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Projections in the Convex Hull of Three Isometries on Absolutely Continuous Function Spaces

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Abstract

In this paper, we prove that any projection in the convex hull of three surjective linear isometries on $\mathrm{AC}(X)$ is a generalized bi-circular projection, where $\mathrm{AC}(X)$ denotes the Banach space of all absolutely continuous functions on a compact subset of $\mathbb{R}$ with at least two points. We also show that the trivial projections are the only projections on $\mathrm{AC}(X)$ which can be represented as the average of three surjective linear isometries.

Keywords Convex combination of isometries · Absolutely continuous function · Generalized bi-circular projection · Surjective linear isometry

Mathematics Subject Classification Primary 47B38; Secondary 46J10 · 47B33

1 Introduction

The study of projections in the convex hull of surjective linear isometries was initiated by Botelho [2], who proved that any projection on $C(X)$ expressed as the convex combination of two surjective linear isometries is a generalized bi-circular projection, where $C(X)$ is the Banach space of all complex-valued continuous functions on a compact connected Hausdorff space X. Let us recall that a projection P on a Banach space is said to be a *generalized bi-circular projection* if there is a unimodular scalar λ, different from 1, such that $P + \lambda(\mathrm{Id} - P)$ is an isometry, where Id is the identity operator [5] Motivated by [2] such projections were studied on spaces of Banach-

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valued continuous functions [3], spaces of Lipschitz functions [4], spaces of Hilbert-valued analytic functions [6] and spaces of absolutely continuous functions [9].

Meantime, some work has been done to characterize projections on Banach spaces which are in the convex hull of three surjective linear isometries. One can see the results dealing with such projections defined on continuous function spaces $C(X)$ for a compact connected Hausdorff space X in [1] and Lipschitz function spaces $\mathrm{Lip}(X)$ and $\mathrm{lip}(X^\alpha)$ with $0<\alpha<1$ for a compact 1-connected metric space X with diameter at most 2 in [4].

In this paper, we study projections in the convex hull of three surjective linear isometries on absolutely continuous function spaces $\mathrm{AC}(X)$ for a subset X of the real line with at least two points. More precisely, it is shown that such a projection must be a generalized bi-circular projection. We also prove that the trivial projections are the only projections on $\mathrm{AC}(X)$ given by the average of three surjective linear isometries.

2 Preliminaries

In this section, we recall some definitions and fix notations. Throughout the rest of the paper, X is a subset of $\mathbb{R}$ with at least two points.

A function $f\colon X\to\mathbb{C}$ is said to be *absolutely continuous* if given $\epsilon>0$, there exists a $\delta>0$ such that for every finite family of non-overlapping open intervals $\{(a_i,b_i)\colon i=1,\ldots,n\}$ whose extreme points belong to X and $\sum_{i=1}^n(b_i-a_i)<\delta$, we have $\sum_{i=1}^n|f(b_i)-f(a_i)|<\epsilon$.

We denote by $\mathrm{AC}(X)$ the Banach space of all absolutely continuous functions $f\colon X\to\mathbb{C}$, equipped with the sum norm

$$\|f\|_\Sigma:=\|f\|_\infty+\mathcal{V}(f),$$

where

$$\|f\|_\infty:=\sup\{|f(x)|\colon x\in X\}$$

and $\mathcal{V}(f)$ is the *total variation of* f defined by

$$\mathcal{V}(f):=\sup\left\{\sum_{i=1}^n|f(x_i)-f(x_{i-1})|\colon\right.$$
$$\left.n\in\mathbb{N},\ x_0,x_1,\ldots,x_n\in X,\ x_0<x_1<\cdots<x_n\right\}.$$

It is worth pointing out that for every closed subset $F\subseteq X$ and every point $x\in X\backslash F$, there exists a function $f\in\mathrm{AC}(X)$ such that $f(x)\neq 0$ and $f|_F=0$, where $f|_F$ denotes the restriction function of f to F.

Now, let us mention that by id_X and 1_X we mean the identity function and the function constantly 1 on X, respectively. We denote by Id the identity operator on

AC(X). Meantime, let MH(X) denote the set of all monotonic absolutely continuous homeomorphisms of X onto itself. As usual, $\mathbb{T}$ stands for the set of all unimodular complex numbers.

Next, we state a result due to Pathak [11] which characterizes surjective linear isometries of absolutely continuous function spaces AC(X) (see also [10, Example 7] and [8, Corollary 4.3]).

Theorem 2.1 [11, Theorem 2.10 and Lemma 2.3] *A map* $T\colon \mathrm{AC}(X) \to \mathrm{AC}(X)$ *is a surjective linear isometry if and only if there exist a map* $\varphi \in \mathrm{MH}(X)$ *and a scalar* $\lambda \in \mathbb{T}$ *such that* $Tf = \lambda f \circ \varphi$ *for all* $f \in \mathrm{AC}(X)$. □

Finally, we bring the following result saying that generalized bi-circular projections are the only projections on AC(X) written as the convex combination of two surjective linear isometries.

Theorem 2.2 [9, Remark 3.3] *A projection* P *on* AC(X) *is in the convex hull of two surjective linear isometries if and only if there exist a number* $\lambda \in \{1, -1\}$ *and a map* $\varphi \in \mathrm{MH}(X)$ *with* $\varphi^2 = \mathrm{id}_X$ *such that* $Pf = (1/2)[f + \lambda f \circ \varphi]$ *for all* $f \in \mathrm{AC}(X)$. □

3 Results

Our purpose is to characterize those projections on AC(X) which belong to the convex hull of three surjective linear isometries.

Throughout this section, we assume that, unless explicitly stated, $P = \sum_{i=1}^{3} \alpha_i T_i$ is a projection on AC(X), where $0 < \alpha_i < 1$ with $\sum_{i=1}^{3} \alpha_i = 1$ and $T_i f = \lambda_i f \circ \varphi_i$ for all $f \in \mathrm{AC}(X)$, with $\lambda_i \in \mathbb{T}$ and $\varphi_i \in \mathrm{MH}(X)$ for $i = 1, 2, 3$ (see Theorem 2.1).

Since P is a projection, we have

$$\begin{aligned}
&\alpha_1\lambda_1 \Big[\alpha_1\lambda_1 f(\varphi_1^2(x)) + \alpha_2\lambda_2 f(\varphi_2(\varphi_1(x))) + \alpha_3\lambda_3 f(\varphi_3(\varphi_1(x)))\Big] \\
&\quad + \alpha_2\lambda_2 \Big[\alpha_1\lambda_1 f(\varphi_1(\varphi_2(x))) + \alpha_2\lambda_2 f(\varphi_2^2(x)) + \alpha_3\lambda_3 f(\varphi_3(\varphi_2(x)))\Big] \\
&\quad + \alpha_3\lambda_3 \Big[\alpha_1\lambda_1 f(\varphi_1(\varphi_3(x))) + \alpha_2\lambda_2 f(\varphi_2(\varphi_3(x))) + \alpha_3\lambda_3 f(\varphi_3^2(x))\Big] \\
&\quad = \alpha_1\lambda_1 f(\varphi_1(x)) + \alpha_2\lambda_2 f(\varphi_2(x)) + \alpha_3\lambda_3 f(\varphi_3(x)).
\end{aligned} \tag{3.1}$$

for all $f \in \mathrm{AC}(X)$ and $x \in X$.

For the proofs of our main results, we will need a series of lemmas.

Lemma 3.1 *The following assertions hold:*

(1) *We have either* $\sum_{i=1}^{3} \alpha_i\lambda_i = 1$, *or* $\sum_{i=1}^{3} \alpha_i\lambda_i = 0$. *Moreover, if* $\sum_{i=1}^{3} \alpha_i\lambda_i = 1$, *then* $\lambda_1 = \lambda_2 = \lambda_3 = 1$.
(2) *For any* $i, j \in \{1, 2, 3\}$ *with* $i \neq j$, *we have* $0 \neq \alpha_i\lambda_i + \alpha_j\lambda_j \neq 1$.
(3) *For any* $i, j \in \{1, 2, 3\}$, *we have* $\alpha_i\lambda_i - \alpha_j\lambda_j \neq 1$.
(4) *If* $P \neq 0$, *then* $\alpha_i \leq 1/2$ *for all* $i \in \{1, 2, 3\}$.

Proof (1) If we take $f = 1_X$ in Eq. (3.1), then $(\sum_{i=1}^{3} \alpha_i \lambda_i)^2 = \sum_{i=1}^{3} \alpha_i \lambda_i$, which implies that either $\sum_{i=1}^{3} \alpha_i \lambda_i = 1$ or $\sum_{i=1}^{3} \alpha_i \lambda_i = 0$. Moreover, since $\mathbb{C}$ is strictly convex, it is clear that $\lambda_1 = \lambda_2 = \lambda_3 = 1$ if $\sum_{i=1}^{3} \alpha_i \lambda_i = 1$.

(2) Let us assume that $\alpha_1 \lambda_1 + \alpha_2 \lambda_2 = 0$. It follows from Part (1) that $\alpha_3 = 0$ or $\alpha_3 = 1$, a contradiction. Furthermore, if $\alpha_1 \lambda_1 + \alpha_2 \lambda_2 = 1$, then $1 \leq \alpha_1 + \alpha_2$, which yields $\alpha_3 = 0$, and it is impossible.

(3) If $\alpha_i \lambda_i - \alpha_j \lambda_j = 1$ for some $i, j \in \{1, 2, 3\}$ (clearly, $i \neq j$), then we get $1 \leq \alpha_i + \alpha_j$, which is impossible as above.

