides with the space of all Lipschitz functions in $H(\mathbb{D})$. We refer to the chapter 4 of the monograph [6] by Fleming and Jamison for a survey on the isometries of these and other spaces of holomorphic functions.

In the words of Cowen and MacCluer [4], *the study of composition operators on spaces of holomorphic functions provides a rich arena in which to explore the connections between operator theory and function theory.* The investigation of composition operators on $S^p(\mathbb{D})$ was initiated by Roan [21]. Namely, he dealed with their boundedness and compactness, which were later characterized in terms of Carleson measures by MacCluer [15]. Contreras and Hernández-Díaz [3] extended this study to weighted composition operators on $S^p(\mathbb{D})$.

Once the problem of describing the isometry group of these spaces is solved, an interesting question is to address the (topological) algebraic reflexivity problem as to when every (approximate) local isometry is a surjective isometry. Given a continuous linear operator $T\colon E \to E$, let us recall that T is a local isometry whenever T coincides at every point of E with some element of the isometry group $\mathcal{G}(E)$ of E, and that T is an approximate local isometry if T can be norm-approximated at every point of E by an element of $\mathcal{G}(E)$.

Although we are interested here in the reflexivity problem for isometries, the study of the algebraic and topological reflexivity of the sets of derivations and automorphisms on operator algebras and function algebras is a classical problem which follows attracting the attention of numerous researchers. Molnár's monograph [17] gives an interesting account of these developments.

In the context of spaces of holomorphic functions, Cabello Sánchez and Molnár [2] showed that the isometry group and the automorphism group of the disc algebra $A(\overline{\mathbb{D}})$ and the space $H^\infty(\Omega)$ (with Ω a simply connected domain of $\mathbb{C}$) are topologically reflexive as well as the isometry group of the space $H^p(\mathbb{D})$ with $p \neq 2$. Botelho and Jamison [1] established the topological reflexivity of different spaces of holomorphic functions, among them the spaces $S^p(\mathbb{D})$ with the norm

$$\|f\|_\sigma = |f(0)| + \|f'\|_p .$$

In contrast, Samei [22] stated the non-reflexivity of the space of bounded derivations from A into A^* with A being any Banach algebra of holomorphic functions on a connected domain of the plane.

Our goal here is to see that approximate local isometries on $S^p(\mathbb{D})$ are surjective isometries. Applying this fact, we also deduce that the sets of isometric reflections and generalized bi-circular projections on $S^p(\mathbb{D})$ are topologically reflexive. For this we first state a complete description of both kinds of maps on $S^p(\mathbb{D})$.

On the other hand, the class of 2-local isometries on function algebras has been investigated recently by different authors (see [8, 9, 10, 11, 13, 14], among others). The 2-locality problem for surjective isometries (without assuming linearity) on $S^\infty(\mathbb{D})$ has been studied by Oi [20]. We also establish here that the set of surjective linear isometries on $S^p(\mathbb{D})$ and the sets of isometric reflections and generalized bi-circular projections on $S^p(\mathbb{D})$ are 2-topologically reflexive.

2. Preliminaries. In order to address these questions, we recall the concepts of algebraic and topological reflexivity. Given two Banach spaces E and F, let F^E be the set of all the maps from E to F and let $\mathcal{B}(E,F)$ be the Banach space of all bounded linear maps in F^E. Given a nonempty set $\mathcal{S}\subset\mathcal{B}(E,F)$, define the sets

$$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})=\{T\in\mathcal{B}(E,F)\colon \forall e\in E,\ \exists S_e\in\mathcal{S}\mid S_e(e)=T(e)\}$$

and

$$\mathrm{ref}_{\mathrm{top}}(\mathcal{S})=\left\{T\in\mathcal{B}(E,F)\colon \forall e\in E,\ \exists\{S_{e,n}\}_{n\in\mathbb{N}}\subset\mathcal{S}\mid \lim_{n\to\infty}S_{e,n}(e)=T(e)\right\}.$$

Their respective elements are called local $\mathcal{S}$-maps and approximate local $\mathcal{S}$-maps.

Consider also the sets $2\text{-}\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$ and $2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$ given, respectively, by

$$\left\{\Delta\in F^E\colon \forall e,u\in E,\ \exists S_{e,u}\in\mathcal{S}\mid S_{e,u}(e)=\Delta(e),\ S_{e,u}(u)=\Delta(u)\right\}$$

and

$$\left\{\Delta\in F^E\colon \forall e,u\in E,\ \exists\{S_{e,u,n}\}_{n\in\mathbb{N}}\subset\mathcal{S}\mid \lim_{n\to\infty}S_{e,u,n}(e)=\Delta(e),\ \lim_{n\to\infty}S_{e,u,n}(u)=\Delta(u)\right\}.$$

Their members are referred to as 2-local $\mathcal{S}$-maps and approximate 2-local $\mathcal{S}$-maps, respectively.

