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# Approximate local isometries of Banach algebras of differentiable functions

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ABSTRACT

Let $X$ and $Y$ be compact subsets of $\mathbb{R}$ such that $X$ and $Y$ coincide with the closures of their interiors. For any $n \in \mathbb{N}$, let $C^{(n)}(X)$ be the Banach algebra of all $n$-times continuously differentiable complex-valued functions $f$ on $X$, with the norm $\|f\|_C = \max_{x\in X}(\sum_{k=0}^{n}(|f^{(k)}(x)|/k!))$. We prove that every approximate local isometry of $C^{(n)}(X)$ to $C^{(n)}(Y)$ is an isometric linear algebra monomorphism multiplied by a fixed $n$-times continuously differentiable unimodular function. This description allows us to establish the algebraic and 2-algebraic reflexivity of the set of linear isometries of $C^{(n)}(X)$ onto $C^{(n)}(Y)$. Furthermore, this algebraic reflexivity becomes topological whenever $X$ and $Y$ are compact intervals of $\mathbb{R}$. Another application of our main result shows that the sets of isometric reflections and generalized bi-circular projections of $C^{(n)}(X)$ are topologically and 2-topologically reflexive.



## 1. Introduction

Let $n \geq 1$ be any integer and let $X$ be a compact subset of $\mathbb{R}$ such that $X$ coincides with the closure of its interior. Let $C^{(n)}(X)$ be the space of all $n$-times continuously differentiable complex-valued functions on $X$, with the C-norm given by

$$\|f\|_C = \max_{x\in X}\left(\sum_{k=0}^{n}\frac{|f^{(k)}(x)|}{k!}\right) \qquad (f \in C^{(n)}(X)).$$

It is known that $C^{(n)}(X)$ is a unital semisimple commutative Banach algebra.

Linear isometries of these spaces have been studied by different authors. Pathak [19] proved that any surjective linear isometry of $C^{(n)}([0,1])$ is induced by a surjective isometry of $[0,1]$. Previously, Cambern [3] had obtained Pathak's result for $n = 1$. Wang [22] extended this result for $C_0^{(n)}(X)$, the Banach space of

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complex-valued functions which have up to $n$-th continuous derivatives and vanish at infinity on a locally compact subset $X$ of $\mathbb{R}$ such that $X$ is contained in the closure of its interior (see also the paper [4] by Cambern and Pathak for the case $n = 1$). We denote by $\mathcal{G}(C^{(n)}(X))$ the set of all surjective linear isometries of $C^{(n)}(X)$.

The purpose of the present note is to study the approximate local isometries between $C^{(n)}(X)$-spaces. Let us recall that a local isometry of $C^{(n)}(X)$ is a continuous linear map $T$ of $C^{(n)}(X)$ into itself which agrees at every point of $C^{(n)}(X)$ with some element of $\mathcal{G}(C^{(n)}(X))$, that is, for every $f \in C^{(n)}(X)$, there exists a $T_f \in \mathcal{G}(C^{(n)}(X))$, possibly depending on $f$, such that $T(f) = T_f(f)$. Also, we can consider approximate local isometries on $C^{(n)}(X)$ which are those continuous linear maps $T$ satisfying that for every $f \in C^{(n)}(X)$ and $\varepsilon > 0$, there is a $T_f \in \mathcal{G}(C^{(n)}(X))$ such that $\|T(f) - T_f(f)\|_C < \varepsilon$. Similarly, the concepts of local automorphism and local derivation on $C^{(n)}(X)$ and their approximate versions can be introduced.

The study of local automorphisms and local derivations on operator algebras began by Larson [13], Kadison [11] and Larson and Sourour [14]. The investigation concerning (approximate) local isometries on operator algebras and function algebras was initiated by Molnár [16], Molnár and Zalar [18] and Cabello Sánchez and Molnár [2]. The main problem that arises in these papers is to study whether every local derivation, local automorphism or local isometry of an algebra is a derivation, an automorphism or a surjective isometry, respectively, or equivalently, to establish the algebraic reflexivity of the sets of derivations, automorphisms and isometries of such algebras. One can also deal with the approximate version of this question, that is, when the sets of derivations, automorphisms and isometries of an algebra are topologically reflexive. The consideration of approximate local maps instead of local maps is more general and allows us to address the problems of algebraic reflexivity and topological reflexivity of some sets of linear transformations on $C^{(n)}(X)$ simultaneously.

Concerning to derivations on such spaces, Johnson [10] proved that the space of all bounded derivations from $C^{(1)}([0,1])$ into $C^{(1)}([0,1])^*$ is algebraically reflexive, but Samei [20] showed that $C^{(n)}([0,1])$ does not enjoy this property for all $n \geq 2$.