(4) As $P \neq 0$, take $f \in \mathrm{AC}(X)$ such that $\|f\|_\Sigma = 1$ and $Pf = 0$. Then $\alpha_1 T_1 f + \alpha_2 T_2 f = -\alpha_3 T_3 f$. Hence, by taking norms, it follows that $\alpha_1 + \alpha_2 \geq \alpha_3$. Similarly, $\alpha_2 + \alpha_3 \geq \alpha_1$ and $\alpha_1 + \alpha_3 \geq \alpha_2$. Thus, as $\sum_{i=1}^{3} \alpha_i = 1$, it is easily inferred that $\alpha_i \leq 1/2$ for all $i \in \{1, 2, 3\}$. □

Here let us fix some notation. For each $x \in X$, we set

$$\mathcal{S}_x = \{\varphi_1(x), \varphi_2(x), \varphi_3(x)\}.$$

If $\mathrm{Card}(S_x)$ denotes the cardinality of S_x, we put

$$X_0 := \{x \in X \colon \varphi_1(x) = \varphi_2(x) = \varphi_3(x)\} = \{x \in X \colon \mathrm{Card}(\mathcal{S}_x) = 1\}.$$

We also express the set $X \backslash X_0$ as a partition of the following sets:

$$\begin{aligned}
A_i &= \left\{x \in X \colon x \neq \varphi_i(x) \neq \varphi_j(x) = \varphi_k(x) \neq x\right\}, \\
B_i &= \left\{x \in X \colon x = \varphi_j(x) = \varphi_k(x) \neq \varphi_i(x)\right\}, \\
C_i &= \left\{x \in X \colon x = \varphi_i(x) \neq \varphi_j(x) = \varphi_k(x)\right\}, \\
D_i &= \left\{x \in X \colon x = \varphi_i(x) \neq \varphi_j(x) \neq \varphi_k(x) \neq x\right\}, \\
E &= \{x \in X \colon \mathrm{Card}(\mathcal{S}_x \cup \{x\}) = 4\},
\end{aligned}$$

where $i, j, k \in \{1, 2, 3\}$ and $k \neq i \neq j \neq k$.

Lemma 3.2 *If $X = X_0$, then $P = 0$ or $P = \mathrm{Id}$.*

Proof For simplicity, take $\varphi = \varphi_1$. Since $\varphi_1 = \varphi_2 = \varphi_3$, we have $Pf = (\sum_{i=1}^{3} \alpha_i \lambda_i) f \circ \varphi$ for all $f \in \mathrm{AC}(X)$. In view of Lemma 3.1, if $\sum_{i=1}^{3} \alpha_i \lambda_i = 0$, then $P = 0$. Otherwise, if $\sum_{i=1}^{3} \alpha_i \lambda_i = 1$, we have $Pf = f \circ \varphi$ for all $f \in \mathrm{AC}(X)$. Since P is a projection, we get that $f \circ \varphi^2 = f \circ \varphi$ for all $f \in \mathrm{AC}(X)$. Hence, considering $f = \mathrm{id}_X$, from the above relation it is deduced that $\varphi^2 = \varphi$. Now, taking into account the injectivity of φ, it follows that $\varphi = \mathrm{id}_X$, and thus, $P = \mathrm{Id}$. □

To avoid discussing on trivial cases, in the next lemmas it is assumed that P is a proper projection.

Lemma 3.3 *We have $E = \emptyset$ and $A_i = \emptyset$ for every $i \in \{1, 2, 3\}$.*

Proof First, it is shown that $A_i = \emptyset$ for $i \in \{1, 2, 3\}$. We assume, without loss of generality, that $i = 3$. Suppose, on the contrary, that $A_3 \neq \emptyset$. Let $x \in A_3$. Then we have $x \neq \varphi_1(x) = \varphi_2(x) \neq \varphi_3(x) \neq x$. Since $\varphi_1(x)$ does not belong to the set

$$F = \left\{\varphi_3(x), \varphi_1(\varphi_3(x)), \varphi_2(\varphi_3(x)), \varphi_1^2(x), \varphi_2^2(x)\right\},$$

we can choose $f \in \mathrm{AC}(X)$ with $f(\varphi_1(x)) = 1$ and $f|_F = 0$. By considering f in Eq. (3.1), we get

$$\begin{aligned}\alpha_1\lambda_1 \left[\alpha_3\lambda_3 f(\varphi_3(\varphi_1(x)))\right] &+ \alpha_2\lambda_2 \left[\alpha_3\lambda_3 f(\varphi_3(\varphi_2(x)))\right]\\ &+ \alpha_3\lambda_3 \left[\alpha_3\lambda_3 f(\varphi_3^2(x))\right] = \alpha_1\lambda_1 + \alpha_2\lambda_2,\end{aligned}$$

that is,

$$\alpha_3\lambda_3 \left[\alpha_1\lambda_1 + \alpha_2\lambda_2\right] f(\varphi_3(\varphi_1(x))) + \alpha_3^2\lambda_3^2 f(\varphi_3^2(x)) = \alpha_1\lambda_1 + \alpha_2\lambda_2.$$

Since $\alpha_1\lambda_1 + \alpha_2\lambda_2 \neq 0$ by Lemma 3.1, the above relation shows that at least one of the two points $\varphi_3(\varphi_1(x))$ and $\varphi_3^2(x)$ must be equal to $\varphi_1(x)$.

If $\varphi_3^2(x) \neq \varphi_1(x) = \varphi_3(\varphi_1(x))$, then by taking a function $f \in \mathrm{AC}(X)$ with $0 \leq f \leq 1$, $f(\varphi_1(x)) = 1$ and $f = 0$ on $F \cup \{\varphi_3^2(x)\}$, it follows that $\alpha_3\lambda_3[\alpha_1\lambda_1 + \alpha_2\lambda_2] = \alpha_1\lambda_1 + \alpha_2\lambda_2$, which yields $\alpha_3\lambda_3 = 1$, and consequently, $\alpha_3 = 1$, a contradiction.

Now, if $\varphi_3^2(x) = \varphi_1(x) \neq \varphi_3(\varphi_1(x))$, similarly by choosing a suitable function we can conclude that $\alpha_3^2\lambda_3^2 = \alpha_1\lambda_1 + \alpha_2\lambda_2$. Hence, from Lemma 3.1, we have either $\alpha_3^2\lambda_3^2 = -\alpha_3\lambda_3$, or $\alpha_3^2 = 1 - \alpha_3$. The first case yields $\alpha_3 = 1$, which is a contradiction. The second case implies that $\alpha_3 = (-1+\sqrt{5})/2 > 1/2$, and it is impossible by Lemma 3.1. Thus, $\varphi_3^2(x) = \varphi_1(x) = \varphi_3(\varphi_1(x))$, which especially yields $\varphi_3(x) = \varphi_1(x)$ because of the injectivity of φ_3, a contradiction. Therefore, $A_3 = \emptyset$.

We now prove that $E = \emptyset$. Contrary to what we claim, let $x \in X$ with $\mathrm{Card}(\mathcal{S}_x \cup \{x\}) = 4$. We assume, without loss of generality, that $\alpha_1 = \max\{\alpha_1, \alpha_2, \alpha_3\}$. Since $\varphi_1(x)$ does not belong to the set

$$A = \left\{\varphi_2(x), \varphi_3(x), \varphi_1^2(x), \varphi_1(\varphi_2(x)), \varphi_1(\varphi_3(x))\right\},$$

we can select a function $f \in \mathrm{AC}(X)$ with $0 \leq f \leq 1$ such that $f(\varphi_1(x)) = 1$ and $f|_A = 0$. Then Eq. (3.1) becomes

$$\begin{aligned}\alpha_1\lambda_1 &\left[\alpha_2\lambda_2 f(\varphi_2(\varphi_1(x))) + \alpha_3\lambda_3 f(\varphi_3(\varphi_1(x)))\right]\\ &+ \alpha_2\lambda_2 \left[\alpha_2\lambda_2 f(\varphi_2^2(x)) + \alpha_3\lambda_3 f(\varphi_3(\varphi_2(x)))\right]\\ &+ \alpha_3\lambda_3 \left[\alpha_2\lambda_2 f(\varphi_2(\varphi_3(x))) + \alpha_3\lambda_3 f(\varphi_3^2(x))\right]\\ &= \alpha_1\lambda_1. \qquad (3.2)\end{aligned}$$

If $\alpha_1 = \alpha_2 = \alpha_3$, Eq. (3.2) reduces to $\sum_{k=1,j=2}^{3} \lambda_k\lambda_j f(\varphi_j(\varphi_k(x))) = 3\lambda_1$, which clearly implies that at least three points in the set

$$B = \left\{\varphi_2(\varphi_1(x)), \varphi_3(\varphi_1(x)), \varphi_2^2(x), \varphi_3(\varphi_2(x)), \varphi_2(\varphi_3(x)), \varphi_3^2(x)\right\}$$

must be equal to $\varphi_1(x)$, but this is impossible because B does not contain three equal points.

Now, assume that the relation $\alpha_1 = \alpha_2 = \alpha_3$ is not valid, which especially yields $\alpha_1 > \min\{\alpha_2, \alpha_3\}$. Hence, taking into account that $\alpha_1 = \max\{\alpha_1, \alpha_2, \alpha_3\}$ and $\alpha_i < 1$ for each $i \in \{1, 2, 3\}$, from (3.2) one can easily conclude that at least two points (in fact, exactly two) from the set B must be equal to $\varphi_1(x)$. Since $x \in E$, it is deduced that $\varphi_1(x) = \varphi_2(\varphi_i(x)) = \varphi_3(\varphi_j(x))$ with $i, j \in \{1, 2, 3\}$. Thus, from Eq. (3.2) it follows that $\alpha_1 \leq \alpha_2\alpha_i + \alpha_3\alpha_j$, which taking into account that $\alpha_1 > \min\{\alpha_2, \alpha_3\}$, implies that $\alpha_1 < \alpha_1^2 + \alpha_1^2 = 2\alpha_1^2$, whence $\alpha_1 > 1/2$, a contradiction by Lemma 3.1. □

Lemma 3.4 (1) *If for some $i \in \{1, 2, 3\}$, $B_i \neq \emptyset$, then $\alpha_i = 1/2$. Moreover, $\lambda_i = 1$ (resp., $\lambda_i = -1$) when $\sum_{i=1}^{3} \alpha_i\lambda_i = 1$ (resp., $\sum_{i=1}^{3} \alpha_i\lambda_i = 0$), $\alpha_j\lambda_j + \alpha_k\lambda_k = 1/2$ for $j, k \in \{1, 2, 3\}$ with $k \neq i \neq j \neq k$.*

(2) *If for some $i \in \{1, 2, 3\}$, $C_i \neq \emptyset$, then $\alpha_i = 1/2$ and $\lambda_i = 1$. Further, $\varphi_i \circ \varphi_j = \varphi_j$, and we have either $\lambda_j = \lambda_k = 1$ and $\alpha_j\lambda_j + \alpha_k\lambda_k = 1/2$, or $\lambda_j = \lambda_k = -1$ and $\alpha_j\lambda_j + \alpha_k\lambda_k = -1/2$ for $j, k \in \{1, 2, 3\}$ with $k \neq i \neq j \neq k$.*

Proof (1) With no loss of generality, suppose that $i = 3$. Let $x \in B_3$. Then $x = \varphi_1(x) = \varphi_2(x) \neq \varphi_3(x)$. In the following, we show that $\varphi_3(x) = \varphi_2(\varphi_3(x)) = \varphi_1(\varphi_3(x))$.

