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Results in Mathematics

Approximate Local Isometries on Spaces of Absolutely Continuous Functions

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Abstract. Let $\mathrm{AC}(X)$ be the Banach algebra of all absolutely continuous complex-valued functions f on a compact subset $X \subset \mathbb{R}$ with at least two points under the norm $\|f\|_\Sigma = \|f\|_\infty + \mathrm{V}(f)$, where $\mathrm{V}(f)$ denotes the total variation of f. We prove that every approximate local isometry from $\mathrm{AC}(X)$ to $\mathrm{AC}(Y)$ admits a Banach–Stone type representation as an isometric weighted composition operator. Using this description, we prove that the set of linear isometries from $\mathrm{AC}(X)$ onto $\mathrm{AC}(Y)$ is algebraically reflexive and 2-algebraically reflexive. Moreover, it is shown that although the topological reflexivity and 2-topological reflexivity do not necessarily hold for the isometry group of $\mathrm{AC}(X)$, but they hold for the sets of isometric reflections and generalized bi-circular projections of $\mathrm{AC}(X)$.

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1. Introduction

Let $\mathcal{B}(E)$ be the Banach space of all continuous linear operators of a Banach space E and let $\mathcal{G}(E)$ be its group of surjective linear isometries. Let us recall that a local isometry of E is a map $T \in \mathcal{B}(E)$ such that for every $e \in E$, there exists a $T_e \in \mathcal{G}(E)$, possibly depending on e, such that $T_e(e) = T(e)$. Moreover, an approximate local isometry of E is a map $T \in \mathcal{B}(E)$ satisfying that for every $e \in E$, there is a sequence $\{T_{e,n}\}_{n\in\mathbb{N}}$ in $\mathcal{G}(E)$ such that $\lim_{n\to\infty} T_{e,n}(e) = T(e)$.

The main question addressed by the authors is for which Banach spaces E, every local isometry of E is a surjective isometry or, equivalently, which

Banach spaces E have an algebraically reflexive isometry group. One can also study the topological version of this question, that is, when every approximate local isometry of E is a surjective isometry or, with other words, when the isometry group of E is topologically reflexive.

In [17], Molnár initiated the study of 2-local isometries of Banach spaces. Namely, a map $\Delta\colon E \to E$ (which is not assumed to be linear) is a 2-local isometry if for any $e, u \in E$, there exists a $T_{e,u} \in \mathcal{G}(E)$, depending in general on e and u, such that $T_{e,u}(e) = \Delta(e)$ and $T_{e,u}(u) = \Delta(u)$. Moreover, $\Delta\colon E \to E$ is an approximate 2-local isometry if for every $e, u \in E$, there is a sequence $\{T_{e,u,n}\}_{n\in\mathbb{N}}$ in $\mathcal{G}(E)$ such that $\lim_{n\to\infty} T_{e,u,n}(e) = \Delta(e)$ and $\lim_{n\to\infty} T_{e,u,n}(u) = \Delta(u)$. In the 2-local setting, the main problem is to study for which Banach spaces E, every 2-local isometry (respectively, approximate 2-local isometry) of E is a surjective linear isometry or, equivalently, when the isometry group of E is 2-algebraically reflexive (respectively, 2-topologically reflexive).

An extensive research has been done on (approximate) local isometries [4–7,13,18] and (approximate) 2-local isometries [10,11,14,16] for different spaces of continuous scalar-valued functions. In this paper, we deal with approximate local (2-local) isometries on spaces of absolutely continuous complex-valued functions on an arbitrary bounded time scale $X \subset \mathbb{R}$. These spaces are a fundamental tool to solve boundary value problems in ordinary and partial differential equations and difference equations (see, for example, [2,21]).

Let X be a compact subset of $\mathbb{R}$ with at least two points. A finite increasing subset of X is a set $P = \{x_0, x_1, \ldots, x_{n-1}, x_n\} \subseteq X$, where $x_0 < x_1 < \cdots < x_{n-1} < x_n$. Given a function $f\colon X \to \mathbb{C}$ and a finite increasing subset $P = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ of X, define

$$\mathrm{V}(P, f) := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \, .$$

We define the total variation of f on X as

$$\mathrm{V}(f, X) := \sup\{\mathrm{V}(P, f)\colon P \text{ is a finite increasing subset of } X\} \in [0, \infty].$$

If $\mathrm{V}(f, X) < \infty$, we say that f is a function of bounded variation on X. We shall simply write $\mathrm{V}(f)$ when there is no confusion.

A function $f\colon X \to \mathbb{C}$ is said to be absolutely continuous if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\{(a_i, b_i)\}_{i=1}^n$, with $a_i, b_i \in X$, is a finite pairwise disjoint of open intervals satisfying $\sum_{i=1}^n (b_i - a_i) < \delta$, then $\sum_{i=1}^n |f(b_i) - f(a_i)| < \varepsilon$. It is known that every absolutely continuous function $f\colon X \to \mathbb{C}$ has bounded variation on X.

Let $\mathrm{AC}(X)$ denote the space of all absolutely continuous functions $f\colon X \to \mathbb{C}$, with the norm

$$\|f\|_\Sigma = \|f\|_\infty + \mathrm{V}(f),$$

where $\|f\|_\infty$ denotes the supremum of $|f|$ on X. It is known that $\mathrm{AC}(X)$ is a unital semisimple commutative Banach algebra which contains the constant functions and separates the points of X. We refer the reader to [22] for the study of the classical spaces $\mathrm{AC}([a,b])$ and to [1,9] for the general case $\mathrm{AC}(X)$.

The topological reflexivity does not hold true for the isometry group of any $\mathrm{AC}(X)$-space. We present in this paper an easy counterexample. However, we can describe the elements of the topological reflexive closure of the set $\mathcal{G}(\mathrm{AC}(X),\mathrm{AC}(Y))$ of all linear isometries of $\mathrm{AC}(X)$ onto $\mathrm{AC}(Y)$. To be more precise, we prove that every approximate local isometry T from $\mathrm{AC}(X)$ to $\mathrm{AC}(Y)$ is an isometric weighted composition operator induced by a monotonic absolutely continuous function ϕ from Y onto X. Using this description, we prove that the set $\mathcal{G}(\mathrm{AC}(X),\mathrm{AC}(Y))$ is algebraically reflexive. Moreover, it is shown that every (approximate) 2-local isometry from $\mathrm{AC}(X)$ to $\mathrm{AC}(Y)$ is a (an approximate) local isometry and, as a consequence, we establish the 2-algebraic reflexivity of $\mathcal{G}(\mathrm{AC}(X),\mathrm{AC}(Y))$. As mentioned above, the isometry group $\mathrm{AC}(X)$ is usually neither topologically reflexive nor 2-topologically reflexive, but we show that the sets of isometric reflections and generalized bi-circular projections of $\mathrm{AC}(X)$ are.