In particular, when $\mathcal{S}$ is the set of all linear isometries from E onto E, the elements of $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$, $\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$, $2\text{-}\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$ and $2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$ are named local isometries, approximate local isometries, 2-local isometries and approximate 2-local isometries of E, respectively. The terminology local map, approximate local map, 2-local map and approximate 2-local map of E, replacing the word *map* by the terms *isometric reflection* and *generalized bi-circular projection* should be clear.

Finally, the set $\mathcal{S}$ is said to be algebraically reflexive (topologically reflexive) if $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})=\mathcal{S}$ (respectively, $\mathrm{ref}_{\mathrm{top}}(\mathcal{S})=\mathcal{S}$). Similarly, the set $\mathcal{S}$ is called 2-algebraically reflexive (2-topologically reflexive) if $2\text{-}\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})=\mathcal{S}$ (respectively, $2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathcal{S})=\mathcal{S}$).

We now fix some notation. Let $\mathbb{T}$ be the unit circle of $\mathbb{C}$. The symbols $\mathbf{1}$ and id stand for the function constantly equal to 1 on $\mathbb{D}$ and the identity function on $\mathbb{D}$. Given a Banach space E, we denote by $\mathcal{G}(E)$ the set of all surjective linear isometries of E. We denote the identity operator on a Banach space by Id.

3. Results. In the sequel, let $1 \le p \le \infty$ and let $S^p(\mathbb{D})$ be the derivative Hardy space of holomorphic functions f on $\mathbb{D}$ such that $f' \in H^p(\mathbb{D})$, with the norm

$$\|f\|_\Sigma = \|f\|_\infty + \|f'\|_p .$$

It is known that $S^p(\mathbb{D})$ is a unital subalgebra of the disc algebra on $\overline{\mathbb{D}}$ (see [5, Theorem 3.11]).

The following description of (non necessarily surjective) linear isometries of spaces $S^p(\mathbb{D})$ for $1 \le p < \infty$ was stated by Novinger and Oberlin [19].

THEOREM 3.1. ([19, Theorem 3.1]) *Let $1 \le p < \infty$ and let T be a linear isometry from $(S^p(\mathbb{D}), \|\cdot\|_\Sigma)$ into $(S^p(\mathbb{D}), \|\cdot\|_\Sigma)$. Then there exist a number $\lambda \in \mathbb{T}$ and a conformal automorphism $\phi\colon \mathbb{D} \to \mathbb{D}$ such that*

$$T(f)(z) = \lambda f(\phi(z)) \qquad (z \in \mathbb{D},\ f \in S^p(\mathbb{D})) .$$

If $p > 1$, ϕ is necessarily a rotation of $\mathbb{D}$.

Hence every linear isometry of $S^p(\mathbb{D})$ with $1 \le p < \infty$ is surjective. Moreover, a similar result holds in the case $p = \infty$.

THEOREM 3.2. (see [16, Theorem 1]) *A linear map $T\colon S^\infty(\mathbb{D}) \to S^\infty(\mathbb{D})$ is a surjective isometry with respect to the Σ-norm if and only if there exist constants $\lambda, \tau \in \mathbb{T}$ such that*

$$T(f)(z) = \lambda f(\tau z) \qquad (z \in \mathbb{D},\ f \in S^\infty(\mathbb{D})) .$$

Let us recall that every conformal automorphism ϕ from $\mathbb{D}$ onto itself must have the form $\phi = \tau\phi_a$ for some $\tau \in \mathbb{T}$ and some $a \in \mathbb{D}$, where

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z} \qquad (z \in \mathbb{D}) .$$

The set of conformal automorphisms of $\mathbb{D}$ forms a group $\mathrm{Aut}(\mathbb{D})$ under composition, and the subgroup of them that fix the origin coincides with the set of rotations of the complex plane about the origin.

The following result is an immediate consequence of the preceding results.

COROLLARY 3.3. *Let T be a surjective linear isometry of $S^p(\mathbb{D})$. Then $\|T(f)\|_\infty = \|f\|_\infty$ and $\|T(f)'\|_p = \|f'\|_p$ for all $f \in S^p(\mathbb{D})$.*

Proof. Notice that $T(f)(z) = \lambda f(\phi(z))$ for all $z \in \mathbb{D}$ and $f \in S^p(\mathbb{D})$, with λ and ϕ being as in Theorem 3.1. Let $f \in S^p(\mathbb{D})$. An easy calculation yields

$$\|T(f)\|_\infty = \sup_{z\in\mathbb{D}} |T(f)(z)| = \sup_{z\in\mathbb{D}} |\lambda f(\phi(z))| = \sup_{w\in\mathbb{D}} |f(w)| = \|f\|_\infty .$$

Since

$$\|T(f)\|_\infty + \|T(f)'\|_p = \|T(f)\|_\Sigma = \|f\|_\Sigma = \|f\|_\infty + \|f'\|_p ,$$

we infer that $\|T(f)'\|_p = \|f'\|_p$. □

We first establish the topological reflexivity of the isometry group of $S^p(\mathbb{D})$.