Our goal in this paper is to provide a Banach–Stone type representation of approximate local isometries between $C^{(n)}(X)$-spaces. More concretely, we prove that every approximate local isometry of $C^{(n)}(X)$ to $C^{(n)}(Y)$ is an isometric linear algebra monomorphism induced by an $n$-times continuously differentiable surjection $\sigma\colon Y \to X$ and multiplied by a fixed $n$-times continuously differentiable unimodular function $\theta\colon Y \to \mathbb{C}$. Furthermore, this monomorphism is an isomorphism if and only if $\sigma$ is injective. As an application, we state the algebraic reflexivity of the set $\mathcal{G}(C^{(n)}(X), C^{(n)}(Y))$ of all linear isometries of $C^{(n)}(X)$ onto $C^{(n)}(Y)$, and this algebraic reflexivity becomes topological when $X$ and $Y$ are compact intervals of $\mathbb{R}$. It is interesting to note that the obtained representation also permits us to deduce that the sets of isometric reflections and generalized bi-circular projections on $C^{(n)}(X)$ are topologically reflexive. Our approach requires first to state characterizations for these classes of maps on $C^{(n)}(X)$.

It is worth noting that the proof of our main result strongly depends on the application of a spherical reformulation of the Gleason–Kahane–Żelazko theorem, obtained by Li, Peralta, L. Wang and Y.-S. Wang [15]. This result applies only to unital complex Banach algebras and therefore the technique used in this paper cannot be applied to study the problem in the case of $C_0^{(n)}(X)$-spaces.

We are also interested in the notion of 2-locality which is due to Šemrl [21] who stated the first results on 2-local automorphisms and 2-local derivations on operator algebras. Motivated by these results, Molnár [17] studied 2-local isometries on such algebras. 2-local isometries on function algebras have been investigated by different authors (see, for example, [7,15]). Recently, the 2-locality problem for surjective linear isometries on $C^{(n)}([0,1])$ with the C-norm has been addressed by Kawamura, Koshimizu and Miura [12], and without assuming linearity by Hatori and Oi [8]. The first author of this paper also studied 2-local isometries on $C^{(n)}([0,1])$ in [9] but with a norm different to the C-norm.

Applying now a spherical variant of the Kowalski–Słodkowski theorem (see [15]), we establish here that the set of surjective linear isometries from $C^{(n)}(X)$ onto $C^{(n)}(Y)$ is 2-algebraically reflexive. This reflexivity becomes topological whenever $X$ and $Y$ are compact intervals of $\mathbb{R}$ and we extend the main result concerning the 2-algebraically reflexivity of the isometry group of $C^{(n)}([0,1])$ stated in [12]. Furthermore, we prove that the sets of isometric reflections and generalized bi-circular projections of $C^{(n)}(X)$ are 2-topologically reflexive.

## 2. Preliminaries

We recall the concepts of algebraic and topological reflexivity. Let $E$ and $F$ be Banach spaces, let $F^E$ be the set of all maps from $E$ into $F$ and let $\mathcal{S}$ be a nonempty subset of the Banach space $\mathcal{B}(E,F)$ of all bounded linear maps from $E$ to $F$. Define the algebraic reflexive closure and the topological reflexive closure of the set of $\mathcal{S}$ by

$$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \{T \in \mathcal{B}(E,F)\colon \forall e \in E,\ \exists S_e \in \mathcal{S} \mid S_e(e) = T(e)\}$$

and

$$\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \left\{T \in \mathcal{B}(E,F)\colon \forall e \in E,\ \exists \{S_{e,i}\}_{i\in\mathbb{N}} \subset \mathcal{S} \mid \lim_{i\to\infty} S_{e,i}(e) = T(e)\right\},$$

respectively. The elements of $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$ and $\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$ are known as local $\mathcal{S}$-maps and approximate local $\mathcal{S}$-maps, respectively.

Consider also the 2-algebraic reflexive closure of $\mathcal{S}$, 2-$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$, defined as

$$\left\{\Delta \in F^E\colon \forall e,u \in E,\ \exists S_{e,u} \in \mathcal{S} \mid S_{e,u}(e) = \Delta(e),\ S_{e,u}(u) = \Delta(u)\right\}$$

and the 2-topological reflexive closure of $\mathcal{S}$, 2-$\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$, given by

$$\left\{\Delta \in F^E\colon \forall e,u \in E,\ \exists \{S_{e,u,i}\}_{i\in\mathbb{N}} \subset \mathcal{S} \ \mid \lim_{n\to\infty} S_{e,u,i}(e) = \Delta(e),\ \lim_{i\to\infty} S_{e,u,i}(u) = \Delta(u)\right\}.$$

The elements of 2-$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$ and 2-$\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$ are referred to as 2-local $\mathcal{S}$-maps and approximate 2-local $\mathcal{S}$-maps, respectively.