The starting point of our study is an appropriate description of surjective linear isometries between $\mathrm{AC}(X)$-spaces in terms of a weighted composition operator. Such isometries of $\mathrm{AC}(X)$ equipped with the Σ-norm were characterized by Pathak [19]. Previously, the isometries of $\mathrm{AC}([0,1])$ were investigated by Cambern [8] and Rao and Roy [20]. Moreover, Jarosz and Pathak [15] developed a technical scheme to verify that surjective linear isometries between some classical function spaces are induced by homeomorphisms between corresponding Hausdorff compact spaces. In particular, this holds for $\mathrm{AC}(X)$-spaces.

The first author stated in [13] the algebraic reflexivity of the set of onto linear isometries between spaces of absolutely continuous vector-valued functions with a norm different to the Σ-norm. Our approach here is quite different because our objective is to study the topological reflexivity, or at least to describe the elements of the topological reflexive closure, and it essentially depends on the application of two spherical reformulations of the known theorems of Gleason–Kahane–Żelazko and of Kowalski–Słodkowski, stated by Li, Peralta, Wang and Wang [16] who applied them to study the 2-locality problem of the isometry group of uniform algebras and Lipschitz algebras.

We thus extend here some results established by the first author in [13] concerning the algebraic reflexivity of the sets of onto linear isometries, isometric reflections and generalized bi-circular projections of $\mathrm{AC}(X)$-spaces.

2. Preliminaries

Let us recall the concepts of reflexivity addressed in this paper. Let E and F be Banach spaces, F^E the set of all maps of E to F and $\mathcal{B}(E,F)$ the set of all continuous linear maps of E to F. Let $\mathcal{S}$ be a nonempty subset of $\mathcal{B}(E,F)$. Define the algebraic reflexive and topological reflexive closures of $\mathcal{S}$, respectively, by

$$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \{T \in \mathcal{B}(E,F) \colon T(e) \in \mathcal{S}(e),\ \forall e \in E\},$$
$$\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \left\{T \in \mathcal{B}(E,F) \colon T(e) \in \overline{\mathcal{S}(e)},\ \forall e \in E\right\},$$

where $\mathcal{S}(e) = \{L(e) \colon L \in \mathcal{S}\}$ and $\overline{\mathcal{S}(e)}$ denotes its norm-closure in F. The set $\mathcal{S}$ is said to be algebraically reflexive (topologically reflexive) if $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \mathcal{S}$ (respectively, $\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \mathcal{S}$).

Consider now the 2-algebraic reflexive closure of $\mathcal{S}$, 2-$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$, given by

$$\left\{\Delta \in F^E \colon \forall e,u \in E,\ \exists S_{e,u} \in \mathcal{S} \,|\, S_{e,u}(e) = \Delta(e),\ S_{e,u}(u) = \Delta(u)\right\},$$

and the 2-topological reflexive closure of $\mathcal{S}$, 2-$\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$, defined by

$$\Big\{\Delta \in F^E \colon \forall e,u \in E,\ \exists \{S_{e,u,n}\}_{n\in\mathbb{N}} \subset \mathcal{S} \,|\, \lim_{n\to\infty} S_{e,u,n}(e) = \Delta(e),\ \lim_{n\to\infty} S_{e,u,n}(u) = \Delta(u)\Big\}.$$

We say that the set $\mathcal{S}$ is 2-algebraically reflexive (2-topologically reflexive) if 2-$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \mathcal{S}$ (respectively, 2-$\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \mathcal{S}$).

In the sequel, X and Y are two compact subsets of $\mathbb{R}$ with at least two points. We denote $m_X = \min(X)$, $M_X = \max(X)$, and similarly, $m_Y = \min(Y)$, $M_Y = \max(Y)$.

We denote by $\mathrm{MH}(Y,X)$ the set of all monotonic absolutely continuous homeomorphisms of Y onto X. In particular, we write $\mathrm{MH}(X)$ instead of $\mathrm{MH}(X,X)$ and $\mathrm{MH}^2(X)$ represents the set of functions $\phi \in \mathrm{MH}(X)$ such that $\phi^2 = \mathrm{id}_X$, where id_X is the identity map of X. The symbol 1_X stands for the function constantly 1 on X and $|X|$ for the cardinality of X. Given a Banach space E, we denote by Id_E the identity operator of E. As usual, $\mathbb{T}$ denotes the unit circle of $\mathbb{C}$.

The following functions will frequently be used through the paper. For each $x_0 \in X$ and $r > 0$, the function $h_{x_0,r} \colon X \to [0,1]$ defined by

$$h_{x_0,r}(x) = \max\left\{0, 1 - \frac{|x - x_0|}{r}\right\},$$

is Lipschitz with $h^{-1}(\{1\}) = \{x_0\}$ and $h^{-1}(\{0\}) = \{x \in X \colon |x - x_0| \geq r\}$. Notice that every Lipschitz function is absolutely continuous.

3. Results

For our purposes, we first need to complete a description of onto linear isometries between $\mathrm{AC}(X)$-spaces stated by Pathak (see Theorem 2.10 and Lemma 2.3 in [19]). We deduce this description by applying a more general result of Jarosz and Pathak (see [15, Example 7]).