If $\varphi_2(\varphi_3(x)) \neq \varphi_3(x) \neq \varphi_1(\varphi_3(x))$, then by considering $f \in \mathrm{AC}(X)$ with $f(\varphi_3(x)) = 1$ and $f|_F = 0$, where

$$F = \left\{\varphi_1(x), \varphi_1(\varphi_3(x)), \varphi_2(\varphi_3(x)), \varphi_3^2(x)\right\},$$

in Eq. (3.1) it follows that $\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_2\lambda_2\alpha_3\lambda_3 = \alpha_3\lambda_3$, which yields $\alpha_1\lambda_1 + \alpha_2\lambda_2 = 1$, a contradiction by Lemma 3.1. Then $\varphi_3(x) = \varphi_1(\varphi_3(x))$, or $\varphi_3(x) = \varphi_2(\varphi_3(x))$.

If $\varphi_2(\varphi_3(x)) \neq \varphi_3(x) = \varphi_1(\varphi_3(x))$, then choose a function $f \in \mathrm{AC}(X)$ with $f(\varphi_3(x)) = 1$ and $f(\varphi_1(x)) = f(\varphi_3^2(x)) = f(\varphi_2(\varphi_3(x))) = 0$. Hence, from Eq. (3.1), we have

$$\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_2\lambda_2\alpha_3\lambda_3 + \alpha_3\lambda_3\alpha_1\lambda_1 = \alpha_3\lambda_3,$$

which gives $2\alpha_1\lambda_1 + \alpha_2\lambda_2 = 1$. Now, if $\sum_{i=1}^{3} \alpha_i\lambda_i = 0$, then from the latter equation it follows that $\alpha_1\lambda_1 - \alpha_3\lambda_3 = 1$, which is impossible by Lemma 3.1. Therefore, $\sum_{i=1}^{3} \alpha_i\lambda_i = 1$. Thus, $\alpha_3\lambda_3 = \alpha_1\lambda_1$, which shows that $\alpha_3 = \alpha_1$. On the other hand, by choosing $f \in \mathrm{AC}(X)$ with $f(x) = 1$ and $f(\varphi_3(x)) = 0$, Eq. (3.1) reduces to

$$\begin{aligned}
&\alpha_1\lambda_1 \left[\alpha_1\lambda_1 + \alpha_2\lambda_2\right] + \alpha_2\lambda_2 \left[\alpha_1\lambda_1 + \alpha_2\lambda_2\right] \\
&\quad + \alpha_3\lambda_3 \left[\alpha_2\lambda_2 f(\varphi_2(\varphi_3(x))) + \alpha_3\lambda_3 f(\varphi_3^2(x))\right] \\
&\quad = \alpha_1\lambda_1 + \alpha_2\lambda_2,
\end{aligned}$$

and so

$$[\alpha_1\lambda_1 + \alpha_2\lambda_2]^2 + \alpha_3\lambda_3 \left[\alpha_2\lambda_2 f(\varphi_2(\varphi_3(x))) + \alpha_3\lambda_3 f(\varphi_3^2(x))\right] = \alpha_1\lambda_1 + \alpha_2\lambda_2.$$

So, taking into account that $\varphi_2(\varphi_3(x)) \neq \varphi_2(x) = x$ and $\varphi_3^2(x) \neq \varphi_3(x)$, it follows that $\varphi_3^2(x) = x$. Then we choose an $f \in \mathrm{AC}(X)$ with $f(x) = 1$ and $f(\varphi_3(x)) = f(\varphi_2(\varphi_3(x))) = 0$ and obtain

$$[\alpha_1\lambda_1 + \alpha_2\lambda_2]^2 + \alpha_3^2\lambda_3^2 = \alpha_1\lambda_1 + \alpha_2\lambda_2.$$

As $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we have $(1-\alpha_3)^2 + \alpha_3^2 = 1 - \alpha_3$, which yields $\alpha_3 = 1/2$, and since $\alpha_1 = \alpha_3$, we obtain $\alpha_2 = 0$, a contradiction.

Similarly, it is proved that the relation $\varphi_2(\varphi_3(x)) = \varphi_3(x) \neq \varphi_1(\varphi_3(x))$ leads to a contradiction. Now, from the above discussion one concludes that $\varphi_3(x) = \varphi_2(\varphi_3(x)) = \varphi_1(\varphi_3(x))$, as desired.

Moreover, taking $f \in \mathrm{AC}(X)$ with $f(\varphi_3(x)) = 1$ and $f(\varphi_1(x)) = f(\varphi_3^2(x)) = 0$, from Eq. (3.1) we have

$$\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_2\lambda_2\alpha_3\lambda_3 + \alpha_3\lambda_3[\alpha_1\lambda_1 + \alpha_2\lambda_2] = \alpha_3\lambda_3,$$

and so $\alpha_1\lambda_1 + \alpha_2\lambda_2 = 1/2$. Now, if $\sum_{i=1}^{3} \alpha_i\lambda_i = 0$, then $\alpha_3\lambda_3 = -1/2$, which implies that $\alpha_3 = 1/2$ and $\lambda_3 = -1$. Furthermore, if $\sum_{i=1}^{3} \alpha_i\lambda_i = 1$, then $\alpha_3 = 1/2$ and $\lambda_3 = 1$.

(2) Let $x \in C_3$. Then $x = \varphi_3(x) \neq \varphi_1(x) = \varphi_2(x)$. Select a function $f \in \mathrm{AC}(X)$ with $f(\varphi_1(x)) = 1$ and $f|_A = 0$, where

$$A = \left\{\varphi_3(x), \varphi_1^2(x), \varphi_2^2(x)\right\}.$$

Then, from Eq. (3.1), it follows that

$$\begin{aligned}
&\alpha_1\lambda_1 \left[\alpha_3\lambda_3 f(\varphi_3(\varphi_1(x)))\right] + \alpha_2\lambda_2 \left[\alpha_3\lambda_3 f(\varphi_3(\varphi_1(x)))\right] \\
&\quad + \alpha_3\lambda_3 \left[\alpha_1\lambda_1 + \alpha_2\lambda_2\right] = \alpha_1\lambda_1 + \alpha_2\lambda_2,
\end{aligned}$$

and so $\alpha_3\lambda_3 f(\varphi_3(\varphi_1(x))) + \alpha_3\lambda_3 = 1$ because $\alpha_1\lambda_1 + \alpha_2\lambda_2 \neq 0$ by Lemma 3.1. Since $0 < \alpha_3 < 1$, from the above it follows that $\varphi_3(\varphi_1(x)) = \varphi_1(x)$, whence $2\alpha_3\lambda_3 = 1$. The latter equation especially shows that $\alpha_3 = 1/2$ and $\lambda_3 = 1$.

If $\sum_{i=1}^{3} \alpha_i\lambda_i = 1$, then $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\alpha_1\lambda_1 + \alpha_2\lambda_2 = 1 - \alpha_3\lambda_3 = 1/2$. Meantime, if $\sum_{i=1}^{3} \alpha_i\lambda_i = 0$, then $\alpha_1\lambda_1 + \alpha_2\lambda_2 = -\alpha_3\lambda_3 = -1/2$, which yields

$-1 = 2\alpha_1\lambda_1 + 2\alpha_2\lambda_2$. Since $\alpha_1 + \alpha_2 = 1/2$, we infer that $\lambda_1 = \lambda_2 = -1$ by the strict convexity of $\mathbb{C}$. □

Lemma 3.5 *If $i \in \{1, 2, 3\}$ and $x \in D_i$, then $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$.*

Proof We prove the result by an argument similar to that of [1, Lemma 2.3]. Assume that $x \in D_1$. Then we have $x = \varphi_1(x) \neq \varphi_2(x) \neq \varphi_3(x) \neq x$ and we can rewrite Eq. (3.1) as follows:

$$\begin{aligned}
&\alpha_1\lambda_1\left[\alpha_1\lambda_1 f(x) + \alpha_2\lambda_2 f(\varphi_2(x)) + \alpha_3\lambda_3 f(\varphi_3(x))\right] \\
&\quad + \alpha_2\lambda_2\left[\alpha_1\lambda_1 f(\varphi_1(\varphi_2(x))) + \alpha_2\lambda_2 f(\varphi_2^2(x)) + \alpha_3\lambda_3 f(\varphi_3(\varphi_2(x)))\right] \\
&\quad + \alpha_3\lambda_3\left[\alpha_1\lambda_1 f(\varphi_1(\varphi_3(x))) + \alpha_2\lambda_2 f(\varphi_2(\varphi_3(x))) + \alpha_3\lambda_3 f(\varphi_3^2(x))\right] \\
&\quad = \alpha_1\lambda_1 f(x) + \alpha_2\lambda_2 f(\varphi_2(x)) + \alpha_3\lambda_3 f(\varphi_3(x)).
\end{aligned} \tag{3.3}$$