In the case that $\mathcal{S}$ is the set of all linear isometries from $E$ onto $F$, we refer to the elements of $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$, $\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$, 2-$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S})$ and 2-$\mathrm{ref}_{\mathrm{top}}(\mathcal{S})$ as local isometries, approximate local isometries, 2-local isometries and approximate 2-local isometries of $E$ to $F$, respectively. The terminology local map, approximate local map, 2-local map and approximate 2-local map, substituting the word "map" by isometric reflection and generalized bi-circular projection should be self-explanatory.

The set $\mathcal{S}$ is said to be algebraically reflexive (topologically reflexive) if $\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \mathcal{S}$ (respectively, $\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \mathcal{S}$). Similarly, the set $\mathcal{S}$ is called 2-algebraically reflexive (2-topologically reflexive) if 2-$\mathrm{ref}_{\mathrm{alg}}(\mathcal{S}) = \mathcal{S}$ (respectively, 2-$\mathrm{ref}_{\mathrm{top}}(\mathcal{S}) = \mathcal{S}$).

Throughout this paper, $\mathbb{T}$ denotes be the unit circle of $\mathbb{C}$. Given a set $X \subseteq \mathbb{R}$, the symbols $1_X$ and $\mathrm{id}_X$ stand for the function constantly 1 and the identity function on $X$, respectively. Moreover, $\mathrm{cl}(X)$ and $\mathrm{int}(X)$ denote the closure and the interior of $X$, respectively.

## 3. Results

Our starting point is the following characterization of surjective linear isometries between $C_0^{(n)}(X)$-spaces equipped with the C-norms, obtained by Wang [22].

**Theorem 1.** *[22, Theorem 4.4] Let $n \in \mathbb{N}$ and $X$ and $Y$ be locally compact subsets of $\mathbb{R}$ with $X \subseteq \mathrm{cl}(\mathrm{int}(X))$ and $Y \subseteq \mathrm{cl}(\mathrm{int}(Y))$. A map $T\colon C_0^{(n)}(X) \to C_0^{(n)}(Y)$ is a surjective linear isometry with respect to the C-norms if and only if there exist a function $\theta\colon Y \to \mathbb{C}$ with $|\theta(y)| = 1$ and $\theta'(y) = 0$ for all $y \in Y$, and a homeomorphism $\sigma\colon Y \to X$ with $|\sigma'(y)| = 1$ and $\sigma''(y) = 0$ for all $y \in Y$ such that*

$$T(f)(y) = \theta(y)f(\sigma(y)) \qquad (y \in Y,\ f \in C_0^n(X)). \quad \square$$

From now on we shall suppose that $X$ and $Y$ are compact subsets of $\mathbb{R}$ such that $X = \mathrm{cl}(\mathrm{int}(X))$ and $Y = \mathrm{cl}(\mathrm{int}(Y))$. Moreover, $I$ and $J$ will denote two compact intervals of $\mathbb{R}$.

To simplify the writing, for any $n \in \mathbb{N}$ we introduce the following sets of functions:

$$\mathcal{G}(C^{(n)}(X), C^{(n)}(Y)) = \left\{T\colon C^{(n)}(X) \to C^{(n)}(Y)\colon T \text{ is a surjective linear isometry}\right\},$$
$$A_n(Y) = \{\theta\colon Y \to \mathbb{C}\colon |\theta(y)| = 1,\ \theta'(y) = 0,\ \forall y \in Y\},$$
$$B_n(Y, X) = \{\sigma\colon Y \to X\colon \sigma \text{ is a homeomorphism},\ |\sigma'(y)| = 1,\ \sigma''(y) = 0,\ \forall y \in Y\}.$$

The main result of this paper is the following description of the elements of the topological reflexive closure of the set $\mathcal{G}(C^{(n)}(X), C^{(n)}(Y))$.