Theorem 1. ([15,19]) *A map* $T\colon \mathrm{AC}(X)\to\mathrm{AC}(Y)$ *is a surjective linear isometry if and only if there exist a number* $\lambda\in\mathbb{T}$ *and a map* $\phi\in\mathrm{MH}(Y,X)$ *such that*

$$T(f)(y)=\lambda f(\phi(y)) \qquad (f\in\mathrm{AC}(X),\ y\in Y).$$

Proof. Assume that T is a linear isometry from $\mathrm{AC}(X)$ onto $\mathrm{AC}(Y)$. By [15, Proposition], T is of the form

$$T(f)(y)=\chi(y)f(\phi(y)) \qquad (f\in\mathrm{AC}(X),\ y\in Y),$$

where $\phi\colon Y\to X$ is a homeomorphism and $\chi\in\mathrm{AC}(Y)$ is a unimodular function with $\mathrm{V}(\chi)=0$. Hence χ is a constant function, therefore there is a $\lambda\in\mathbb{T}$ such that $\chi(y)=\lambda$ for all $y\in Y$, and thus

$$T(f)(y)=\lambda f(\phi(y)) \qquad (f\in\mathrm{AC}(X),\ y\in Y).$$

Clearly, $\phi=\overline{\lambda}T(\mathrm{id}_X)\in\mathrm{AC}(Y)$. Since $T^{-1}\colon\mathrm{AC}(Y)\to\mathrm{AC}(X)$ has the form

$$T^{-1}(g)(x)=\overline{\lambda}g(\phi^{-1}(x)) \qquad (g\in\mathrm{AC}(Y),\ x\in X),$$

it follows that $\phi^{-1}=\lambda T^{-1}(\mathrm{id}_Y)\in\mathrm{AC}(X)$ and so ϕ is an absolutely continuous homeomorphism. Moreover, taking into account the representation of T, the monotonicity of ϕ can be obtained by an argument similar to the proof of [12, Lemma3.15].

Conversely, assume that $T\colon\mathrm{AC}(X)\to\mathrm{AC}(Y)$ is a map having the form

$$T(f)(y)=\lambda f(\phi(y)) \qquad (f\in\mathrm{AC}(X),\ y\in Y)$$

with λ and ϕ being as in the statement of the theorem. The linearity of T is immediate. To prove its surjectivity, given $g\in\mathrm{AC}(Y)$, take $f=\overline{\lambda}(g\circ\phi^{-1})$. Since ϕ^{-1} is absolutely continuous and strictly monotonic, it follows easily that $f\in\mathrm{AC}(X)$ and clearly $T(f)=g$. Finally, given $f\in\mathrm{AC}(X)$, an easy verification shows that $\|T(f)\|_\infty=\|f\|_\infty$ and $\mathrm{V}(T(f))=\mathrm{V}(f)$, and therefore T is an isometry with respect to the Σ-norms. □

For a later reference we deduce from the proof of Theorem 1 the following fact.

Corollary 1. *Let* T *be a linear isometry from* $\mathrm{AC}(X)$ *onto* $\mathrm{AC}(Y)$*. Then* $\|T(f)\|_\infty=\|f\|_\infty$ *and* $\mathrm{V}(T(f))=\mathrm{V}(f)$ *for all* $f\in\mathrm{AC}(X)$*.* □

The following example, borrowed from [3], shows that $\mathrm{AC}([0,1])$ is neither topologically reflexive nor 2-topologically reflexive. In what follows, we shall use the following notation. Given a function $f\colon X\to\mathbb{C}$, a subset $A\subseteq X$ with $|A|\geq 2$ and a partition P of A, we set $\mathrm{V}(P,f,A):=\mathrm{V}(P,f|_A)$ and $\mathrm{V}(f,A):=\mathrm{V}(f|_A,A)$.

Example 1. For each $n\geq 2$, define the functions $\phi,\phi_n\colon[0,1]\to[0,1]$ by

$$\phi(x)=\begin{cases} 0 & x\in[0,\frac{1}{2}],\\ 2x-1 & x\in[\frac{1}{2},1]\end{cases}$$

$$\phi_n(x) = \begin{cases} \frac{2}{n+1}x & x \in [0, \frac{n+1}{2n}], \\ \\ 2x-1 & x \in [\frac{n+1}{2n}, 1]. \end{cases}$$

Moreover, define $T, T_n \colon \mathrm{AC}([0,1]) \to \mathrm{AC}([0,1])$ by $T(f) = f \circ \phi$ and $T_n(f) = f \circ \phi_n$. Clearly, each T_n is a surjective linear isometry.

We next show that $T(f) = \lim_{n\to\infty} T_n(f)$ for each $f \in \mathrm{AC}([0,1])$, whence T is an approximate local isometry. Let $f \in \mathrm{AC}([0,1])$. Since one can see that $\{\|T_n(f) - T(f)\|_\infty\}_{n\in\mathbb{N}} \to 0$ by Dini's theorem, it is enough to show that $\{\mathrm{V}(T_n(f) - T(f))\}_{n\in\mathbb{N}} \to 0$. For this purpose, let $\epsilon > 0$ and choose $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \frac{\epsilon}{4},$$

for every finite family of non-overlapping open intervals $\{(a_i, b_i) \colon i = 1, \ldots, n\}$ whose extreme points belong to $[0,1]$ and $\sum_{i=1}^{n}(b_i - a_i) < \delta$.

Choose $n_0 \in \mathbb{N}$ such that $1/n_0 < \min\{\delta, 1/2\}$, and $0 < (n+1)/2n - 1/2 < \delta$ for all $n \geq n_0$. From the additive property of the total variation we have

$$\mathrm{V}\left(T_n(f) - T(f)\right) = \mathrm{V}\left(T_n(f) - T(f), \left[0, \frac{1}{2}\right]\right) + \mathrm{V}\left(T_n(f) - T(f), \left[\frac{1}{2}, 1\right]\right).$$

If $P = \{x_0, \ldots, x_m\}$ is a partition of $[0, 1/2]$, then for each $n \geq n_0$

$$\begin{aligned} \mathrm{V}\left(P, T_n(f) - T(f), \left[0, \frac{1}{2}\right]\right) &= \sum_{i=1}^{m} |(T_n(f) - T(f))(x_i) \\ &\quad - (T_n(f) - T(f))(x_{i-1})| \\ &= \sum_{i=1}^{m} |f(\phi_n(x_i)) - f(\phi(x_i)) \\ &\quad - f(\phi_n(x_{i-1})) + f(\phi(x_{i-1}))| \\ &= \sum_{i=1}^{m} \left| f\left(\frac{2}{n+1}x_i\right) - f\left(\frac{2}{n+1}x_{i-1}\right)\right| < \frac{\epsilon}{4}, \end{aligned}$$

because

$$\sum_{i=1}^{m} \left(\frac{2}{n+1}x_i - \frac{2}{n+1}x_{i-1}\right) = \frac{2}{n+1}\frac{1}{2} = \frac{1}{n+1} < \delta.$$

Thus, for all $n \geq n_0$, we have

$$\mathrm{V}\left(T_n(f) - T(f), \left[0, \frac{1}{2}\right]\right) \leq \frac{\epsilon}{4}.$$

Now, suppose that $n \geq n_0$ and $P = \{x_0, \ldots, x_m\}$ is a partition of $[1/2, 1]$. We assume, with no loss of generality, that $x_k = (n+1)/2n$ for some $1 \leq k < m$.