Choose a function $f \in \mathrm{AC}(X)$ such that $f(x) = 1$ and $f(\varphi_2(x)) = f(\varphi_3(x)) = f(\varphi_1(\varphi_2(x))) = f(\varphi_1(\varphi_3(x))) = 0$. Then Eq. (3.3) reduces to

$$\begin{aligned}
&\alpha_1^2\lambda_1^2 + \alpha_2\lambda_2\left[\alpha_2\lambda_2 f(\varphi_2^2(x)) + \alpha_3\lambda_3 f(\varphi_3(\varphi_2(x)))\right] \\
&\quad + \alpha_3\lambda_3\left[\alpha_2\lambda_2 f(\varphi_2(\varphi_3(x))) + \alpha_3\lambda_3 f(\varphi_3^2(x))\right] \\
&\quad = \alpha_1\lambda_1.
\end{aligned} \tag{3.4}$$

Taking into account that $\alpha_1 < 1$, from the above relation it follows that at least one point in the set

$$D = \left\{\varphi_2^2(x), \varphi_3(\varphi_2(x)), \varphi_2(\varphi_3(x)), \varphi_3^2(x)\right\}$$

is equal to x. Meantime, note that at most two elements in D are equal. Then, noting at the following claim, which is proved at the end of the proof, we deduce that x is equal to exactly two elements in D. Hence, one of the following cases may happen:

$$\begin{cases}
x = \varphi_2^2(x) = \varphi_3(\varphi_2(x)), \\
x = \varphi_2(\varphi_3(x)) = \varphi_3(\varphi_2(x)), \\
x = \varphi_2^2(x) = \varphi_3^2(x).
\end{cases} \tag{3.5}$$

Claim 3.6 (1) *If $x = \varphi_2(\varphi_i(x))$ for $i \in \{2, 3\}$, then $x = \varphi_3(\varphi_j(x))$ for $j \in \{2, 3\}$.*
(2) *If $x = \varphi_3(\varphi_i(x))$ for $i \in \{2, 3\}$, then $x = \varphi_2(\varphi_j(x))$ for $j \in \{2, 3\}$.*

By considering $f \in \mathrm{AC}(X)$ with $f(\varphi_2(x)) = 1$ and $f(x) = f(\varphi_3(x)) = f(\varphi_2^2(x)) = f(\varphi_2(\varphi_3(x))) = 0$ in Eq. (3.3), we get

$$\begin{aligned}&\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_2\lambda_2[\alpha_1\lambda_1 f(\varphi_1(\varphi_2(x))) + \alpha_3\lambda_3 f(\varphi_3(\varphi_2(x)))]\\ &\quad + \alpha_3\lambda_3\left[\alpha_1\lambda_1 f(\varphi_1(\varphi_3(x))) + \alpha_3\lambda_3 f(\varphi_3^2(x))\right]\\ &\quad = \alpha_2\lambda_2.\end{aligned} \tag{3.6}$$

Moreover, if we choose $f \in \mathrm{AC}(X)$ with $f(\varphi_3(x)) = 1$ and $f(x) = f(\varphi_2(x)) = f(\varphi_3^2(x)) = f(\varphi_3(\varphi_2(x))) = 0$ in Eq. (3.3), then

$$\begin{aligned}&\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_2\lambda_2\left[\alpha_1\lambda_1 f(\varphi_1(\varphi_2(x))) + \alpha_2\lambda_2 f(\varphi_2^2(x))\right]\\ &\quad + \alpha_3\lambda_3[\alpha_1\lambda_1 f(\varphi_1(\varphi_3(x))) + \alpha_2\lambda_2 f(\varphi_2(\varphi_3(x)))]\\ &\quad = \alpha_3\lambda_3.\end{aligned} \tag{3.7}$$

Now, from (3.4)–(3.7) we can see that one of following disjoint situations may happen for the points of D_1:

$$\begin{aligned}D_{11} &= \Big\{x \in D_1 :\ x = \varphi_2^2(x) = \varphi_3(\varphi_2(x)),\ \varphi_2(x) = \varphi_3^2(x) = \varphi_1(\varphi_2(x)),\\ &\qquad \varphi_3(x) = \varphi_1(\varphi_3(x)) = \varphi_2(\varphi_3(x))\},\\ D_{12} &= \Big\{x \in D_1 :\ x = \varphi_2^2(x) = \varphi_3(\varphi_2(x)),\ \varphi_2(x) = \varphi_3^2(x) = \varphi_1(\varphi_3(x)),\\ &\qquad \varphi_3(x) = \varphi_1(\varphi_2(x)) = \varphi_2(\varphi_3(x))\},\\ D_{13} &= \Big\{x \in D_1 :\ x = \varphi_2(\varphi_3(x)) = \varphi_3(\varphi_2(x)),\ \varphi_2(x) = \varphi_3^2(x) = \varphi_1(\varphi_2(x)),\\ &\qquad \varphi_3(x) = \varphi_1(\varphi_3(x)) = \varphi_2^2(x)\Big\},\\ D_{14} &= \Big\{x \in D_1 :\ x = \varphi_2(\varphi_3(x)) = \varphi_3(\varphi_2(x)),\ \varphi_2(x) = \varphi_3^2(x) = \varphi_1(\varphi_3(x)),\\ &\qquad \varphi_3(x) = \varphi_1(\varphi_2(x)) = \varphi_2^2(x)\Big\},\\ D_{15} &= \Big\{x \in D_1 :\ x = \varphi_2^2(x) = \varphi_3^2(x),\ \varphi_2(x) = \varphi_1(\varphi_2(x)) = \varphi_3(\varphi_2(x)),\\ &\qquad \varphi_3(x) = \varphi_1(\varphi_3(x)) = \varphi_2(\varphi_3(x))\},\\ D_{16} &= \Big\{x \in D_1 :\ x = \varphi_2^2(x) = \varphi_3^2(x),\ \varphi_2(x) = \varphi_1(\varphi_3(x)) = \varphi_3(\varphi_2(x)),\\ &\qquad \varphi_3(x) = \varphi_1(\varphi_2(x)) = \varphi_2(\varphi_3(x))\}.\end{aligned}$$

For each $x \in D_{11}$, Eq. (3.3) becomes

$$\begin{aligned}&\left[\alpha_1^2\lambda_1^2 + \alpha_2\lambda_2(\alpha_2\lambda_2 + \alpha_3\lambda_3)\right] f(x) + \left[2\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3^2\lambda_3^2\right] f(\varphi_2(x))\\ &\quad + [2\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_3\lambda_3\alpha_2\lambda_2]\, f(\varphi_3(x))\\ &\quad = \alpha_1\lambda_1 f(x) + \alpha_2\lambda_2 f(\varphi_2(x)) + \alpha_3\lambda_3 f(\varphi_3(x)).\end{aligned}$$

Since $x \neq \varphi_2(x) \neq \varphi_3(x) \neq x$, by taking a suitable function in the above equation it follows that $2\alpha_1\lambda_1 + \alpha_2\lambda_2 = 1$, which taking into account Lemma 3.1, implies that $\sum_{i=1}^{3} \alpha_i\lambda_i = 1$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Then one can obtain easily the following two equations

$$\begin{cases} 1 = 2\alpha_1 + \alpha_2, \\ \alpha_1 = \alpha_1^2 + \alpha_2(\alpha_2 + \alpha_3). \end{cases} \tag{3.8}$$

For $x \in D_{12}$, we have

$$\begin{aligned}&\left[\alpha_1^2\lambda_1^2 + \alpha_2\lambda_2(\alpha_2\lambda_2 + \alpha_3\lambda_3)\right] f(x)\\ &\quad + [\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3(\alpha_1\lambda_1 + \alpha_3\lambda_3)]\, f(\varphi_2(x))\\ &\quad + [\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2]\, f(\varphi_3(x))\\ &\quad = \alpha_1\lambda_1 f(x) + \alpha_2\lambda_2 f(\varphi_2(x)) + \alpha_3\lambda_3 f(\varphi_3(x)),\end{aligned}$$

which, by choosing proper functions, implies that

$$\begin{cases} \alpha_1 \leq \alpha_1^2 + \alpha_2(\alpha_2 + \alpha_3), \\ \alpha_2 \leq \alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_3), \\ \alpha_3 \leq \alpha_3\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_3. \end{cases} \tag{3.9}$$

For $x \in D_{13}$, we have

$$\begin{aligned}&\left[\alpha_1^2\lambda_1^2 + 2\alpha_2\alpha_3\lambda_2\lambda_3\right] f(x)\\ &\quad + [2\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3^2\lambda_3^2] f(\varphi_2(x))\\ &\quad + \left[\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_2^2\lambda_2^2 + \alpha_3\lambda_3\alpha_1\lambda_1\right] f(\varphi_3(x))\\ &\quad = \alpha_1\lambda_1 f(x) + \alpha_2\lambda_2 f(\varphi_2(x)) + \alpha_3\lambda_3 f(\varphi_3(x)),\end{aligned}$$

which similarly shows that

$$\begin{cases} \alpha_1 \leq \alpha_1^2 + 2\alpha_1\alpha_3, \\ \alpha_2 \leq 2\alpha_1\alpha_2 + \alpha_3^2, \\ \alpha_3 \leq 2\alpha_1\alpha_3 + \alpha_2^2. \end{cases} \tag{3.10}$$

If $x \in D_{14}$, then

$$\begin{aligned}&\left[\alpha_1^2\lambda_1^2 + 2\alpha_2\alpha_3\lambda_2\lambda_3\right] f(x)\\&\quad + \left[\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_3^2\lambda_3^2\right] f(\varphi_2(x))\\&\quad + \left[\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_2\lambda_2\alpha_1\lambda_1 + \alpha_2^2\lambda_2^2\right] f(\varphi_3(x))\\&\quad = \alpha_1\lambda_1 f(x) + \alpha_2\lambda_2 f(\varphi_2(x)) + \alpha_3\lambda_3 f(\varphi_3(x)),\end{aligned}$$

which yields

$$\begin{cases}\alpha_1 \le \alpha_1^2 + 2\alpha_2\alpha_3,\\ \alpha_2 \le \alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_3),\\ \alpha_3 \le \alpha_1\alpha_3 + \alpha_2(\alpha_1 + \alpha_2).\end{cases} \tag{3.11}$$