**Theorem 2.** *Every approximate local isometry $T$ of $C^{(n)}(X)$ to $C^{(n)}(Y)$ is an isometry of the form*

$$T(f)(y) = \theta(y)f(\sigma(y)) \qquad (y \in Y,\ f \in C^{(n)}(X)),$$

*where $\theta$ is a function of $C^{(n)}(Y)$ such that $|\theta(y)| = 1$ and $\theta'(y) = 0$ for all $y \in Y$, and $\sigma\colon Y \to X$ is an $n$-times continuously differentiable surjective function such that $|\sigma'(y)| = 1$ and $\sigma''(y) = 0$ for all $y \in Y$. Moreover, $T$ is surjective if and only if $\sigma$ is injective.*

**Proof.** We prove the result through several steps.

**Step 1.** *$T$ is an isometry with respect to the C-norms.*

Let $f \in C^{(n)}(X)$. Hence there is a sequence $\{T_{f,i}\}_{i\in\mathbb{N}}$ in $\mathcal{G}(C^{(n)}(X), C^{(n)}(Y))$ such that

$$\lim_{i\to\infty} T_{f,i}(f) = T(f).$$

Clearly, we have

$$\lim_{i\to\infty} \|T_{f,i}(f)\|_C = \|T(f)\|_C,$$

and since $\|T_{f,i}(f)\|_C = \|f\|_C$ for all $i \in \mathbb{N}$, we conclude that $\|T(f)\|_C = \|f\|_C$.

**Step 2.** *For every $f \in C^{(n)}(X)$, there exist sequences $\{\theta_{f,i}\}_{i\in\mathbb{N}}$ in $A_n(Y)$ and $\{\sigma_{f,i}\}_{i\in\mathbb{N}}$ in $B_n(Y, X)$ such that $T(f) = \lim_{i\to\infty} \theta_{f,i}(f \circ \sigma_{f,i})$.*

Let $f \in C^{(n)}(X)$. Hence there is a sequence $\{T_{f,i}\}_{i\in\mathbb{N}}$ in $\mathcal{G}(C^{(n)}(X), C^{(n)}(Y))$ such that

$$T(f) = \lim_{i\to\infty} T_{f,i}(f).$$

By Theorem 1, for each $i \in \mathbb{N}$, there are two functions $\theta_{f,i} \in A_n(Y)$ and $\sigma_{f,i} \in B_n(Y, X)$ such that $T_{f,i}(h) = \theta_{f,i}(h \circ \sigma_{f,i})$ for all $h \in C^{(n)}(X)$, and therefore

$$T(f)=\lim_{i\to\infty}\theta_{f,i}(f\circ\sigma_{f,i}).$$

**Step 3.** *$\theta:=T(1_X)$ is a function of $C^{(n)}(Y)$ such that $|\theta(y)|=1$ and $\theta'(y)=0$ for all $y\in Y$.*

Clearly, $T(1_X)\in C^{(n)}(Y)$. By Step 2, there exist sequences $\{\theta_{1_X,i}\}_{i\in\mathbb{N}}$ in $A_n(Y)$ and $\{\sigma_{1_X,i}\}_{i\in\mathbb{N}}$ in $B_n(Y,X)$ such that

$$T(1_X)=\lim_{i\to\infty}\theta_{1_X,i}1_Y.$$

Since the convergence in the C-norm implies pointwise convergence, for each $y\in Y$ we have

$$T(1_X)(y)=\lim_{i\to\infty}\theta_{1_X,i}(y)$$

and thus

$$|T(1_X)(y)|=\lim_{i\to\infty}|\theta_{1_X,i}(y)|=1.$$

That convergence also implies pointwise convergence for the derivatives and therefore

$$T(1_X)'(y)=\lim_{i\to\infty}\theta'_{1_X,i}(y)=0$$

for every $y\in Y$. Hence $T(1_X)\in A_n(Y)$.

**Step 4.** *For each $y\in Y$, the map $S_y\colon C^{(n)}(X)\to\mathbb{C}$ defined by*

$$S_y(f)=\overline{\theta(y)}T(f)(y)\qquad(f\in C^{(n)}(X)),$$

*is a unital multiplicative linear functional.*

Fix $y\in Y$. Clearly, $S_y$ is linear and

$$S_y(1_X)=\overline{\theta(y)}T(1_X)(y)=\overline{\theta(y)}\theta(y)=|\theta(y)|^2=1$$

by Step 3. To prove its multiplicativity, define $T_y\colon C^{(n)}(X)\to\mathbb{C}$ by

$$T_y(f)=T(f)(y)\qquad(f\in C^{(n)}(X)).$$

Since $T_y$ is linear and $|T_y(f)|=|T(f)(y)|\le\|T(f)\|_C=\|f\|_C$ for all $f\in C^{(n)}(X)$ by Step 1, then $T_y$ is continuous. Take now any $f\in C^{(n)}(X)$. By Step 2, there exist sequences $\{\theta_{f,i}\}_{i\in\mathbb{N}}$ in $A_n(Y)$ and $\{\sigma_{f,i}\}_{i\in\mathbb{N}}$ in $B_n(Y,X)$ such that

$$T(f)=\lim_{i\to\infty}\theta_{f,i}(f\circ\sigma_{f,i}).$$

Therefore we have

$$T_y(f)=T(f)(y)=\lim_{i\to\infty}\theta_{f,i}(y)f(\sigma_{f,i}(y))\in\mathbb{T}\sigma(f),$$

where $\sigma(f)$ denotes the spectrum of $f$. Applying a spherical version of the Gleason–Kahane–Żelazko theorem [15, Proposition 2.2], we conclude that $S_y=\overline{T_y(1_X)}T_y$ is multiplicative.