Then we get

$$\begin{aligned}
&\mathrm{V}\left(P, T_n(f)-T(f),\left[\frac{1}{2},1\right]\right)=\sum_{i=1}^{m}|(T_n(f)-T(f))(x_i)-(T_n(f)-T(f))(x_{i-1})|\\
&=\sum_{i=1}^{m}|f(\phi_n(x_i))-f(\phi(x_i))-f(\phi_n(x_{i-1}))+f(\phi(x_{i-1}))|\\
&=\sum_{i=1}^{k}|f\left(\frac{2}{n+1}x_i\right)-f(2x_i-1)-f\left(\frac{2}{n+1}x_{i-1}\right)+f(2x_{i-1}-1)|+0\\
&\leq\sum_{i=1}^{k}\left|f\left(\frac{2}{n+1}x_i\right)-f\left(\frac{2}{n+1}x_{i-1}\right)\right|+\sum_{i=1}^{k}|f(2x_i-1)-f(2x_{i-1}-1)|\\
&<\frac{\epsilon}{4}+\frac{\epsilon}{4}=\frac{\epsilon}{2},
\end{aligned}$$

because

$$\sum_{i=1}^{k}\left(\frac{2}{n+1}x_i-\frac{2}{n+1}x_{i-1}\right)\leq\frac{2}{n+1}\frac{n+1}{2n}=\frac{1}{n}<\delta$$

and

$$\sum_{i=1}^{k}(2x_i-1-2x_{i-1}+1)=2\sum_{i=1}^{k}(x_i-x_{i-1})=2\left(\frac{n+1}{2n}-\frac{1}{2}\right)=\frac{1}{n}<\delta.$$

Then for any $n\geq n_0$, we have

$$\mathrm{V}\left(T_n(f)-T(f),\left[\frac{1}{2},1\right]\right)\leq\frac{\epsilon}{2},$$

and in consequence, $\mathrm{V}(T_n(f)-T(f))<\epsilon$. Therefore, $\{\mathrm{V}(T_n(f)-T(f))\}_{n\in\mathbb{N}}\to 0$, as claimed.

The main result of this paper is the following description of the topological reflexive closure of the set of linear isometries from $\mathrm{AC}(X)$ onto $\mathrm{AC}(Y)$.

Theorem 2. *Let $T\in\mathrm{ref}_{\mathrm{top}}(\mathcal{G}(\mathrm{AC}(X),\mathrm{AC}(Y)))$. Then T is an isometry of the form*

$$T(f)(y)=\lambda f(\phi(y))\qquad(f\in\mathrm{AC}(X),\ y\in Y),$$

where $\lambda\in\mathbb{T}$ and $\phi\colon Y\to X$ is a surjective, monotonic, absolutely continuous function. Moreover, T is surjective if and only if ϕ is injective and ϕ^{-1} is absolutely continuous.

Proof. We have divided the proof into a series of claims. □

Claim 1. $\|T(f)\|_\Sigma=\|f\|_\Sigma$, $\|T(f)\|_\infty=\|f\|_\infty$ *and* $\mathrm{V}(T(f))=\mathrm{V}(f)$ *for all* $f\in\mathrm{AC}(X)$.

Let $f \in \mathrm{AC}(X)$. Then there exists a sequence $\{T_{f,n}\}_{n\in\mathbb{N}}$ in $\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y))$ such that $\lim_{n\to\infty} T_{f,n}(f) = T(f)$. It is clear that $\lim_{n\to\infty} \|T_{f,n}(f)\|_\Sigma = \|T(f)\|_\Sigma$, $\lim_{n\to\infty} \|T_{f,n}(f)\|_\infty = \|T(f)\|_\infty$ and $\lim_{n\to\infty} \mathrm{V}(T_{f,n}(f)) = \mathrm{V}(T(f))$. Since $\|T_{f,n}(f)\|_\Sigma = \|f\|_\Sigma$, $\|T_{f,n}(f)\|_\infty = \|f\|_\infty$ and $\mathrm{V}(T_{f,n}(f)) = \mathrm{V}(f)$ for all $n \in \mathbb{N}$ by Corollary 1, the claim holds.

Claim 2. *For every $f \in \mathrm{AC}(X)$, there exist sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ and $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{MH}(Y, X)$ such that $\lim_{n\to\infty} \lambda_{f,n}(f \circ \phi_{f,n}) = T(f)$.*

Since $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y)))$, the claim is an immediate consequence of Theorem 1.

Claim 3. *There exists a number $\lambda \in \mathbb{T}$ such that $T(1_X) = \lambda 1_Y$.*

Claim 1 yields $\mathrm{V}(T(1_X)) = \mathrm{V}(1_X) = 0$ and $\|T(1_X)\|_\infty = \|1_X\|_\infty = 1$. Hence $T(1_X)$ is a unimodular constant function on Y, and therefore $T(1_X) = \lambda 1_Y$ for some $\lambda \in \mathbb{T}$.

Claim 4. *For each $y \in Y$, the mapping $S_y\colon \mathrm{AC}(X) \to \mathbb{C}$ defined by*

$$S_y(f) = \overline{\lambda} T(f)(y) \qquad (f \in \mathrm{AC}(X)),$$

is a unital multiplicative linear functional.

Let $y \in Y$. Clearly, $S_y(1_X) = 1$ by Claim 3. By the linearity of T, so is S_y. Moreover, S_y is continuous since

$$|S_y(f)| = |T(f)(y)| \leq \|T(f)\|_\infty \leq \|T(f)\|_\Sigma = \|f\|_\Sigma$$

for all $f \in \mathrm{AC}(X)$. To prove that S_y is multiplicative, fix $f \in \mathrm{AC}(X)$. By Claim 2, there exist sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ and $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{MH}(Y, X)$ such that

$$T(f) = \lim_{n\to\infty} \lambda_{f,n}(f \circ \phi_{f,n}).$$

It follows that

$$S_y(f) = \overline{\lambda} T(f)(y) = \lim_{n\to\infty} \overline{\lambda} \lambda_{f,n} f(\phi_{f,n}(y)) \in \mathbb{T}\sigma(f),$$

where $\sigma(f)$ denotes the spectrum of f. Applying the spherical version of the Gleason–Kahane–Żelazko theorem stated in [16, Proposition 2.2], we conclude that $S_y = \overline{S_y(1_X)} S_y$ is multiplicative.