Theorem 3.4. *Every approximate local isometry of $S^p(\mathbb{D})$ is a surjective isometry.*

Proof. Let T be an approximate local isometry of $S^p(\mathbb{D})$. Hence, for each $f \in S^p(\mathbb{D})$, there exists a sequence $\{T_{f,n}\}_{n\in\mathbb{N}}$ in $\mathcal{G}(S^p(\mathbb{D}))$ such that

$$\lim_{n\to\infty} T_{f,n}(f) = T(f).$$

Since $\|T_{f,n}(f)\|_\Sigma = \|f\|_\Sigma$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|T_{f,n}(f)\|_\Sigma = \|T(f)\|_\Sigma ,$$

we have $\|f\|_\Sigma = \|T(f)\|_\Sigma$. Hence T is a linear isometry of $S^p(\mathbb{D})$. In the case $1 \le p < \infty$, Theorem 3.1 yields the surjectivity of T and the proof is finished. For the case $p = \infty$, we give some steps leading to the result.

Step 1. *It holds that $\|T(f)\|_\infty = \|f\|_\infty$ and $\|T(f)'\|_\infty = \|f'\|_\infty$ for all $f \in S^\infty(\mathbb{D})$.*

Let $f \in S^\infty(\mathbb{D})$. Take a sequence $\{T_{f,n}\}_{n\in\mathbb{N}}$ in $\mathcal{G}(S^\infty(\mathbb{D}))$ such that

$$\lim_{n\to\infty} T_{f,n}(f) = T(f).$$

Clearly, we have

$$\lim_{n\to\infty} \|T_{f,n}(f)\|_\infty = \|T(f)\|_\infty ,$$
$$\lim_{n\to\infty} \|T_{f,n}(f)'\|_\infty = \|T(f)'\|_\infty .$$

Since $\|T_{f,n}(f)\|_\infty = \|f\|_\infty$ and $\|T_{f,n}(f)'\|_\infty = \|f'\|_\infty$ for all $n \in \mathbb{N}$ by Corollary 3.3, the equalities of Step 1 hold.

Step 2. *There exists a number $\lambda \in \mathbb{T}$ such that $T(\mathbf{1}) = \lambda\mathbf{1}$.*

Step 1 yields $\|T(\mathbf{1})'\|_\infty = \|\mathbf{1}'\|_\infty = 0$ and $\|T(\mathbf{1})\|_\infty = \|\mathbf{1}\|_\infty = 1$. Hence $T(\mathbf{1})$ is a unimodular constant function on $\mathbb{D}$, and therefore $T(\mathbf{1}) = \lambda\mathbf{1}$ for some $\lambda \in \mathbb{T}$.

Step 3. *For every $f \in S^\infty(\mathbb{D})$, there exist sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ and $\{\tau_{f,n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ such that $\lim_{n\to\infty} \lambda_{f,n}(f \circ \tau_{f,n}\mathrm{id}) = T(f)$.*

Let $f \in S^\infty(\mathbb{D})$. By hypothesis there is a sequence $\{T_{f,n}\}_{n\in\mathbb{N}}$ in $\mathcal{G}(S^\infty(\mathbb{D}))$ such that

$$\lim_{n\to\infty} T_{f,n}(f) = T(f).$$

By Theorem 3.2, for each $n \in \mathbb{N}$, there are constants $\lambda_{f,n}, \tau_{f,n} \in \mathbb{T}$ such that $T_{f,n}(h) = \lambda_{f,n}(h \circ \tau_{f,n}\mathrm{id})$ for all $h \in S^\infty(\mathbb{D})$. Hence $\lim_{n\to\infty} \lambda_{f,n}(f \circ \tau_{f,n}\mathrm{id}) = T(f)$.

Step 4. *For each $z \in \mathbb{D}$, the mapping $S_z\colon S^\infty(\mathbb{D}) \to \mathbb{C}$ defined by*

$$S_z(f) = \overline{\lambda}T(f)(z) \qquad (f \in S^\infty(\mathbb{D})),$$

is a unital multiplicative linear functional.