**Step 5.** *There exists an $n$-times continuously differentiable surjective function $\sigma\colon Y \to X$ with $|\sigma'(y)| = 1$ and $\sigma''(y) = 0$ for all $y \in Y$ such that $T(f)(y) = \theta(y)f(\sigma(y))$ for all $y \in Y$ and $f \in C^{(n)}(X)$.*

Using Step 4, we deduce easily that the map $S\colon C^{(n)}(X) \to C^{(n)}(Y)$ defined by

$$S(f)(y) = \overline{\theta(y)}T(f)(y) \qquad (f \in C^{(n)}(X),\ y \in Y)$$

is a unital algebra homomorphism. By Gelfand theory (see, e.g., [5, Theorem 2.3.25]), $S$ induces a continuous map $\sigma\colon Y \to X$ such that

$$S(f)(y) = f(\sigma(y)) \qquad (f \in C^{(n)}(X),\ y \in Y),$$

and we thus obtain the desired representation of $T$:

$$T(f)(y) = \theta(y)f(\sigma(y)) \qquad (f \in C^{(n)}(X),\ y \in Y).$$

Notice that $\sigma = \overline{\theta}T(\mathrm{id}_X) \in C^{(n)}(Y)$. To show the surjectivity of $\sigma$, assume on the contrary that there exists $x_0 \in X \setminus \sigma(Y)$. Being $\sigma(Y)$ compact, from the $C^{(\infty)}$-Urysohn Lemma (see, e.g., [6, page 245]) we can take a function $f \in C^{(n)}(X)$ such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \in \sigma(Y)$. Hence $T(f)(y) = \theta(y)f(\sigma(y)) = 0$ for all $y \in Y$, but $T$ is injective by Step 1, a contradiction.

By Step 2, we can take sequences $\{\theta_{\mathrm{id}_X,i}\}_{i\in\mathbb{N}}$ in $A_n(Y)$ and $\{\sigma_{\mathrm{id}_X,i}\}_{i\in\mathbb{N}}$ in $B_n(Y,X)$ such that $\lim_{i\to\infty} \theta_{\mathrm{id}_X,i}\sigma_{\mathrm{id}_X,i} = T(\mathrm{id}_X)$, and therefore

$$\lim_{i\to\infty} \theta_{\mathrm{id}_X,i}\sigma_{\mathrm{id}_X,i} = \theta\sigma.$$

For each $y \in Y$, we deduce that

$$\begin{aligned}
\lim_{i\to\infty} \theta_{\mathrm{id}_X,i}(y)\sigma'_{\mathrm{id}_X,i}(y) &= \lim_{i\to\infty} \left[\theta'_{\mathrm{id}_X,i}(y)\sigma_{\mathrm{id}_X,i}(y) + \theta_{\mathrm{id}_X,i}(y)\sigma'_{\mathrm{id}_X,i}(y)\right] \\
&= \lim_{i\to\infty} (\theta_{\mathrm{id}_X,i}\sigma_{\mathrm{id}_X,i})'(y) \\
&= (\theta\sigma)'(y) = \theta'(y)\sigma(y) + \theta(y)\sigma'(y) = \theta(y)\sigma'(y),
\end{aligned}$$

which implies that $|\sigma'(y)| = 1$. Moreover, one can observe that $\lim_{i\to\infty} \|\overline{\theta}\theta_{\mathrm{id}_X,i}\sigma'_{\mathrm{id}_X,i} - \sigma'\|_\infty = 0$ ($\|\cdot\|_\infty$ denotes the uniform norm), which yields the continuity of $\sigma'$. Now, since $\sigma'$ is real-valued, for each $y_0 \in Y$, we can find a $\delta > 0$ such that $\sigma'(y) = \sigma'(y_0)$ for all $y \in Y$ with $|y - y_0| < \delta$, which clearly implies that $\sigma''(y_0) = 0$. Therefore, $\sigma'' = 0$ on $Y$, as claimed.