Claim 5. *There exists a surjective, monotonic, absolutely continuous function $\phi\colon Y \to X$ such that $T(f)(y) = \lambda f(\phi(y))$ for all $y \in Y$ and $f \in \mathrm{AC}(X)$.*

Using Claim 4, we deduce easily that the mapping $S\colon \mathrm{AC}(X) \to \mathrm{AC}(Y)$ defined by

$$S(f)(y) = \overline{\lambda} T(f)(y) \qquad (f \in \mathrm{AC}(X),\ y \in Y)$$

is a unital algebra homomorphism. By Gelfand theory, S is continuous and induces a continuous mapping $\phi\colon Y \to X$ such that $S(f)(y) = f(\phi(y))$ for all $f \in \mathrm{AC}(X)$ and $y \in Y$, and therefore

$$T(f)(y) = \lambda f(\phi(y)) \qquad (f \in \mathrm{AC}(X),\ y \in Y).$$

In fact, $\phi = \mathrm{id}_X \circ \phi = \overline{\lambda} T(\mathrm{id}_X) \in \mathrm{AC}(Y)$. Notice that ϕ is not constant since

$$\mathrm{V}(\phi) = \mathrm{V}(\overline{\lambda} T(\mathrm{id}_X)) = \mathrm{V}(T(\mathrm{id}_X)) = \mathrm{V}(\mathrm{id}_X) > 0.$$

To show the surjectivity of ϕ, assume on the contrary that there exists $x_0 \in X$ with $x_0 \notin \phi(Y)$. Being $\phi(Y)$ compact, we have $r = d(x_0, \phi(Y)) > 0$. Take the function $h_{x_0,r} \in \mathrm{AC}(X)$ and notice that $h_{x_0,r}(x_0) = 1$ and $h_{x_0,r} = 0$ on $\phi(Y)$. Hence $T(h_{x_0,r})(y) = \lambda h_{x_0,r}(\phi(y)) = 0$ for all $y \in Y$, but T is injective by Claim 1, a contradiction.

We next show that ϕ is monotonic. By Claim 2, there exist sequences $\{\lambda_{\mathrm{id}_X,n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ and $\{\phi_{\mathrm{id}_X,n}\}_{n\in\mathbb{N}}$ in $\mathrm{MH}(Y, X)$ such that

$$T(\mathrm{id}_X) = \lim_{n\to\infty} \lambda_{\mathrm{id}_X,n} \phi_{\mathrm{id}_X,n}.$$

We assume, without loss of generality, that $\{\lambda_{\mathrm{id}_X,n}\}_{n\in\mathbb{N}}$ is convergent to a certain $\lambda_{\mathrm{id}_X} \in \mathbb{T}$, and so

$$\phi(y) = \overline{\lambda} T(\mathrm{id}_X)(y) = \overline{\lambda}\lambda_{\mathrm{id}_X} \lim_{n\to\infty} \phi_{\mathrm{id}_X,n}(y) \qquad (y \in Y).$$

Notice that $\overline{\lambda}\lambda_{\mathrm{id}_X} = \pm 1$ because $|\lambda\overline{\lambda}_{\mathrm{id}_X}| = 1$ and $\lambda\overline{\lambda}_{\mathrm{id}_X} = (\lim_{n\to\infty} \phi_{\mathrm{id}_X,n}(y))/\phi(y) \in \mathbb{R}$ for some $y \in Y$ with $\phi(y) \neq 0$. For each $n \in \mathbb{N}$, since $\phi_{\mathrm{id}_X,n} \in \mathrm{MH}(Y, X)$, one of the following cases holds: $\phi_{\mathrm{id}_X,n}(m_Y) = m_X$ and $\phi_{\mathrm{id}_X,n}(M_Y) = M_X$, or $\phi_{\mathrm{id}_X,n}(m_Y) = M_X$ and $\phi_{\mathrm{id}_X,n}(M_Y) = m_X$. Since

$$\phi(m_Y) = \overline{\lambda}\lambda_{\mathrm{id}_X} \lim_{n\to\infty} \phi_{\mathrm{id}_X,n}(m_Y)$$

and

$$\phi(M_Y) = \overline{\lambda}\lambda_{\mathrm{id}_X} \lim_{n\to\infty} \phi_{\mathrm{id}_X,n}(M_Y),$$

one can easily conclude that there exists some $n_0 \in \mathbb{N}$ such that either $\phi_{\mathrm{id}_X,n}(m_Y) = m_X$ and $\phi_{\mathrm{id}_X,n}(M_Y) = M_X$ for all $n \geq n_0$, or $\phi_{\mathrm{id}_X,n}(m_Y) = M_X$ and $\phi_{\mathrm{id}_X,n}(M_Y) = m_X$ for all $n \geq n_0$. It shows that either all $\phi_{\mathrm{id}_X,n}$'s with $n \geq n_0$ are increasing, or all of them are decreasing. Now, taking into account that $\phi(y) = \overline{\lambda}\lambda_{\mathrm{id}_X} \lim_{n\to\infty} \phi_{\mathrm{id}_X,n}(y)$ for all $y \in Y$ and $\overline{\lambda}\lambda_{\mathrm{id}_X} \in \{1, -1\}$, it easily follows that ϕ is monotonic.

Claim 6. *T is surjective if and only if ϕ is injective and ϕ^{-1} is absolutely continuous.*

Assume that T is surjective. Let y_1, y_2 be in Y with $\phi(y_1) = \phi(y_2)$. By Claim 5, we have

$$T(f)(y_1) = \lambda f(\phi(y_1)) = \lambda f(\phi(y_2)) = T(f)(y_2)$$

for all $f \in \mathrm{AC}(X)$, that is, $g(y_1) = g(y_2)$ for all $g \in \mathrm{AC}(Y)$. This implies $y_1 = y_2$ since $\mathrm{AC}(Y)$ separates the points of Y and thus ϕ is injective. Now, using Claim 5, we have $\overline{\lambda} g \circ \phi^{-1} \in \mathrm{AC}(X)$ for all $g \in \mathrm{AC}(Y)$ by the surjectivity of T, and taking $g = \lambda \mathrm{id}_Y$, we conclude that $\phi^{-1} \in \mathrm{AC}(X)$. This proves an implication, and the converse one follows by applying Theorem 1.

Now we prove that every local isometry between $\mathrm{AC}(X)$-spaces is a surjective isometry.