For $x \in D_{15}$, we have

$$\begin{aligned}&\left[\alpha_1^2\lambda_1^2 + \alpha_2^2\lambda_2^2 + \alpha_3^2\lambda_3^2\right] f(x)\\&\quad + [2\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_2\lambda_2\alpha_3\lambda_3] f(\varphi_2(x))\\&\quad + [2\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_3\lambda_3\alpha_2\lambda_2] f(\varphi_3(x))\\&\quad = \alpha_1\lambda_1 f(x) + \alpha_2\lambda_2 f(\varphi_2(x)) + \alpha_3\lambda_3 f(\varphi_3(x)),\end{aligned}$$

which gives $1 = 2\alpha_1\lambda_1 + \alpha_2\lambda_2$. Now from Lemma 3.1 it easily follows that $\sum_{i=1}^3 \alpha_i\lambda_i = 1$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Now, the following equations can be derived immediately

$$\begin{cases}1 = 2\alpha_1 + \alpha_3,\\ 1 = 2\alpha_1 + \alpha_2.\end{cases} \tag{3.12}$$

For $x \in D_{16}$,

$$\begin{aligned}&\left[\alpha_1^2\lambda_1^2 + \alpha_2^2\lambda_2^2 + \alpha_3^2\lambda_3^2\right] f(x)\\&\quad + [\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_2\lambda_2\alpha_3\lambda_3 + \alpha_1\lambda_1\alpha_3\lambda_3] f(\varphi_2(x))\\&\quad + [\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2] f(\varphi_3(x))\\&\quad = \alpha_1\lambda_1 f(x) + \alpha_2\lambda_2 f(\varphi_2(x)) + \alpha_3\lambda_3 f(\varphi_3(x)),\end{aligned}$$

which gives

$$\begin{cases}\alpha_1 \le \alpha_1^2 + \alpha_2^2 + \alpha_3^2,\\ \alpha_2 \le \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1,\\ \alpha_3 \le \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1.\end{cases} \tag{3.13}$$

It is easy to check that the unique solution of the systems (3.8)–(3.13) is $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$. This completes the proof of Lemma 3.5. □

Now, we turn to the proof of Claim 3.6.

Proof We only prove the first part because the second one will be similar. Contrary to what we claim, suppose that $x = \varphi_2(\varphi_{i_2}(x))$ where $i_2 \in \{2, 3\}$, and $x \neq \varphi_3(\varphi_k(x))$ for $k = 2, 3$. Put $j_2 \in \{2, 3\}\setminus\{i_2\}$. If we consider $f \in \mathrm{AC}(X)$ with $0 \leq f \leq 1$, $f(x) = 1$ and $f(\varphi_3(\varphi_2(x))) = f(\varphi_2(\varphi_{j_2}(x))) = f(\varphi_3^2(x)) = 0$ in Eq. (3.4), then we have

$$\alpha_1^2\lambda_1^2 + \alpha_2\lambda_2\alpha_{i_2}\lambda_{i_2} = \alpha_1\lambda_1 \tag{3.14}$$

which yields

$$\alpha_1 \leq \alpha_1^2 + \alpha_2\alpha_{i_2}. \tag{3.15}$$

Moreover, by taking $f \in \mathrm{AC}(X)$ with $0 \leq f \leq 1$, $f(\varphi_2(x)) = 1$ and $f(x) = f(\varphi_3(x)) = f(\varphi_2^2(x)) = f(\varphi_2(\varphi_3(x))) = 0$ in Eq. (3.3) we get

$$\begin{aligned} &\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_2\lambda_2\left[\alpha_1\lambda_1 f(\varphi_1(\varphi_2(x))) + \alpha_3\lambda_3 f(\varphi_3(\varphi_2(x)))\right] \\ &\quad + \alpha_3\lambda_3\left[\alpha_1\lambda_1 f(\varphi_1(\varphi_3(x))) + \alpha_3\lambda_3 f(\varphi_3^2(x))\right] \\ &\quad = \alpha_2\lambda_2. \end{aligned} \tag{3.16}$$

Since $\alpha_1 < 1$, it easily follows from the above that $\varphi_2(x)$ must be equal to at least one of the points in the set

$$\left\{\varphi_1(\varphi_2(x)), \varphi_3(\varphi_2(x)), \varphi_1(\varphi_3(x)), \varphi_3^2(x)\right\}.$$

Assume that $\varphi_2(x) \neq \varphi_3(\varphi_k(x))$ for $k = 2, 3$. Then by taking the preceding f such that $f(\varphi_3(\varphi_k(x))) = 0$ $(k = 2, 3)$ in Eq. (3.16), we obtain that

$$\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_1\lambda_1\alpha_2\lambda_2 f(\varphi_1(\varphi_2(x))) + \alpha_1\lambda_1\alpha_3\lambda_3 f(\varphi_1(\varphi_3(x))) = \alpha_2\lambda_2.$$

The above relation clearly shows that $\varphi_1(\varphi_2(x))$ or $\varphi_1(\varphi_3(x))$ must be equal to $\varphi_2(x)$. Let us assume that $\varphi_2(x) = \varphi_1(\varphi_{i_1}(x))$ where $i_1 \in \{2, 3\}$. It is apparent that $\varphi_2(x) \neq \varphi_1(\varphi_{j_1}(x))$, where $j_1 \in \{2, 3\}\setminus\{i_1\}$. Then from the above relation, we get $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_1\lambda_1\alpha_{i_1}\lambda_{i_1} = \alpha_2\lambda_2$, which implies that

$$\alpha_2 \leq \alpha_1(\alpha_2 + \alpha_{i_1}).$$

If $i_1 = 2$, then $\alpha_2 \leq 2\alpha_1\alpha_2$, and so $\alpha_1 = 1/2$. Thus, according to (3.15), $1/2 \leq 1/4 + \alpha_2\alpha_{i_2} \leq 1/4 + (\max\{\alpha_2, \alpha_3\})^2$, which verifies that $1/4 \leq (\max\{\alpha_2, \alpha_3\})^2$, and hence, $\max\{\alpha_2, \alpha_3\} \geq 1/2$, a contradiction.

Now, assume that $i_1 = 3$. Thus, $\alpha_2 \leq \alpha_1(\alpha_2 + \alpha_3) = \alpha_1(1 - \alpha_1)$. If $\alpha_1 \leq \alpha_2$, then $\alpha_2 \leq \alpha_2(1 - \alpha_1)$ which implies that $1 - \alpha_1 \geq 1$, a contradiction showing that

$\alpha_1 > \alpha_2$. Next, from (3.15) it follows that $\alpha_1 \leq \alpha_1^2 + \alpha_2\alpha_{i_2} < \alpha_1^2 + \alpha_1\alpha_{i_2}$, which yields $1 < \alpha_1 + \alpha_{i_2}$ and it is impossible.

Therefore, from the above discussion, we infer that $\varphi_2(x)$ is equal to (exactly) one of the points $\varphi_3(\varphi_2(x))$ or $\varphi_3^2(x)$. Let us suppose that $\varphi_2(x) = \varphi_3(\varphi_{i_3}(x))$, where i_3 is either 2 or 3. Obviously, $\varphi_2(x) \neq \varphi_3(\varphi_{j_3}(x))$, where $j_3 \in \{2,3\}\setminus\{i_3\}$. Moreover, consider $f \in \mathrm{AC}(X)$ with $0 \leq f \leq 1$, $f(\varphi_3(\varphi_{j_3}(x))) = 1$ and $f(x) = f(\varphi_2(x)) = f(\varphi_3(x))) = 0$ in (3.3), and we have

$$
\begin{aligned}
&\alpha_1\lambda_1\alpha_2\lambda_2 f(\varphi_1(\varphi_2(x))) + \alpha_1\lambda_1\alpha_3\lambda_3 f(\varphi_1(\varphi_3(x))) \\
&\quad + \alpha_2\lambda_2\alpha_{j_2}\lambda_{j_2} f(\varphi_2(\varphi_{j_2}(x))) + \alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3} f(\varphi_3(\varphi_{j_3}(x))) \\
&\quad = \alpha_1\lambda_1\alpha_2\lambda_2 f(\varphi_1(\varphi_2(x))) + \alpha_1\lambda_1\alpha_3\lambda_3 f(\varphi_1(\varphi_3(x))) + \alpha_2\lambda_2\alpha_{j_2}\lambda_{j_2} f(\varphi_2(\varphi_{j_2}(x))) \\
&\quad + \alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3} = 0.
\end{aligned} \tag{3.17}
$$

We next show that

$$\alpha_2\lambda_2\alpha_{j_2}\lambda_{j_2} + \alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3} \neq 0. \tag{3.18}$$

Otherwise, we have

$$\alpha_2\lambda_2\alpha_{j_2}\lambda_{j_2} = -\alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3}. \tag{3.19}$$

If $\sum_{i=1}^3 \alpha_i\lambda_i = 1$, then from Eq. (3.19) we get $\alpha_2\alpha_{j_2} = -\alpha_3\alpha_{j_3}$, which is impossible. Hence, $\sum_{i=1}^3 \alpha_i\lambda_i = 0$.

If $j_2 = j_3$, then from (3.19) we deduce that $\alpha_2\lambda_2 = -\alpha_3\lambda_3$, which is a contradiction by Lemma 3.1. Now, suppose that $j_2 \neq j_3$. Let $j_2 = 2$ and $j_3 = 3$. If $\varphi_1(\varphi_k(x)) \neq \varphi_2(x)$ for $k = 2, 3$, from (3.16) we have $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, which yields $\alpha_1\lambda_1 + \alpha_3\lambda_3 = 1$, a contradiction by Lemma 3.1. Then, we have either $\varphi_1(\varphi_2(x)) = \varphi_2(x)$ or $\varphi_1(\varphi_3(x)) = \varphi_2(x)$. From (3.16), we have two possibilities:

(1) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$; in consequence, $2\alpha_1\lambda_1 + \alpha_3\lambda_3 = 1$, and so $\alpha_1\lambda_1 - \alpha_2\lambda_2 = 1$, a contradiction (Lemma 3.1),

(2) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_1\lambda_1 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, which shows that $\alpha_1\lambda_1\alpha_2\lambda_2 - \alpha_3\lambda_3\alpha_3\lambda_3 = \alpha_2\lambda_2$, and from (3.19) it follows that $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_2^2\lambda_2^2 = \alpha_2\lambda_2$, whence $\alpha_1\lambda_1 + \alpha_2\lambda_2 = 1$, a contradiction (Lemma 3.1).