**Step 6.** *$T\colon C^{(n)}(X) \to C^{(n)}(Y)$ is surjective if and only if $\sigma\colon Y \to X$ is injective.*

We have $T(f) = \theta(f \circ \sigma)$ for all $f \in C^{(n)}(X)$ with $\theta$ and $\sigma$ being as in Step 5. Assume that $T$ is surjective. Let $y_1, y_2 \in Y$ be such that $\sigma(y_1) = \sigma(y_2)$. It follows that

$$\overline{\theta(y_1)}T(f)(y_1) = f(\sigma(y_1)) = f(\sigma(y_2)) = \overline{\theta(y_2)}T(f)(y_2)$$

for all $f \in C^{(n)}(X)$, that is, $\overline{\theta(y_1)}g(y_1) = \overline{\theta(y_2)}g(y_2)$ for all $g \in C^{(n)}(Y)$. Hence $g(y_1) = g(y_2)$ for all $g \in C^{(n)}(Y)$ such that $g \geq 0$. This implies $y_1 = y_2$ since the set $\{g \in C^{(n)}(Y) \colon g \geq 0\}$ separates the points of $Y$. Therefore $\sigma$ is injective.

Conversely, suppose that $\sigma\colon Y \to X$ is injective. Given $g \in C^{(n)}(Y)$, take $f = \overline{\theta}(g \circ \sigma^{-1})$. Since $\sigma \in C^{(n)}(Y)$ and $|\sigma'| = 1$, it easily follows that $\sigma^{-1} \in C^{(n)}(X)$. Consequently, $f \in C^{(n)}(X)$ and $T(f) = g$. This completes the proof of Theorem 2. $\square$

Now we state the algebraic reflexivity of the set $\mathcal{G}(C^{(n)}(X), C^{(n)}(Y))$.

**Corollary 1.** *Every local isometry of $C^{(n)}(X)$ to $C^{(n)}(Y)$ is a surjective isometry.*

**Proof.** Let $T \in \mathrm{ref}_{\mathrm{alg}}(\mathcal{G}(C^{(n)}(X), C^{(n)}(Y)))$. We have

$$T(f)(y) = \theta(y)f(\sigma(y)) \qquad (f \in C^{(n)}(X),\ y \in Y),$$

with $\theta$ and $\sigma$ being as in Theorem 2. We only need to prove that $\sigma\colon Y \to X$ is injective and thus $T$ will be surjective. Fix $x_0 \in X$ and let $y_0 \in \sigma^{-1}(\{x_0\})$. Consider $f_{x_0}\colon X \to \mathbb{R}^+$ defined by

$$f_{x_0}(x) = \frac{x - \min(X) + 1}{x_0 - \min(X) + 1} \qquad (x \in X).$$

Clearly, $f_{x_0} \in C^{(n)}(X)$ with $f_{x_0}^{-1}(\{1\}) = \{x_0\}$. Since $T$ is a local isometry, Theorem 1 provides two functions $\theta_{f_{x_0}} \in A_n(Y)$ and $\sigma_{f_{x_0}} \in B_n(Y, X)$ such that

$$T(f_{x_0})(y) = \theta_{f_{x_0}}(y)f_{x_0}(\sigma_{f_{x_0}}(y)) \qquad (y \in Y).$$

In particular, we obtain

$$\theta(y_0) = \theta(y_0)f_{x_0}(\sigma(y_0)) = T(f_{x_0})(y_0) = \theta_{f_{x_0}}(y_0)f_{x_0}(\sigma_{f_{x_0}}(y_0)),$$

hence $f_{x_0}(\sigma_{f_{x_0}}(y_0)) = 1$, and therefore $\sigma_{f_{x_0}}(y_0) = x_0$. Since $y_0$ was arbitrary, we have proved that

$$\sigma^{-1}(\{x_0\}) \subseteq \sigma_{f_{x_0}}^{-1}(\{x_0\}).$$

It follows that $\sigma$ is injective because so is $\sigma_{f_{x_0}}$. $\square$

In the following case we can assert the topological reflexivity of the set $\mathcal{G}(C^{(n)}(I), C^{(n)}(J))$.

**Corollary 2.** *Every approximate local isometry of $C^{(n)}(I)$ to $C^{(n)}(J)$ is a surjective isometry.*

**Proof.** Let $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}(C^{(n)}(I), C^{(n)}(J)))$. By Theorem 2, we have

$$T(f)(y) = \theta(y)f(\sigma(y)) \qquad (y \in J,\ f \in C^{(n)}(I)),$$

where $\theta \in A_n(J)$ and $\sigma\colon J \to I$ is an $n$-times continuously differentiable surjective function such that $|\sigma'(y)| = 1$ for all $y \in J$. Now, an easy argument shows that there exists $c \in \mathbb{R}$ such that either $\sigma(y) = y + c$ for all $y \in J$, or $\sigma(y) = -y + c$ for all $y \in J$. In any case, $\sigma$ is injective and therefore $T$ is surjective. $\square$

We now establish a relationship between the 2-topological (2-algebraic) reflexive closure and the topological (algebraic) reflexive closure of $\mathcal{G}(C^{(n)}(X), C^{(n)}(Y))$.