Corollary 2. *The set $\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y))$ is algebraically reflexive.*

Proof. Let $T \in \mathrm{ref}_{\mathrm{alg}}(\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y)))$. By Theorem 2 we have a number $\lambda \in \mathbb{T}$ and a surjective, monotonic, absolutely continuous function $\phi\colon Y \to X$ such that

$$T(f)(y) = \lambda f(\phi(y)) \qquad (f \in \mathrm{AC}(X),\ y \in Y).$$

We need to show that $\phi \in \mathrm{MH}(Y, X)$ to assure that T is surjective by Theorem 1. Define $h(x) = x - m_X + 1$ for all $x \in X$. Since $T \in \mathrm{ref}_{\mathrm{alg}}(\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y)))$, Theorem 1 asserts the existence of a number $\lambda_h \in \mathbb{T}$ and a map $\phi_h \in \mathrm{MH}(Y, X)$ for which

$$T(h)(y) = \lambda_h h(\phi_h(y)) = \lambda_h(\phi_h(y) - m_X + 1) \qquad (y \in Y).$$

It follows that $\lambda(\phi(y) - m_X + 1) = \lambda_h(\phi_h(y) - m_X + 1)$ for each $y \in Y$, which taking into account that $|\lambda| = |\lambda_h| = 1$ implies that $\phi(y) - m_X + 1 = \phi_h(y) - m_X + 1$. Hence $\phi = \phi_h$. Thus $T \in \mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y))$, and the proof is complete. $\square$

We also show that every (approximate) 2-local isometry between $\mathrm{AC}(X)$-spaces is an (approximate) local isometry.

Theorem 3. *The following inclusions are fulfilled:*

$$\mathit{2}\text{-}\mathrm{ref}_{\mathrm{top}}(\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y))) \subseteq \mathrm{ref}_{\mathrm{top}}(\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y))),$$
$$\mathit{2}\text{-}\mathrm{ref}_{\mathrm{alg}}(\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y))) \subseteq \mathrm{ref}_{\mathrm{alg}}(\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y))),$$

Proof. Let $\Delta \in 2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y)))$. We first prove that for each $y \in Y$, the complex-valued function Δ_y on $\mathrm{AC}(X)$ defined by

$$\Delta_y(f) = \Delta(f)(y) \qquad (f \in \mathrm{AC}(X)),$$

is linear. According to the spherical version of the Kowalski–Słodkowski theorem [16], it suffices to show that Δ_y is 1-homogeneous and satisfies that $\Delta_y(f) - \Delta_y(g) \in \mathbb{T}\sigma(f - g)$ for all $f, g \in \mathrm{AC}(X)$. The 1-homogeneity follows immediately since Δ is an approximate 2-local isometry. For the spectral condition, let $f, g \in \mathrm{AC}(X)$ and take $\{\lambda_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ and $\{\phi_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathrm{MH}(Y, X)$ such that

$$\lim_{n\to\infty} \lambda_{f,g,n} f(\phi_{f,g,n}(y)) = \Delta(f)(y),$$
$$\lim_{n\to\infty} \lambda_{f,g,n} g(\phi_{f,g,n}(y)) = \Delta(g)(y).$$

Thus

$$\Delta_y(f) - \Delta_y(g) = \lim_{n\to\infty} \lambda_{f,g,n}(f-g)(\phi_{f,g,n}(y)) \in \mathbb{T}\sigma(f-g).$$

Hence Δ is linear by the arbitrariness of y and, consequently, $\Delta \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y)))$.

An analogous proof gives the second inclusion of the statement. □

From Theorem 3 and Corollary 2 we immediately obtain the following theorem, which was a part of the main result in [14].

Theorem 4. *The set $\mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y))$ is 2-algebraically reflexive.* □

We next study the topological reflexivity of other distinguished subsets of linear transformations of $\mathrm{AC}(X)$.

Let E be a Banach space. Let us recall that an isometric reflection of E is a linear isometry $T\colon E \to E$ which is involutive, that is, $T^2 = \mathrm{Id}_E$; and a generalized bi-circular projection of E is a linear projection $P\colon E \to E$ such that $P + \tau(\mathrm{Id}_E - P)$ is a linear surjective isometry for some $\tau \in \mathbb{T}$ with $\tau \neq 1$. Note that each isometric reflection of E is surjective. The symbols $\mathcal{G}^2(E)$ and $\mathrm{GBP}(E)$ stand for the sets of isometric reflections and generalized bi-circular projections of E, respectively.

Characterizations of both types of maps on $\mathrm{AC}(X)$ were stated in [13] when $\mathrm{AC}(X)$ is equipped with the maximum norm

$$\|f\|_M = \max\{\|f\|_\infty, \mathrm{V}(f)\} \qquad (f \in \mathrm{AC}(X)).$$

According to Corollary 1, it easily follows that each $T \in \mathcal{G}(\mathrm{AC}(X), \mathrm{AC}(Y))$ is an isometry with respect to $\|\cdot\|_M$ such that $T(1_X)$ is a unimodular constant function. Then one can obtain the form of isometric reflections and generalized bi-circular projections on $\mathrm{AC}(X)$ endowed with the Σ-norm by [13, Theorem 2.2 (1) and Corollary 3.3]. But we include the proofs to give a self-contained paper.

Theorem 5. *A map $T\colon \mathrm{AC}(X) \to \mathrm{AC}(X)$ is an isometric reflection if and only if there exist a number $\lambda \in \{-1, 1\}$ and a map $\phi \in \mathrm{MH}^2(X)$ such that*

$$T(f) = \lambda(f \circ \phi) \qquad (f \in \mathrm{AC}(X)).$$

Proof. Assume first that $T \in \mathcal{G}^2(\mathrm{AC}(X))$. By Theorem 1, there are a number $\lambda \in \mathbb{T}$ and a map $\phi \in \mathrm{MH}(X)$ such that $T(f) = \lambda(f \circ \phi)$ for all $f \in \mathrm{AC}(X)$. In particular, $T(1_X) = \lambda 1_X$ and $T^2(1_X) = \lambda^2 1_X$. Since $T^2 = \mathrm{Id}_{\mathrm{AC}(X)}$, it follows that $1_X = \lambda^2 1_X$ and so $\lambda \in \{\pm 1\}$. Moreover,

$$\phi^2(x) = \lambda T(\phi)(x) = \lambda T(\lambda T(\mathrm{id}_X))(x) = \lambda^2 T^2(\mathrm{id}_X)(x) = x \qquad (x \in X),$$

as desired. The converse is clear. □

Theorem 6. *A map* $P\colon \mathrm{AC}(X) \to \mathrm{AC}(X)$ *is a generalized bi-circular projection if and only if there exist a number* $\lambda \in \{-1, 1\}$ *and a map* $\phi \in \mathrm{MH}^2(X)$ *such that*

$$P(f) = \frac{1}{2}\,[f + \lambda(f \circ \phi)] \qquad (f \in \mathrm{AC}(X)).$$