If $j_2 = 3$ and $j_3 = 2$, then from (3.19), $\alpha_2\lambda_2\alpha_3\lambda_3 = -\alpha_3\lambda_3\alpha_2\lambda_2$, which is impossible. Therefore, Eq. (3.18) holds.

Now, taking into account (3.17), (3.18) and the fact that $\varphi_1(\varphi_2(x)) \neq \varphi_1(\varphi_3(x))$, it is easy to see that (exactly) one of the points $\varphi_1(\varphi_2(x))$ or $\varphi_1(\varphi_3(x))$ must be equal to $\varphi_3(\varphi_{j_3}(x))$. Put $\varphi_1(\varphi_{r_1}(x)) = \varphi_3(\varphi_{j_3}(x))$ and $\varphi_1(\varphi_{t_1}(x)) \neq \varphi_3(\varphi_{j_3}(x))$ ($\{r_1, t_1\} = \{2, 3\}$). Let us rewrite (3.16) and (3.17) as

$$\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_1\lambda_1\alpha_{t_1}\lambda_{t_1}\varphi_1(\varphi_{t_1}(x)) + \alpha_3\lambda_3\alpha_{i_3}\lambda_{i_3} = \alpha_2\lambda_2, \tag{3.20}$$

and

$$(\alpha_1\lambda_1\alpha_{r_1}\lambda_{r_1} + \alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3}) f(\varphi_3(\varphi_{j_3}(x))) + \alpha_2\lambda_2\alpha_{j_2}\lambda_{j_2} f(\varphi_2(\varphi_{j_2}(x)))$$
$$= \alpha_1\lambda_1\alpha_{r_1}\lambda_{r_1} + \alpha_2\lambda_2\alpha_{j_2}\lambda_{j_2} f(\varphi_2(\varphi_{j_2}(x))) + \alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3} = 0. \tag{3.21}$$

Next, it is shown that

$$\alpha_1\lambda_1\alpha_{r_1}\lambda_{r_1} + \alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3} \neq 0. \tag{3.22}$$

Otherwise,

$$\alpha_1\lambda_1\alpha_{r_1}\lambda_{r_1} = -\alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3}. \tag{3.23}$$

If $\sum_{i=1}^3 \alpha_i\lambda_i = 1$, then from (3.23) we get $\alpha_{r_1} = -\alpha_3\alpha_{j_3}$, a contradiction. Hence, $\sum_{i=1}^3 \alpha_i\lambda_i = 0$. If $r_1 = j_3$, then again from (3.23), $\alpha_1\lambda_1 = -\alpha_3\lambda_3$, a contradiction (Lemma 3.1). Thus, $r_1 \neq j_3$. If $r_1 = 3$ and $j_3 = 2$, then $\alpha_1\lambda_1\alpha_3\lambda_3 = -\alpha_3\lambda_3\alpha_2\lambda_2$ by (3.23), which leads to a contradiction. If $r_1 = 2$ and $j_3 = 3$, then from (3.23) we have

$$\alpha_1\lambda_1\alpha_2\lambda_2 = -\alpha_3^2\lambda_3^2. \tag{3.24}$$

Based on the fact that $\varphi_1(\varphi_{t_1}(x)) \neq \varphi_2(x)$ or $\varphi_1(\varphi_{t_1}(x)) = \varphi_2(x)$, from (3.20) we have either

(1) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, whence $\alpha_1\lambda_1 + \alpha_3\lambda_3 = 1$, a contradiction, or
(2) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, and so from (3.24), $-\alpha_3^2\lambda_3^2 + \alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, and hence, $\alpha_3\lambda_3[-\alpha_3\lambda_3 + \alpha_2\lambda_2 + \alpha_1\lambda_1] = \alpha_2\lambda_2$, which yields $-2\alpha_3^2\lambda_3^2 = \alpha_2\lambda_2$. Consequently, from (3.24) we get $2\alpha_1\lambda_1\alpha_2\lambda_2 = \alpha_2\lambda_2$, which gives $\alpha_1 = 1/2$ and $\lambda_1 = 1$. Moreover, again from (3.24) it follows that $\alpha_3^2 = \alpha_1\alpha_2$ and $-\lambda_3^2 = \lambda_1\lambda_2$. On the other hand, from (3.14), $\alpha_1^2\lambda_1^2 + \alpha_2\lambda_2\alpha_{i_2}\lambda_{i_2} = \alpha_1\lambda_1$, and then putting $\alpha_1 = 1/2$ and $\lambda_1 = 1$, we get $\alpha_2\lambda_2\alpha_{i_2}\lambda_{i_2} = 1/4$. If $i_2 = 3$, then $\alpha_2\lambda_2\alpha_3\lambda_3 = 1/4$, whence $\alpha_2\alpha_3 = 1/4$. Thus, $\alpha_3^3 = \alpha_2\alpha_1\alpha_3 = \alpha_1/4 = 1/8$, and so $\alpha_3 = 1/2$, a contradiction. Also, if $i_2 = 2$, then $\alpha_2^2 = 1/4$, and $\alpha_2 = 1/2$, which is impossible.

Now we deduce (3.22).

Finally, we claim that

$$\alpha_1\lambda_1\alpha_{r_1}\lambda_{r_1} + \alpha_2\lambda_2\alpha_{j_2}\lambda_{j_2} + \alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3} \neq 0. \tag{3.25}$$

Otherwise,

$$\alpha_1\lambda_1\alpha_{r_1}\lambda_{r_1} + \alpha_2\lambda_2\alpha_{j_2}\lambda_{j_2} + \alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3} = 0.$$

If $\sum_{i=1}^3 \alpha_i\lambda_i = 1$, then $\alpha_1\alpha_{r_1} + \alpha_2\alpha_{j_2} + \alpha_{j_3}\alpha_{j_3} = 0$, which is impossible. Hence, $\sum_{i=1}^3 \alpha_i\lambda_i = 0$.

If $r_1 = j_2 = j_3$, then $t_1 = i_2 = i_3$. If $\varphi_1(\varphi_{t_1}(x)) = \varphi_2(x)$, from (3.20) we have $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_{i_3}\alpha_{i_3}(\alpha_1\lambda_1 + \alpha_3\lambda_3) = \alpha_2\lambda_2$, and consequently, $\alpha_1\lambda_1 - \alpha_{i_3}\alpha_{i_3} = 1$, a contradiction.

Now, assume that $\varphi_1(\varphi_{t_1}(x)) \neq \varphi_2(x)$. From (3.20) we have $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_{i_3}\alpha_{i_3} = \alpha_2\lambda_2$.

If $i_3 = 2$, then from the above relation we get $\alpha_1\lambda_1 + \alpha_3\lambda_3 = 1$, a contradiction. Now, suppose that $i_3 = 3$. Then (3.20) and (3.14) become

$$\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3^2\lambda_3^2 = \alpha_2\lambda_2, \tag{3.26}$$

and

$$\alpha_1^2\lambda_1^2 + \alpha_2\lambda_2\alpha_3\lambda_3 = \alpha_1\lambda_1. \tag{3.27}$$

By adding (3.26) and (3.27) together, we obtain $\alpha_1\lambda_1\alpha_2\lambda_2+\alpha_1^2\lambda_1^2+\alpha_3^2\lambda_3^2+\alpha_2\lambda_2\alpha_3\lambda_3 = -\alpha_3\lambda_3$, and so $\alpha_1\lambda_1(-\alpha_3\lambda_3) + \alpha_3\lambda_3(-\alpha_1\lambda_1) = -\alpha_3\lambda_3$; thus, $\alpha_1\lambda_1 = 1 - \alpha_1\lambda_1$, which verifies that $\alpha_1 = 1/2$ and $\lambda_1 = 1$. Now, from (3.27) one can conclude that $1/4 + \alpha_2\lambda_2\alpha_3\lambda_3 = 1/2$, whence

$$\alpha_2\lambda_2\alpha_3\lambda_3 = \frac{1}{4}. \tag{3.28}$$

On the other hand, from (3.26), $(1/2)\alpha_2\lambda_2 + \alpha_3^2\lambda_3^2 = \alpha_2\lambda_2$, which yields

$$\frac{1}{2}\alpha_2\lambda_2 = \alpha_3^2\lambda_3^2. \tag{3.29}$$

Combining (3.27) and (3.29), we get $1/4 + 2\alpha_3^3\lambda_3^3 = 1/2$, and so $\alpha_3^3\lambda_3^3 = 1/8$; consequently, $\alpha_3 = 1/2$, which is impossible.

Assume that $r_1 \neq j_2 = j_3$. Thus, $\alpha_1\lambda_1\alpha_{r_1}\lambda_{r_1} = -(\alpha_2\lambda_2+\alpha_3\lambda_3)\alpha_{j_2}\lambda_{j_2}$. As before, one can see $\sum_{i=1}^{3}\alpha_i\lambda_i = 0$, which easily implies that $\alpha_1\lambda_1\alpha_{r_1}\lambda_{r_1} = \alpha_1\lambda_1\alpha_{j_2}\lambda_{j_2}$, and so

$$\alpha_{r_1}\lambda_{r_1} = \alpha_{j_2}\lambda_{j_2},\ \alpha_{r_1} = \alpha_{j_2},\ \lambda_{r_1} = \lambda_{j_2}. \tag{3.30}$$

Let $j_2 = j_3 = 3$ and $r_1 = 2$. Based on the fact that $\varphi_1(\varphi_3(x)) \neq \varphi_2(x)$ or $\varphi_1(\varphi_3(x)) = \varphi_2(x)$, from (3.20) we have either

(1) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, a contradiction, or
(2) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, which taking into account (3.30) it follows that $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, and so $\alpha_1\lambda_1 + \alpha_1\lambda_1 + \alpha_3\lambda_3 = \alpha_1\lambda_1 - \alpha_2\lambda_2 = 1$, a contradiction.