Let $z \in \mathbb{D}$. Clearly, S_z is linear and, by Step 2, unital. To prove its multiplicativity, define $T_z\colon S^\infty(\mathbb{D}) \to \mathbb{C}$ by

$$T_z(f) = T(f)(z) \qquad (f \in S^\infty(\mathbb{D})).$$

Since T_z is linear and

$$|T_z(f)| = |T(f)(z)| \leq \|T(f)\|_\infty \leq \|T(f)\|_\Sigma = \|f\|_\Sigma$$

for all $f \in S^\infty(\mathbb{D})$, it follows that T_z is continuous. Take now any $f \in S^\infty(\mathbb{D})$. By Step 3, there exist sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ and $\{\tau_{f,n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ such that

$$T(f) = \lim_{n\to\infty} \lambda_{f,n}(f \circ \tau_{f,n}\mathrm{id}).$$

Since the convergence in the Σ-norm implies pointwise convergence, we have

$$T_z(f) = T(f)(z) = \lim_{n\to\infty} \lambda_{f,n} f(\tau_{f,n} z) \in \overline{\mathbb{T}\sigma(f)} = \mathbb{T}\sigma(f),$$

where $\sigma(f)$ denotes the spectrum of f and $\overline{\mathbb{T}\sigma(f)}$ the closure of $\mathbb{T}\sigma(f)$. Applying a spherical variant of the Gleason–Kahane–Żelazko theorem stated in [14, Proposition 2.2], we conclude that $S_z = \overline{T_z(\mathbf{1})}T_z$ is multiplicative.

Step 5. *There exists a constant $\tau \in \mathbb{T}$ such that $T(f)(z) = \lambda f(\tau z)$ for all $z \in \mathbb{D}$ and $f \in S^\infty(\mathbb{D})$.*

Using Step 4, we deduce easily that the mapping $S\colon S^\infty(\mathbb{D}) \to S^\infty(\mathbb{D})$ defined by

$$S(f)(z) = \overline{\lambda} T(f)(z) \qquad (f \in S^\infty(\mathbb{D}),\ z \in \mathbb{D})$$

is a unital algebra homomorphism. Since the maximal ideal space of $S^\infty(\mathbb{D})$ is homeomorphic to $\overline{\mathbb{D}}$ (see a proof in [20, Theorem 17]), Gelfand theory shows that S is automatically continuous and induces a continuous map $\widehat{\phi}\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that

$$S(f)(z) = f(\widehat{\phi}(z)) \qquad (f \in S^\infty(\mathbb{D}),\ z \in \mathbb{D}).$$

Let ϕ be the restriction map of $\widehat{\phi}$ to $\mathbb{D}$. Hence

$$T(f)(z) = \lambda f(\phi(z)) \qquad (f \in S^\infty(\mathbb{D}),\ z \in \mathbb{D}).$$

Clearly, $\phi = \overline{\lambda} T(\mathrm{id}) \in S^\infty(\mathbb{D})$. We now prove that ϕ is a rotation of $\mathbb{D}$. By Step 3, we can take sequences $\{\lambda_{\mathrm{id},n}\}_{n\in\mathbb{N}}$ and $\{\tau_{\mathrm{id},n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ such that

$$\lim_{n\to\infty} \lambda_{\mathrm{id},n}\tau_{\mathrm{id},n}\mathrm{id} = T(\mathrm{id}).$$

Taking subsequences, we can suppose that $\{\lambda_{\mathrm{id},n}\}_{n\in\mathbb{N}} \to \lambda_{\mathrm{id}}$ and $\{\tau_{\mathrm{id},n}\}_{n\in\mathbb{N}} \to \tau_{\mathrm{id}}$ for some $\lambda_{\mathrm{id}}, \tau_{\mathrm{id}} \in \mathbb{T}$. We have

$$\phi(z) = \overline{\lambda} T(\mathrm{id})(z) = \overline{\lambda} \lim_{n\to\infty} \lambda_{\mathrm{id},n}\tau_{\mathrm{id},n}\mathrm{id}(z) = \overline{\lambda}\lambda_{\mathrm{id}}\tau_{\mathrm{id}} z$$

for all $z \in \mathbb{D}$, that is, $\phi = \tau\,\mathrm{id}$ with $\tau = \overline{\lambda}\lambda_{\mathrm{id}}\tau_{\mathrm{id}} \in \mathbb{T}$. The proof of Theorem 3.4 is now complete in view of Theorem 3.2. □

Using Theorem 3.4 we now deduce that the set $\mathcal{G}(S^p(\mathbb{D}))$ is 2-topologically reflexive.