Proof. Assume that $P \in \mathrm{GBP}(\mathrm{AC}(X))$. Then $P+\tau(\mathrm{Id}_{\mathrm{AC}(X)}-P) \in \mathcal{G}(\mathrm{AC}(X))$ for some $\tau \in \mathbb{T} \setminus \{1\}$. By Theorem 1, we can find a constant $\lambda \in \mathbb{T}$ and a map $\phi \in \mathrm{MH}(X)$ such that

$$[P + \tau(\mathrm{Id}_{\mathrm{AC}(X)} - P)](f)(x) = \lambda f(\phi(x)) \qquad (f \in \mathrm{AC}(X),\ x \in X),$$

which gives the following formula for P:

$$P(f)(x) = (1-\tau)^{-1}[-\tau f(x) + \lambda f(\phi(x))] \qquad (f \in \mathrm{AC}(X),\ x \in X).$$

Since $P^2 = P$, we have the following equation:

$$\tau f(x) - (\tau + 1)\lambda f(\phi(x)) + \lambda^2 f(\phi^2(x)) = 0 \qquad (f \in \mathrm{AC}(X),\ x \in X).$$

Suppose that there exists $x_0 \in X$ such that $x_0 \neq \phi(x_0)$ and $x_0 \neq \phi^2(x_0)$. Take

$$r = \min\left\{|x_0 - \phi(x_0)|, |x_0 - \phi^2(x_0)|\right\}$$

and consider the function $h_{x_0,r} \in \mathrm{AC}(X)$. Observe that $h_{x_0,r}(x_0) = 1$ and $h_{x_0,r}(\phi(x_0)) = 0 = h_{x_0,r}(\phi^2(x_0))$. Taking $f = h_{x_0,r}$ and $x = x_0$ in the equation above, we obtain $\tau = 0$, a contradiction. Hence $\phi(x) = x$ or $\phi^2(x) = x$ for all $x \in X$. In any case we conclude that $\phi^2 = \mathrm{id}_X$.

We now distinguish two cases. If $\phi \neq \mathrm{id}_X$, choose $x_0 \in X$ such that $x_0 \neq \phi(x_0)$ and consider $h_{x_0,s} \in \mathrm{AC}(X)$ with $s = |x_0 - \phi(x_0)|$. Substituting in the equation, first x_0 and $h_{x_0,s}$, and after 1_X, we infer that $\tau + \lambda^2 = 0$ and $\tau - (\tau + 1)\lambda + \lambda^2 = 0$, respectively. Hence $\tau = -1$ and $\lambda^2 = 1$. Hence $\lambda \in \{-1, 1\}$, and the formula of P yields

$$P(f)(x) = \frac{1}{2}\,[f(x) + \lambda f(\phi(x))] \qquad (f \in \mathrm{AC}(X),\ x \in X).$$

In the another case, if $\phi = \mathrm{id}_X$, taking $f = 1_X$ in the equation we deduce $\tau - (\tau + 1)\lambda + \lambda^2 = 0$. Hence $\lambda = \tau$ or $\lambda = 1$. Using the formula, it follows that

$$P(f)(x) = 0 = \frac{1}{2}\,[f(x) - f(\phi(x))] \qquad (f \in \mathrm{AC}(X),\ x \in X)$$

or

$$P(f)(x) = f(x) = \frac{1}{2}\,[f(x) + f(\phi(x))] \qquad (f \in \mathrm{AC}(X),\ x \in X).$$

Conversely, if P is given as the average of the identity operator with an isometric reflection on $\mathrm{AC}(X)$, an easy verification shows that P is in $\mathrm{GBP}(\mathrm{AC}(X))$. $\square$

It is interesting to note that although the isometry group of $\mathrm{AC}(X)$ is not necessarily topologically reflexive, but below it is shown the sets of isometric reflections and generalized bi-circular projections of $\mathrm{AC}(X)$ are topologically reflexive. In particular, the next two corollaries improve the conclusions obtained in Remarks 2.5 and 3.5 in [13] concerning the algebraic reflexivity of $\mathcal{G}^2(\mathrm{AC}([0,1]))$ and $\mathrm{GBP}(\mathrm{AC}([0,1]))$, respectively.

Corollary 3. *The set* $\mathcal{G}^2(\mathrm{AC}(X))$ *is topologically reflexive.*

Proof. Let $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(\mathrm{AC}(X)))$. By Theorem 5, for every $f \in \mathrm{AC}(X)$, there exist $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ in $\{-1,1\}$ and $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{MH}^2(X)$ such that

$$\lim_{n\to\infty} \lambda_{f,n}(f \circ \phi_{f,n}) = T(f).$$

In view of Theorem 1, $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}(\mathrm{AC}(X)))$, and therefore according to Theorem 2, there exist a number $\lambda \in \mathbb{T}$ and a surjective, monotonic, absolutely continuous function $\phi\colon X \to X$ such that

$$T(f) = \lambda(f \circ \phi) \qquad (f \in \mathrm{AC}(X)).$$

Hence $\lambda 1_X = T(1_X) = \lim_{n\to\infty} \lambda_{1_X,n} 1_X$ and therefore $\lambda = \lim_{n\to\infty} \lambda_{1_X,n}$. Since $\lambda_{1_X,n} \in \{-1,1\}$ for all $n \in \mathbb{N}$, it is deduced easily that $\lambda \in \{-1,1\}$.