If $j_2 = j_3 = 2$ and $r_1 = 3$, similarly, based on the fact that $\varphi_1(\varphi_2(x)) \neq \varphi_2(x)$ or $\varphi_1(\varphi_2(x)) = \varphi_2(x)$, from (3.30) and (3.20) we have either

(1) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_2^2\lambda_2^2 = \alpha_2\lambda_2$, and so $\alpha_1\lambda_1 + \alpha_2\lambda_2 = 1$, which is impossible, or

(2) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3^2\lambda_3^2 = \alpha_2\lambda_2$, and so from (3.30) $2\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_2^2\lambda_2^2 = \alpha_2\lambda_2$, which gives and so $\alpha_1\lambda_1 - \alpha_3\lambda_3 = 1$, which is impossible.

Now, assume that $r_1 = j_2 \neq j_3$. If $(\alpha_1\lambda_1 + \alpha_2\lambda_2)\alpha_{j_2}\lambda_{j_2} = -\alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3}$, then as before, $\sum_{i=1}^{3} \alpha_i\lambda_i = 0$, and consequently, the above relation reduces $-\alpha_3\lambda_3\alpha_{j_2}\lambda_{j_2} = -\alpha_3\lambda_3\alpha_{j_3}\lambda_{j_3}$ and so

$$\alpha_{j_2}\lambda_{j_2} = \alpha_{j_3}\lambda_{j_3}. \tag{3.31}$$

If $j_3 = 2$ and $j_2 = r_1 = 3$, similarly to above, based on the fact that $\varphi_1(\varphi_2(x)) \neq \varphi_2(x)$ or $\varphi_1(\varphi_2(x)) = \varphi_2(x)$ from (3.31) and (3.20) we get

(1) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, a contradiction, or
(2) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, and then, $2\alpha_1\lambda_1 + \alpha_3\lambda_3 = \alpha_1\lambda_1 - \alpha_2\lambda_2 = 1$, a contradiction.

Now, if $r_1 = j_2 = 2$ and $j_3 = 3$, then analogously from (3.31) and (3.20), it follows that either

(1) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, and in consequence, $\alpha_1\lambda_1 + \alpha_3\lambda_3 = 1$, a contradiction, or
(2) $\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2 = \alpha_3\lambda_3$, and so $\alpha_1\lambda_1 - \alpha_3\lambda_3 = 1$, a contradiction.

Next, suppose that $j_3 = r_1 \neq j_2$. Clearly, $\sum_{i=1}^{3} \alpha_i\lambda_i = 0$ as above, and we get $(\alpha_1\lambda_1 + \alpha_3\lambda_3)\alpha_{j_3}\lambda_{j_3} = -\alpha_2\lambda_2\alpha_{j_2}\lambda_{j_2}$, and so

$$\alpha_{j_3}\lambda_{j_3} = \alpha_{j_2}\lambda_{j_2}. \tag{3.32}$$

If $j_2 = 3$ and $j_3 = 2$, from (3.32) and (3.20) we have either

(1) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3^2\lambda_3^2 = \alpha_2\lambda_2$, and taking into account (3.32), $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_2^2\lambda_2^2 = \alpha_2\lambda_2$, and so $\alpha_1\lambda_1 + \alpha_2\lambda_2 = 1$, a contradiction, or
(2) $\alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_1\lambda_1\alpha_3\lambda_3 + \alpha_3^2\lambda_3^2 = \alpha_2\lambda_2$, and so from (3.32) $\alpha_1\lambda_1 - \alpha_2\lambda_2 = 1$, a contradiction.

Similarly, if $j_2 = 2$ and $j_3 = 3$, from (3.32) and (3.20) we have either

(1) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, whence $\alpha_1\lambda_1 + \alpha_3\lambda_3 = 1$, or
(2) $\alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_1\lambda_1\alpha_2\lambda_2 + \alpha_3\lambda_3\alpha_2\lambda_2 = \alpha_2\lambda_2$, which implies that $\alpha_1\lambda_1 + \alpha_1\lambda_1 + \alpha_3\lambda_3 = 1$, and so $\alpha_1\lambda_1 - \alpha_2\lambda_2 = 1$, a contradiction.

Finally, (3.25) is also valid.

Now, taking into account (3.18), (3.22), (3.25), one can easily see that Eq. (3.21) leads to a contradiction. This contradiction proves Claim 3.6. □

The following theorem states that each projection on $\mathrm{AC}(X)$ expressed as the average of three surjective linear isometries is a trivial projection.

Theorem 3.7 *If P is a projection on $\mathrm{AC}(X)$ given by the average of three surjective linear isometries, then $P = 0$ or $P = \mathrm{Id}$.*

Proof Assume that $P = (1/3)(T_1+T_2+T_3)$, where $T_i f = \lambda_i f \circ \varphi_i$ for all $f \in AC(X)$, with $\varphi_i \in MH(X)$ and $\lambda_i \in \mathbb{T}$ for all $i \in \{1, 2, 3\}$, according to Theorem 2.1.

Step 1 *We have either all φ_i's are increasing, or all of them are decreasing.*

Since for each $i \in \{1, 2, 3\}$, φ_i is a monotonic homeomorphism, we have either $\varphi_i(m_X) = m_X$ or $\varphi_i(m_X) = M_X$, where $m_X = \min(X)$ and $M_X = \max(X)$. Hence, we conclude that $\text{Card}(\mathcal{S}_{m_X}) \in \{1, 2\}$. If $\text{Card}(\mathcal{S}_{m_X}) = 2$, then from Lemma 3.3 it follows that $m_X \in B_i$ or $m_X \in C_i$ for some $i \in \{1, 2, 3\}$, which is impossible by Lemma 3.4 (here we have $\alpha_i = 1/3$). Therefore, $\text{Card}(\mathcal{S}_{m_X}) = 1$, which especially proves the claim in Step 1.

Step 2 $X = X_0$.

On the contrary, assume that $x \in X \backslash X_0$. Taking into account Lemmas 3.3 and 3.4, we immediately conclude that $x \in D_k$ for some $k \in \{1, 2, 3\}$. Let us assume that $k = 1$. Thus, we have $x = \varphi_1(x) \neq \varphi_2(x) \neq \varphi_3(x) \neq x$. Based on the order of the points, we are in one of the following situations (the other cases are similar by permutating indices 2 and 3):

(a) $x = \varphi_1(x) < \varphi_2(x) < \varphi_3(x)$,
(b) $\varphi_2(x) < \varphi_3(x) < x = \varphi_1(x)$,
(c) $\varphi_2(x) < x = \varphi_1(x) < \varphi_3(x)$.

First, let us rewrite Eq. (3.1):

$$\begin{aligned}
&\lambda_1 \left[\lambda_1 f(x) + \lambda_2 f(\varphi_2(x)) + \lambda_3 f(\varphi_3(x))\right] \\
&\quad + \lambda_2 \left[\lambda_1 f(\varphi_1(\varphi_2(x))) + \lambda_2 f(\varphi_2^2(x)) + \lambda_3 f(\varphi_3(\varphi_2(x)))\right] \\
&\quad + \lambda_3[\lambda_1 f(\varphi_1(\varphi_3(x))) + \lambda_2 f(\varphi_2(\varphi_3(x))) + \lambda_3 f(\varphi_3^2(x))] \\
&\quad = 3\left[\lambda_1 f(\varphi_1(x)) + \lambda_2 f(\varphi_2(x)) + \lambda_3 f(\varphi_3(x))\right].
\end{aligned} \tag{3.33}$$

Taking $f \in AC(X)$ with $0 \leq f \leq 1$, $f(x) = 1$ and $f(\varphi_2(x)) = f(\varphi_3(x)) = 0$, Eq. (3.33) becomes

$$\begin{aligned}
&\lambda_1^2 + \lambda_2 \left[\lambda_1 f(\varphi_1(\varphi_2(x))) + \lambda_2 f(\varphi_2^2(x)) + \lambda_3 f(\varphi_3(\varphi_2(x)))\right] \\
&\quad + \lambda_3 \left[\lambda_1 f(\varphi_1(\varphi_3(x))) + \lambda_2 f(\varphi_2(\varphi_3(x))) + \lambda_3 f(\varphi_3^2(x))\right] = 3\lambda_1.
\end{aligned}$$

Since $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{T}$, from the above equation it follows that at least two points in the set

$$\left\{\varphi_1(\varphi_2(x)), \varphi_2^2(x), \varphi_3(\varphi_2(x)), \varphi_1(\varphi_3(x)), \varphi_2(\varphi_3(x)), \varphi_3^2(x)\right\}$$

must be equal to x. Clearly, one of the following cases may occur:

(i) $x = \varphi_2^2(x) = \varphi_3(\varphi_2(x))$,
(ii) $x = \varphi_2^2(x) = \varphi_3^2(x)$,

(iii) $x = \varphi_3(\varphi_2(x)) = \varphi_2(\varphi_3(x))$,
(iv) $x = \varphi_3^2(x) = \varphi_2(\varphi_3(x))$.

In the following, we show that none of those four cases can happen. According to Step 1, either all φ_i's are increasing, or all φ_i's are decreasing.

Firstly, we assume that all of them are increasing. Consider Case (a). Since φ_2 is increasing, we obtain

$$x = \varphi_1(x) < \varphi_2(x) < \varphi_2^2(x) < \varphi_2(\varphi_3(x)),$$

which clearly implies that none of the cases (i)–(iv) can occur.

The argument in Case (b) is similar to Case (a).