Corollary 3.5. *Every approximate 2-local isometry of $S^p(\mathbb{D})$ is a surjective linear isometry.*

Proof. Let $\Delta \in 2\text{-ref}_{\mathrm{top}}(\mathcal{G}(S^p(\mathbb{D})))$. We claim that for each $z \in \mathbb{D}$, the functional $\Delta_z \colon S^p(\mathbb{D}) \to \mathbb{C}$ given by

$$\Delta_z(f) = \Delta(f)(z) \qquad (f \in S^p(\mathbb{D})),$$

is linear. Applying the spherical version of the Kowalski–Słodkowski theorem [14, Proposition 3.2], it is sufficient to show that Δ_z is homogeneous and satisfies that $\Delta_z(f) - \Delta_z(g) \in \mathbb{T}\sigma(f-g)$ for every $f, g \in S^p(\mathbb{D})$. The homogeneity follows immediately since Δ is an approximate 2-local isometry. For the spectral condition, let $f, g \in S^p(\mathbb{D})$, take two sequences $\{\lambda_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ and $\{\phi_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Aut}(\mathbb{D})$ such that

$$\lim_{n\to\infty} \lambda_{f,g,n} f(\phi_{f,g,n}(z)) = \Delta(f)(z),$$
$$\lim_{n\to\infty} \lambda_{f,g,n} g(\phi_{f,g,n}(z)) = \Delta(g)(z),$$

and we obtain that

$$\begin{aligned}\Delta_z(f) - \Delta_z(g) &= \lim_{n\to\infty} \lambda_{f,g,n}\left[f(\phi_{f,g,n}(z)) - g(\phi_{f,g,n}(z))\right] \\ &= \lim_{n\to\infty} \lambda_{f,g,n}(f-g)(\phi_{f,g,n}(z)) \in \overline{\mathbb{T}\sigma(f-g)} = \mathbb{T}\sigma(f-g).\end{aligned}$$

This proves our claim. Since z was arbitrary in $\mathbb{D}$, it follows that Δ is linear. It is obvious then that $\Delta \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}(S^p(\mathbb{D})))$, and therefore $\Delta \in \mathcal{G}(S^p(\mathbb{D}))$ by Theorem 3.4. This proves the corollary. □

Theorem 3.4 can be applied to state the topological reflexivity of other classes of linear transformations on $S^p(\mathbb{D})$.

Let us recall that an isometric reflection of a Banach space E is a linear isometry $T \colon E \to E$ such that $T^2 = \mathrm{Id}$. We denote by $\mathcal{G}^2(E)$ the set of all isometric reflections of E.

The next theorem provides a characterization of isometric reflections on $S^p(\mathbb{D})$.

Theorem 3.6. *A map $T \colon S^p(\mathbb{D}) \to S^p(\mathbb{D})$ is an isometric reflection if and only if there exist a constant $\lambda \in \{-1, 1\}$ and a map $\phi \in \mathrm{Aut}(\mathbb{D})$ with $\phi^2 = \mathrm{id}$ such that*

$$T(f)(z) = \lambda f(\phi(z)) \qquad (f \in S^p(\mathbb{D}),\ z \in \mathbb{D}).$$

Proof. Let $T \in \mathcal{G}^2(S^p(\mathbb{D}))$. By Theorems 3.1 and 3.2, there are a number $\lambda \in \mathbb{T}$ and a map $\phi \in \mathrm{Aut}(\mathbb{D})$ such that $T(f) = \lambda(f \circ \phi)$ for all $f \in S^p(\mathbb{D})$. Therefore $T(\mathbf{1}) = \lambda\mathbf{1}$ and $T^2(\mathbf{1}) = \lambda^2\mathbf{1}$. Since $T^2 = \mathrm{Id}$, it follows that $\mathbf{1} = \lambda^2\mathbf{1}$ and so $\lambda \in \{\pm 1\}$. Moreover, since $T(\mathrm{id}) = \lambda\phi$ and $T(\phi) = \lambda\phi^2$, we have

$$\mathrm{id} = T^2(\mathrm{id}) = T(\lambda\phi) = \lambda T(\phi) = \lambda^2\phi^2 = \phi^2.$$

The sufficiency is obvious. □

REMARK 3.7. The set of all involutive conformal automorphisms of $\mathbb{D}$ is $\{\phi_a : a \in \mathbb{D}\} \cup \{\mathrm{id}\}$. Indeed, let $\phi = \lambda\phi_a$ with $\lambda \in \mathbb{T}$ and $a \in \mathbb{D}$ such that $\phi^2 = \mathrm{id}$. We have

$$a = \phi^2(a) = \lambda\phi_a(\lambda\phi_a(a)) = \lambda\phi_a(0) = \lambda a,$$

which implies $\lambda = 1$ or $a = 0$. In the first case, we obtain $\phi = \phi_a$. In the other one, $\phi = \lambda\phi_0 = \lambda(-\mathrm{id})$ and since $\phi^2 = \mathrm{id}$, we infer that $\lambda^2\mathrm{id} = \mathrm{id}$, hence $\lambda = \pm 1$ and thus $\phi = \pm\mathrm{id}$. Therefore

$$\left\{\phi \in \mathrm{Aut}(\mathbb{D}) \colon \phi^2 = \mathrm{id}\right\} \subseteq \{\phi_a : a \in \mathbb{D}\} \cup \{\mathrm{id}\}$$

and the converse inclusion is immediate.