We next prove that $\phi^2 = \mathrm{id}_X$. Define the function $h(x) = x - m_X + 1$ for all $x \in X$. We can take a sequence $\{\lambda_{h,n}\}_{n\in\mathbb{N}}$ in $\{-1,1\}$ converging to some $\lambda_h \in \{-1,1\}$ and a sequence $\{\phi_{h,n}\}_{n\in\mathbb{N}}$ in $\mathrm{MH}^2(X)$ such that

$$T(h) = \lim_{n\to\infty} \lambda_{h,n}(h \circ \phi_{h,n}) = \lambda_h \lim_{n\to\infty} (h \circ \phi_{h,n}).$$

Therefore we obtain that

$$\lambda(h \circ \phi) = \lambda_h \lim_{n\to\infty} (h \circ \phi_{h,n}).$$

On a hand, since the convergence in the Σ-norm implies pointwise convergence, we infer that

$$\lambda(\phi(x) - m_X + 1) = \lambda_h \lim_{n\to\infty} (\phi_{h,n}(x) - m_X + 1) \qquad (x \in X),$$

which clearly yields

$$\phi(x) = \lim_{n\to\infty} \phi_{h,n}(x) \qquad (x \in X),$$

and, consequently, $\lambda = \lambda_h$. On the other hand, since the convergence in the Σ-norm implies uniform convergence, we have that $\{\phi_{h,n}\}_{n\in\mathbb{N}}$ converges uniformly to ϕ. Finally, taking into account that $\phi_{h,n} \in \mathrm{MH}^2(X)$ for all $n \in \mathbb{N}$, we deduce easily that $\phi^2(x) = x$ for all $x \in X$. Hence $\phi^2 = \mathrm{id}_X$, which especially yields $\phi = \phi^{-1}$. Now, since ϕ is a monotonic absolutely continuous function, it is immediately inferred that $\phi \in \mathrm{MH}^2(X)$. Therefore, $T \in \mathcal{G}^2(\mathrm{AC}(X))$ by Theorem 5, as desired. $\square$

Corollary 4. *The set* $\mathrm{GBP}(\mathrm{AC}(X))$ *is topologically reflexive.*

Proof. Let $P \in \mathrm{ref}_{\mathrm{top}}(\mathrm{GBP}(\mathrm{AC}(X)))$. By Theorem 6, for every $f \in \mathrm{AC}(X)$, there are sequences $\{\lambda_{f,n}\}_{n\in\mathbb{N}}$ in $\{-1,1\}$ and $\{\phi_{f,n}\}_{n\in\mathbb{N}}$ in $\mathrm{MH}^2(X)$ such that

$$\lim_{n\to\infty} \frac{1}{2}\left[f + \lambda_{f,n}(f \circ \phi_{f,n})\right] = P(f).$$

Hence, for every $f \in \mathrm{AC}(X)$, we have

$$\lim_{n\to\infty} \lambda_{f,n}(f \circ \phi_{f,n}) = 2P(f) - f,$$

and so $2P - \mathrm{Id}_{\mathrm{AC}(X)} \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(\mathrm{AC}(X)))$. Hence $2P - \mathrm{Id}_{\mathrm{AC}(X)} \in \mathcal{G}^2(\mathrm{AC}(X))$ by Corollary 3, and therefore $P \in \mathrm{GBP}(\mathrm{AC}(X))$. □

We also may apply the preceding corollaries to obtain the 2-topological reflexivity of the sets of isometric reflections and generalized bi-circular projections on $\mathrm{AC}(X)$-spaces.

Corollary 5. *The sets* $\mathcal{G}^2(\mathrm{AC}(X))$ *and* $\mathrm{GBP}(\mathrm{AC}(X))$ *are 2-topologically reflexive.*

Proof. Let $\Delta \in 2\text{-ref}_{\mathrm{top}}(\mathcal{G}^2(\mathrm{AC}(X)))$. We first prove that for each $y \in Y$, the functional Δ_y defined by

$$\Delta_y(f) = \Delta(f)(y) \qquad (f \in \mathrm{AC}(X)),$$

is linear. According to the spherical version of the Kowalski–Słodkowski theorem [16], it suffices to show that Δ_y is 1-homogeneous and satisfies that $\Delta_y(f) - \Delta_y(g) \in \mathbb{T}\sigma(f-g)$ for all $f, g \in \mathrm{AC}(X)$. The 1-homogeneity follows immediately since Δ is an approximate 2-local isometry. For the spectral condition, let $f, g \in \mathrm{AC}(X)$ and take $\{\lambda_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathbb{T}$ and $\{\phi_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathrm{MH}^2(X)$ such that

$$\lim_{n\to\infty} \lambda_{f,g,n} f(\phi_{f,g,n}(y)) = \Delta(f)(y),$$
$$\lim_{n\to\infty} \lambda_{f,g,n} g(\phi_{f,g,n}(y)) = \Delta(g)(y).$$

Thus

$$\Delta_y(f) - \Delta_y(g) = \lim_{n\to\infty} \lambda_{f,g,n}(f-g)(\phi_{f,g,n}(y)) \in \mathbb{T}\sigma(f-g).$$

Hence Δ is linear by the arbitrariness of y. Consequently, $\Delta \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(\mathrm{AC}(X)))$, and therefore $\Delta \in \mathcal{G}^2(\mathrm{AC}(X))$ by Corollary 3. This proves that $\mathcal{G}^2(\mathrm{AC}(X))$ is 2-topologically reflexive.

To prove the 2-topological reflexivity of $\mathrm{GBP}(\mathrm{AC}(X))$, let $\Delta \in 2\text{-ref}_{\mathrm{top}}(\mathrm{GBP}(\mathrm{AC}(X)))$. For any $f, g \in \mathrm{AC}(X)$, there are sequences $\{\lambda_{f,g,n}\}_{n\in\mathbb{N}}$ in $\{-1,1\}$ and $\{\phi_{f,g,n}\}_{n\in\mathbb{N}}$ in $\mathrm{MH}^2(X)$ such that

$$\lim_{n\to\infty} \frac{1}{2}\left[f + \lambda_{f,g,n}(f \circ \phi_{f,g,n})\right] = \Delta(f),$$
$$\lim_{n\to\infty} \frac{1}{2}\left[g + \lambda_{f,g,n}(g \circ \phi_{f,g,n})\right] = \Delta(g).$$

Hence, for every $f, g \in \mathrm{AC}(X)$, we have

$$\lim_{n\to\infty} \lambda_{f,g,n}(f \circ \phi_{f,g,n}) = 2\Delta(f) - f,$$
$$\lim_{n\to\infty} \lambda_{f,g,n}(g \circ \phi_{f,g,n}) = 2\Delta(g) - g,$$

and this says that $2\Delta - \mathrm{Id}_{\mathrm{AC}(X)} \in 2\text{-ref}_{\mathrm{top}}(\mathcal{G}^2(\mathrm{AC}(X)))$. Since $\mathcal{G}^2(\mathrm{AC}(X))$ is 2-topologically reflexive, it follows that $2\Delta - \mathrm{Id}_{\mathrm{AC}(X)} \in \mathcal{G}^2(\mathrm{AC}(X))$, and thus $\Delta \in \mathrm{GBP}(\mathrm{AC}(X))$. $\square$

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