**Theorem 3.**

(1) *Every approximate 2-local isometry of $C^{(n)}(X)$ to $C^{(n)}(Y)$ is an approximate local isometry.*
(2) *Every 2-local isometry of $C^{(n)}(X)$ to $C^{(n)}(Y)$ is a local isometry.*

**Proof.** Let $\Delta \in 2\text{-ref}_{\text{top}}(\mathcal{G}(C^{(n)}(X), C^{(n)}(Y)))$. We first show that for each $y \in Y$, the functional $\Delta_y : C^{(n)}(X) \to \mathbb{C}$ defined by

$$\Delta_y(f) = \Delta(f)(y) \qquad (f \in C^{(n)}(X)),$$

is linear. According to the spherical version of the Kowalski–Słodkowski theorem [15], it suffices to show that $\Delta_y$ is 1-homogeneous and satisfies that $\Delta_y(f) - \Delta_y(g) \in \mathbb{T}\sigma(f-g)$ for all $f, g \in C^{(n)}(X)$. The 1-homogeneity follows immediately since $\Delta$ is an approximate 2-local isometry. For the spectral condition, let $f, g \in C^{(n)}(X)$ and take $\{\theta_{f,g,i}\}_{i\in\mathbb{N}}$ in $A_n(Y)$ and $\{\sigma_{f,g,i}\}_{i\in\mathbb{N}}$ in $B_n(Y, X)$ such that

$$\lim_{i\to\infty} \theta_{f,g,i} f(\sigma_{f,g,i}(y)) = \Delta(f)(y),$$
$$\lim_{i\to\infty} \theta_{f,g,i} g(\sigma_{f,g,i}(y)) = \Delta(g)(y).$$

Thus

$$\Delta_y(f) - \Delta_y(g) = \lim_{i\to\infty} \theta_{f,g,i}(f-g)(\sigma_{f,g,i}(y)) \in \mathbb{T}\sigma(f-g).$$

Hence $\Delta$ is linear by the arbitrariness of $y$. Therefore $\Delta \in \text{ref}_{\text{top}}(\mathcal{G}(C^{(n)}(X), C^{(n)}(Y)))$. This proves (1), and (2) is obtained with an analogous proof. □

We next see that $\mathcal{G}(C^{(n)}(X), C^{(n)}(Y))$ is 2-algebraically reflexive and this reflexivity becomes 2-topological whenever $X$ and $Y$ are compact intervals of $\mathbb{R}$. We extend in this way the 2-algebraically reflexivity of $\mathcal{G}(C^{(n)}([0,1]))$ stated in [12, Theorem 2.1]. Our result follows by applying Theorem 3 and Corollaries 1 and 2.

**Corollary 3.**

(1) *Every 2-local isometry of $C^{(n)}(X)$ to $C^{(n)}(Y)$ is a surjective linear isometry.*
(2) *Every approximate 2-local isometry of $C^{(n)}(I)$ to $C^{(n)}(J)$ is a surjective linear isometry.* □

We next shall study the algebraic and topological reflexivity of two classes of linear transformations on $C^{(n)}(X)$: the isometric reflections and the generalized bi-circular projections.

We recall that an isometric reflection of a Banach space $E$ is a linear isometry $T : E \to E$ such that $T^2 = \text{id}_E$. By $\mathcal{G}^2(E)$ we denote the set of all isometric reflections of $E$.

The next theorem provides a characterization of isometric reflections of $C^{(n)}(X)$.

**Theorem 4.** *A map $T : C^{(n)}(X) \to C^{(n)}(X)$ is an isometric reflection if and only if there exist a function $\theta \in A_n(X)$ with $\theta(x) \in \{\pm 1\}$ for all $x \in X$, and a function $\sigma \in B_n(X, X)$ with $\sigma^2(x) = x$ for all $x \in X$ such that*

$$T(f)(x) = \theta(x) f(\sigma(x)) \qquad (x \in X,\ f \in C^{(n)}(X)).$$

**Proof.** Let $T \in \mathcal{G}^2(C^{(n)}([0,1]))$. By Theorem 1, there are functions $\theta \in A_n(X)$ and $\sigma \in B_n(X, X)$ such that

$$T(f)(x) = \theta(x) f(\sigma(x)) \qquad (f \in C^{(n)}(X),\ x \in X).$$

Since $T^2 = \text{id}_{C^{(n)}(X)}$, it follows that

$$f(x) = T^2(f)(x) = T(T(f))(x) = [\theta(x)]^2 f(\sigma^2(x)) \qquad (f \in C^{(n)}(X),\ x \in X).$$

Taking above $f = 1_X$, we deduce that $[\theta(x)]^2 = 1$ for all $x \in X$ and thus $\theta(x) \in \{\pm 1\}$ for all $x \in X$. Substituting also $f = \mathrm{id}_X$, we obtain that $x = [\theta(x)]^2\sigma^2(x) = \sigma^2(x)$ for all $x \in X$.