We next prove that the set $\mathcal{G}^2(S^p(\mathbb{D}))$ is topologically reflexive.

COROLLARY 3.8. *Every approximate local isometric reflection of $S^p(\mathbb{D})$ is an isometric reflection.*

Proof. Let $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(S^p(\mathbb{D})))$. By Theorem 3.6, for every $f \in S^p(\mathbb{D})$, there are two sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ in $\{-1, 1\}$ and $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Aut}(\mathbb{D})$ with $\phi_{f,n}^2 = \mathrm{id}$ for all $n \in \mathbb{N}$ satisfying

$$\lim_{n\to\infty} \lambda_{f,n}(f \circ \phi_{f,n}) = T(f).$$

Hence $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}(S^p(\mathbb{D})))$ and, by Theorem 3.4, $T \in \mathcal{G}(S^p(\mathbb{D})))$. So, by Theorems 3.1 and 3.2, there exist a number $\lambda \in \mathbb{T}$ and a map $\phi \in \mathrm{Aut}(\mathbb{D})$ such that

$$T(f) = \lambda(f \circ \phi) \qquad (f \in S^p(\mathbb{D})).$$

Hence $\lambda\mathbf{1} = T(\mathbf{1}) = \lim_{n\to\infty} \lambda_{\mathbf{1},n}\mathbf{1}$ and therefore $\lambda = \lim_{n\to\infty} \lambda_{\mathbf{1},n}$. Since $\lambda_{\mathbf{1},n} \in \{\pm 1\}$ for all $n \in \mathbb{N}$, it is deduced easily that $\lambda \in \{\pm 1\}$.

We now prove that $\phi^2 = \mathrm{id}$. We have two sequences $\{\lambda_{\mathrm{id},n}\}_{n\in\mathbb{N}}$ in $\{-1, 1\}$ and $\{\phi_{\mathrm{id},n}\}_{n\in\mathbb{N}}$ in $\mathrm{Aut}(\mathbb{D})$ with $\phi_{\mathrm{id},n}^2 = \mathrm{id}$ for all $n \in \mathbb{N}$ satisfying

$$\phi = \lambda T(\mathrm{id}) = \lambda \lim_{n\to\infty} \lambda_{\mathrm{id},n}\phi_{\mathrm{id},n}.$$

Taking a subsequence, we can suppose that $\{\lambda_{\mathrm{id},n}\}_{n\in\mathbb{N}}$ converges to some $\lambda_{\mathrm{id}} \in \{-1, 1\}$ and therefore

$$\phi = \lambda\lambda_{\mathrm{id}} \lim_{n\to\infty} \phi_{\mathrm{id},n}.$$

Assume first $p \neq 1$. Then each $\phi_{\mathrm{id},n}$ is an involutive rotation of $\mathbb{D}$, that is, $\phi_{\mathrm{id},n} = \tau_n \mathrm{id}$ with $\tau_n \in \{-1,1\}$. Taking a subsequence, we can assume that $\lim_{n\to\infty} \tau_n = \tau$ for some $\tau \in \{-1,1\}$ and thus $\lim_{n\to\infty} \phi_{\mathrm{id},n} = \tau\, \mathrm{id}$. Therefore

$$\phi = \lambda\lambda_{\mathrm{id}} \lim_{n\to\infty} \phi_{\mathrm{id},n} = \lambda\lambda_{\mathrm{id}}\tau\, \mathrm{id},$$

hence $\phi = \mathrm{id}$ or $\phi = -\mathrm{id}$, and thus $\phi^2 = \mathrm{id}$. Hence T is an isometric reflection of $S^p(\mathbb{D})$ by Theorem 3.6.