Consider Case (c). Since φ_2 is increasing, $\varphi_2^2(x) < \varphi_2(x) < x$, which shows that (i) and (ii) cannot happen. Moreover, since φ_3 is increasing, then

$$\varphi_3(\varphi_2(x))) < \varphi_3(x) < \varphi_3^2(x).$$

If $x = \varphi_3^2(x)$, then $\varphi_3(x) < x$, which is a contradiction showing that (iv) never happen. Now, suppose that (iii) holds. Again since φ_3 is increasing, we have

$$\varphi_2(x) < x < \varphi_3(x) < \varphi_3^2(x),$$

which yields $\varphi_2(x) \neq \varphi_3^2(x)$. Taking $f \in \mathrm{AC}(X)$ with $0 \leq f \leq 1$, $f(\varphi_2(x)) = 1$ and $f(x) = f(\varphi_3(x)) = f(\varphi_2^2(x)) = f(\varphi_3^2(x)) = 0$, Eq. (3.33) reduces to

$$\lambda_1\lambda_2 + \lambda_1\lambda_2 f(\varphi_1(\varphi_2(x))) + \lambda_1\lambda_3 f(\varphi_1(\varphi_3(x))) = 3\lambda_2,$$

which implies that $\varphi_1(\varphi_2(x)) = \varphi_1(\varphi_3(x))$. Consequently, $\varphi_2(x) = \varphi_3(x)$ because φ_1 is injective, which is a contradiction showing that (iii) cannot occur.

Secondly, we assume that all φ_i's are decreasing. Consider Case (a). Since φ_1 is decreasing, we have

$$\varphi_3(x) > \varphi_2(x) > x > \varphi_1(\varphi_2(x)) > \varphi_1(\varphi_3(x)) \tag{3.34}$$

Taking into account (3.34), if Case (i) holds, we can choose $f \in \mathrm{AC}(X)$ with $0 \leq f \leq 1$, $f(\varphi_3(x)) = 1$ and $f(x) = f(\varphi_2(x)) = f(\varphi_3^2(x)) = f(\varphi_1(\varphi_2(x))) = f(\varphi_1(\varphi_3(x))) = 0$. From Eq. (3.33), we get

$$\lambda_1\lambda_3 + \lambda_2\left[\lambda_2 f(\varphi_2^2(x)) + \lambda_3 f(\varphi_3(\varphi_2(x)))\right] + \lambda_3\lambda_2 f(\varphi_2(\varphi_3(x))) = 3\lambda_3,$$

which shows that $\lambda_1\lambda_3 + \lambda_3\lambda_2 f(\varphi_2(\varphi_3(x))) = 3\lambda_3$ by (i), whence $|3 - \lambda_1| \leq 1$ and it is impossible.

If Case (ii) holds, again taking into account (3.34), we can select $f \in \mathrm{AC}(X)$ with $0 \leq f \leq 1$, $f(\varphi_3(x)) = 1$ and $f(x) = f(\varphi_2(x)) = f(\varphi_3(\varphi_2(x))) = f(\varphi_1(\varphi_2(x))) = f(\varphi_1(\varphi_3(x))) = 0$. Hence, from Eq. (3.33) and (ii), we can get $\lambda_1 + \lambda_2 f(\varphi_2(\varphi_3(x))) = 3$, which is impossible.

For Case (iii), from (3.34), we can choose $f \in \mathrm{AC}(X)$ with $0 \leq f \leq 1$, $f(\varphi_2(x)) = 1$ and $f(x) = f(\varphi_2^2(x)) = f(\varphi_1(\varphi_2(x))) = f(\varphi_1(\varphi_3(x))) = f(\varphi_3(x)) = 0$. Now, from Eq. (3.33) we conclude that $\lambda_1\lambda_2 + \lambda_3^2 f(\varphi_3^2(x)) = 3\lambda_2$, which is impossible.

In Case (iv), from (3.34), we can take $f \in \mathrm{AC}(X)$ with $0 \leq f \leq 1$, $f(\varphi_3(x)) = 1$ and $f(x) = f(\varphi_3(\varphi_2(x))) = f(\varphi_1(\varphi_2(x))) = f(\varphi_1(\varphi_3(x))) = f(\varphi_2(x)) = 0$. Then Eq. (3.33) gives $\lambda_1\lambda_3 + \lambda_2^2 f(\varphi_2^2(x)) = 3\lambda_3$, which yields a contradiction.

The argument for Case (b) is similar to Case (a).

Consider Case (c). Since φ_2 is decreasing, it follows that $\varphi_2^2(x) > \varphi_2(x) > \varphi_2(\varphi_3(x))$. If $\varphi_2(\varphi_3(x)) = x$, then $x > \varphi_2(x) > x$, which is a contradiction showing that (iii) and (iv) are impossible.

If $\varphi_3(\varphi_2(x)) = x$, then $x = \varphi_3(\varphi_2(x)) > \varphi_3(x) > x$ because φ_3 is decreasing, which implies that (i) does not hold.

Since φ_1 and φ_2 are decreasing, it is clear that $\varphi_3(x) > \varphi_1(x) = x > \varphi_1(\varphi_3(x))$ and $\varphi_3(x) > x > \varphi_2(x) > \varphi_2(\varphi_3(x))$. Take now some $f \in \mathrm{AC}(X)$ with $0 \leq f \leq 1$, $f(\varphi_3(x)) = 1$ and $f(x) = f(\varphi_2(x)) = f(\varphi_2(\varphi_3(x))) = f(\varphi_1(\varphi_3(x))) = f(\varphi_3(\varphi_2(x))) = 0$. If (ii) holds, Eq. (3.33) gives $\lambda_1\lambda_3 + \lambda_2\lambda_1 f(\varphi_1(\varphi_2(x))) = 3\lambda_3$, a contradiction.

The above discussion proves Step 2. Now, Theorem 3.7 follows from Lemma 3.2. □

As observed, the trivial projections are the only projections as the average of three surjective linear isometries while, as proved below, it is not true for the projections in the convex hull of three isometries.

Theorem 3.8 *A projection P on $\mathrm{AC}(X)$ is in the convex hull of three surjective linear isometries if and only if P is a generalized bi-circular projection, i.e., $Pf = (1/2)[f + \lambda f \circ \varphi]$ for all $f \in \mathrm{AC}(X)$, where $\lambda \in \{1, -1\}$ and $\varphi \in \mathrm{MH}(X)$ with $\varphi^2 = \mathrm{id}_X$.*

Proof We only prove the necessity because the proof of the sufficiency is easy by [[7], Corollary 3.3]. Let P be a projection on $\mathrm{AC}(X)$ such that $P = \sum_{i=1}^{3} \alpha_i T_i$, where $0 < \alpha_i < 1$ with $\sum_{i=1}^{3} \alpha_i = 1$ and $T_i f = \lambda_i f \circ \varphi_i$ for all $f \in \mathrm{AC}(X)$, with $\lambda_i \in \mathbb{T}$ and $\varphi_i \in \mathrm{MH}(X)$ for all $i \in \{1, 2, 3\}$.

According to Theorem 3.7, the result is valid if $\alpha_1 = \alpha_2 = \alpha_3$. Otherwise, from Lemma 3.4 it follows that any two of the statements $C_1 \neq \emptyset$, $C_2 \neq \emptyset$, $C_3 \neq \emptyset$ (resp., $B_1 \neq \emptyset$, $B_2 \neq \emptyset$, $B_3 \neq \emptyset$) cannot occur simultaneously, and also the conditions $C_i \neq \emptyset$ and $B_j \neq \emptyset$ with $i \neq j$ are incompatible. Then, taking into account Lemmas 3.3 and 3.5, we have the following possibilities:

(1) $X = X_0$.
(2) $X = X_0 \cup B_i$ for $i = 1, 2, 3$.
(3) $X = X_0 \cup C_i$ for $i = 1, 2, 3$.
(4) $X = X_0 \cup B_i \cup C_i$ with $B_i \neq \emptyset \neq C_i$ for $i = 1, 2, 3$.

Case (1). Then $P = 0$ or $P = \mathrm{Id}$ by Lemma 3.2.

Case (2). Here we have $\varphi_j = \varphi_k$ with $j, k \in \{1, 2, 3\}\backslash\{i\}$, and also from Lemma 3.4, it follows that $\lambda_i \in \{1, -1\}$, and

$$Pf = \frac{1}{2}\left[\lambda_i f \circ \varphi_i + f \circ \varphi_j\right] \qquad (f \in \mathrm{AC}(X)).$$

Cases (3) and (4). Similarly to the second case, taking into account Lemma 3.4, we have $\lambda_j \in \{1, -1\}$, $\varphi_j = \varphi_k$ with $j, k \in \{1, 2, 3\}\backslash\{i\}$ and

$$Pf = \frac{1}{2}\left[f \circ \varphi_i + \lambda_j f \circ \varphi_j\right] \qquad (f \in \mathrm{AC}(X)).$$

In the latter three cases, P is expressed as the average of two surjective linear isometries of $\mathrm{AC}(X)$ and, indeed, it is a generalized bi-circular projection by Theorem 2.2. Hence, $Pf = (1/2)[f + \lambda f \circ \varphi]$ for every $f \in \mathrm{AC}(X)$, where $\lambda \in \{1, -1\}$ and $\varphi \in \mathrm{MH}(X)$ with $\varphi^2 = \mathrm{id}_X$. To be more precise, following the proof of Theorem 3.2 in [9], it is proved that in Case (2) we have $\varphi_j = \mathrm{id}_X$, $\varphi = \varphi_i \in \mathrm{MH}(X)$, $\varphi^2 = \mathrm{id}_X$ and $\lambda = \lambda_i \in \{1, -1\}$. In Case (3), $\varphi_i = \mathrm{id}_X$, $\varphi = \varphi_j \in \mathrm{MH}(X)$, $\varphi^2 = \mathrm{id}_X$ and $\lambda = \lambda_j \in \{1, -1\}$. Also, Case (4) will not happen. □

Remark 3.9 We would also like to remark that, taking into account [8, Corollary 4.3] (see also [10, Example 2]), our theorems remain true when $\mathrm{AC}(X)$ is equipped with the maximum norm

$$\|f\|_M = \max\left\{\|f\|_\infty, \mathcal{V}(f)\right\} \qquad (f \in \mathrm{AC}(X)),$$

and we consider the convex hull of three surjective linear isometries of $(\mathrm{AC}(X), \|\cdot\|_M)$ carrying a weighted composition operator form.

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