Assume now $p = 1$. Take $f = \mathrm{id}+\mathbf{1}$. As above, we can take sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ in $\{-1,1\}$ and $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Aut}(\mathbb{D})$ with $\phi_{f,n}^2 = \mathrm{id}$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \lambda_{f,n}(f \circ \phi_{f,n}) = T(f) = \lambda(f \circ \phi),$$

where $\lambda \in \{\pm 1\}$ and $\phi \in \mathrm{Aut}(\mathbb{D})$. We can assume that $\phi_{f,n} = \phi_{a_n}$ with $a_n \in \mathbb{D}$ for all $n \in \mathbb{N}$. Clearly, we have

$$\lim_{n\to\infty} \left\|\lambda_{f,n}(\phi_{a_n} + \mathbf{1}) - \lambda(\phi + \mathbf{1})\right\|_\infty = 0.$$

Hence we can assume that $\lambda\lambda_{f,n} = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \left\|\phi_{a_n} - \phi\right\|_\infty = 0.$$

We claim that $\phi^2 = \mathrm{id}$. Indeed, given $z \in \mathbb{D}$ and $\varepsilon > 0$, since ϕ is continuous and $|\phi_{a_n}(z) - \phi(z)| \to 0$ as $n \to +\infty$, we can take $m \in \mathbb{N}$ such that $\left\|\phi_{a_n} - \phi\right\|_\infty < \varepsilon/2$ and $|\phi(\phi_{a_n}(z)) - \phi^2(z)| < \varepsilon/2$ for all $n \geq m$. Then, for any $n \geq m$, we have

$$\left|\phi_{a_n}^2(z) - \phi^2(z)\right| \leq |\phi_{a_n}(\phi_{a_n}(z)) - \phi(\phi_{a_n}(z))| + \left|\phi(\phi_{a_n}(z)) - \phi^2(z)\right| < \varepsilon.$$

Hence $\left|\phi_{a_n}^2(z) - \phi^2(z)\right| \to 0$ as $n \to +\infty$. Since $\phi_{a_n}^2(z) = z$ for all $n \in \mathbb{N}$, we deduce that $\phi^2(z) = z$ and this proves our claim. □

We now deduce that $\mathcal{G}^2(S^p(\mathbb{D}))$ is 2-topologically reflexive.

Corollary 3.9. *Every approximate 2-local isometric reflection of $S^p(\mathbb{D})$ is an isometric reflection.*

Proof. Let $\Delta \in 2\text{-ref}_{\mathrm{top}}(\mathcal{G}^2(S^p(\mathbb{D})))$. A similar proof to that of Corollary 3.5 shows that Δ is linear. Clearly, $\Delta \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(S^p(\mathbb{D})))$, and therefore $\Delta \in \mathcal{G}^2(S^p(\mathbb{D}))$ by Corollary 3.8. □

Let us recall that a generalized bi-circular projection of a Banach space E is a linear projection $P\colon E \to E$ such that $P + \lambda(\mathrm{Id} - P)$ is a linear surjective isometry for some $\lambda \in \mathbb{T} \setminus \{1\}$. We denote by $\mathcal{GBP}(E)$ the set of all generalized bi-circular projections of E.

The next theorem describes the class of generalized bi-circular projections on $S^p(\mathbb{D})$.

THEOREM 3.10. *A map* $P\colon S^p(\mathbb{D}) \to S^p(\mathbb{D})$ *is a generalized bi-circular projection if and only if there exist a number* $\lambda \in \{-1, 1\}$ *and a map* $\phi \in \mathrm{Aut}(\mathbb{D})$ *with* $\phi^2 = \mathrm{id}$ *such that*

$$P(f)(z) = \frac{1}{2}\left[f(z) + \lambda f(\phi(z))\right] \qquad (z \in \mathbb{D},\ f \in S^p(\mathbb{D}))\,.$$

Proof. The sufficiency follows easily. For the necessity, let $P \in \mathcal{GBP}(S^p(\mathbb{D}))$. Then $P + \lambda(\mathrm{Id} - P)$ is a surjective linear isometry of $S^p(\mathbb{D})$ for some $\lambda \in \mathbb{T}$ with $\lambda \neq 1$. By Theorems 3.1 and 3.2, we can take a constant $\tau \in \mathbb{T}$ and a map $\phi \in \mathrm{Aut}(\mathbb{D})$ such that

$$[P + \lambda(\mathrm{Id} - P)](f)(z) = \tau f(\phi(z)) \qquad (f \in S^p(\mathbb{D}),\ z \in \mathbb{D}),$$

which gives the following formula for P:

$$P(f)(z) = (1 - \lambda)^{-1}[-\lambda f(z) + \tau f(\phi(z))] \qquad (f \in S^p(\mathbb{D}),\ z \in \mathbb{D}).$$

Using the equality $P^2 = P$, we obtain the equation:

$$\lambda f(z) - (\lambda + 1)\tau f(\phi(z)) + \tau^2 f(\phi^2(z)) = 0 \qquad (f \in S^p(\mathbb{D}),\ z \in \mathbb{D}).$$

Let us suppose that there exists a point $z_0 \in \mathbb{D}$ such that $z_0 \neq \phi(z_0)$ and $z_0 \neq \phi^2(z_0)$. We choose a polynomial g such that $g(z_0) = 1$ and $g(\phi(z_0)) = g(\phi^2(z_0)) = 0$. The equation, evaluated at $f = g$ and $z = z_0$, yields $\lambda = 0$, a contradiction. Thus $\phi(z_0) = z_0$ or $\phi^2(z_0) = z_0$. In any case, $\phi^2(z) = z$ for all $z \in \mathbb{D}$.

Taking $f = \mathbf{1}$ in the equation, we obtain $\tau^2 - (\lambda + 1)\tau + \lambda = 0$ and therefore $\tau = \lambda$ or $\tau = 1$. If $\phi = \mathrm{id}$, from the formula of P, we deduce that $P(f)(z) = 0$ (when $\tau = \lambda$) or $P(f)(z) = f(z)$ (when $\tau = 1$) for all $f \in S^p(\mathbb{D})$ and $z \in \mathbb{D}$.

If $\phi \neq \mathrm{id}$, there is some $z_0 \in \mathbb{D}$ for which $\phi(z_0) \neq z_0$. Take a polynomial h such that $h(z_0) = 1$ and $h(\phi(z_0)) = 0$. Substituting $f = h$ and $z = z_0$ in the equation, we get $\lambda + \tau^2 = 0$. Since $\tau = \lambda$ or $\tau = 1$, it follows that $\lambda = -1$ and $\tau^2 = 1$. Then the formula of P shows that $P(f)(z) = (1/2)[f(z) + \tau f(\phi(z))]$ for all $f \in S^p(\mathbb{D})$ and $z \in \mathbb{D}$. □

We now show that the set $\mathcal{GBP}(S^p(\mathbb{D}))$ is both topologically reflexive and 2-topologically reflexive.

COROLLARY 3.11. *Every approximate local (respectively, 2-local) generalized bi-circular projection of* $S^p(\mathbb{D})$ *is a generalized bi-circular projection.*

Proof. Let $P \in \mathrm{ref}_{\mathrm{top}}(\mathcal{GBP}(S^p(\mathbb{D})))$. Theorem 3.10 asserts that for every $f \in S^p(\mathbb{D})$, there are sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ in $\{-1, 1\}$ and $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Aut}(\mathbb{D})$ with $\phi_{f,n}^2 = \mathrm{id}$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{2}\left[f + \lambda_{f,n}(f \circ \phi_{f,n})\right] = P(f).$$

Hence, for every $f \in S^p(\mathbb{D})$, we have

$$\lim_{n\to\infty} \lambda_{f,n}(f \circ \phi_{f,n}) = 2P(f) - f,$$

and so $2P - \mathrm{Id} \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(S^p(\mathbb{D})))$. Hence $2P - \mathrm{Id} \in \mathcal{G}^2(S^p(\mathbb{D}))$ by Corollary 3.8, and therefore $P \in \mathcal{GBP}(S^p(\mathbb{D}))$ by Theorem 3.10.

Let $\Delta \in 2\text{-ref}_{\mathrm{top}}(\mathcal{GBP}(S^p(\mathbb{D})))$. For any $f, g \in S^p(\mathbb{D})$, there are sequences $\{\lambda_{f,g,n}\}_{n\in\mathbb{N}}$ in $\{-1, 1\}$ and $\{\phi_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathrm{Aut}(\mathbb{D})$ with $\phi_{f,g,n}^2 = \mathrm{id}$ for all $n \in \mathbb{N}$ satisfying

$$\lim_{n\to\infty} \frac{1}{2}\left[f + \lambda_{f,g,n}(f \circ \phi_{f,g,n})\right] = \Delta(f),$$
$$\lim_{n\to\infty} \frac{1}{2}\left[g + \lambda_{f,g,n}(g \circ \phi_{f,g,n})\right] = \Delta(g).$$

Hence, for every $f, g \in S^p(\mathbb{D})$, we have

$$\lim_{n\to\infty} \lambda_{f,g,n}(f \circ \phi_{f,g,n}) = 2\Delta(f) - f,$$
$$\lim_{n\to\infty} \lambda_{f,g,n}(g \circ \phi_{f,g,n}) = 2\Delta(g) - g,$$

and this says that $2\Delta - \mathrm{Id} \in 2\text{-ref}_{\mathrm{top}}(\mathcal{G}^2(S^p(\mathbb{D})))$. Since $\mathcal{G}^2(S^p(\mathbb{D}))$ is 2-topologically reflexive by Corollary 3.9, it follows that $2\Delta - \mathrm{Id} \in \mathcal{G}^2(S^p(\mathbb{D}))$, and thus $\Delta \in \mathcal{GBP}(S^p(\mathbb{D}))$. □

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