Conversely, assume that $T$ has the form as in the statement. Then $T \in \mathcal{G}(C^{(n)}(X))$ by Theorem 1, and an easy verification yields

$$T^2(f)(x) = [\theta(x)]^2 f(\sigma^2(x)) = f(x) \qquad (f \in C^{(n)}(X),\ x \in X).$$

Hence $T \in \mathcal{G}^2(C^{(n)}(X))$. □

We can deduce that every approximate local isometric reflection of $C^{(n)}(X)$ is an isometric reflection.

**Corollary 4.** *The set $\mathcal{G}^2(C^{(n)}(X))$ is topologically reflexive.*

**Proof.** Let $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(C^{(n)}(X)))$. By Theorem 4, for $f \in C^{(n)}(X)$, we can take two sequences $\{\theta_{f,i}\}_{i\in\mathbb{N}}$ in $A_n(X)$ with $\theta_{f,i}(x) \in \{\pm 1\}$ for all $x \in X$, and $\{\sigma_{f,i}\}_{i\in\mathbb{N}}$ in $B_n(X, X)$ with $\sigma^2_{f,i}(x) = x$ for all $x \in X$ satisfying

$$\lim_{i\to\infty} \theta_{f,i}(f \circ \sigma_{f,i}) = T(f).$$

Obviously, $T \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}(C^{(n)}(X)))$ and, by Theorem 2, we can find a function $\theta \in A_n(X)$ and an $n$-times continuously differentiable surjective function $\sigma\colon X \to X$ such that $|\sigma'(x)| = 1$ and $\sigma''(x) = 0$ for all $x \in X$ such that

$$T(f) = \theta(f \circ \sigma) \qquad (f \in C^{(n)}(X)).$$

Hence $\theta = T(1_X) = \lim_{i\to\infty} \theta_{1_X,i}$ and since $\theta_{1_X,i}(x) \in \{\pm 1\}$ for all $i \in \mathbb{N}$ and $x \in X$, it is deduced easily that $\theta(x) \in \{\pm 1\}$ for all $x \in X$. Define $g(x) = x - \min(X) + 1$ for all $x \in X$. We have

$$\theta(g \circ \sigma) = T(g) = \lim_{i\to\infty} \theta_{g,i}(g \circ \sigma_{g,i}).$$

Since the convergence in the C-norm implies uniform convergence, and $g(x) > 0$, $\theta(x), \theta_{g,i}(x) \in \{\pm 1\}$ for all $x \in X$, we conclude that $\lim_{i\to\infty} \|g \circ \sigma_{g,i} - g \circ \sigma\|_\infty = 0$. Hence $\lim_{i\to\infty} \|\sigma_{g,i} - \sigma\|_\infty = 0$. Now, given $x \in X$ and $\epsilon > 0$, taking into account that $\sigma$ is continuous and $\lim_{i\to\infty} \sigma_{g,i}(x) = \sigma(x)$, one can find $i_0 \in \mathbb{N}$ such that $\|\sigma_{g,i} - \sigma\|_\infty < \epsilon/2$ and $\left|\sigma(\sigma_{g,i}(x)) - \sigma^2(x)\right| < \epsilon/2$ for all $i \geq i_0$. Then for any $i \geq i_0$, we have

$$\left|\sigma^2_{g,i}(x) - \sigma^2(x)\right| \leq |\sigma_{g,i}(\sigma_{g,i}(x)) - \sigma(\sigma_{g,i}(x))| + \left|\sigma(\sigma_{g,i}(x)) - \sigma^2(x)\right| < \epsilon,$$

which yields $\lim_{i\to\infty} \sigma^2_{g,i}(x) = \sigma^2(x)$. On the other hand, for each $i \in \mathbb{N}$ and $x \in X$ we have $\sigma^2_{g,i}(x) = x$, which finally implies that $\sigma^2(x) = x$. Thus, we infer that $\sigma \in B_n(X, X)$ with $\sigma^2 = \mathrm{id}_X$. Therefore $T \in \mathcal{G}^2(C^{(n)}(X))$ by Theorem 4. □

We now prove that every approximate 2-local isometric reflection of $C^{(n)}(X)$ is an isometric reflection.

**Corollary 5.** *The set $\mathcal{G}^2(C^{(n)}(X))$ is 2-topologically reflexive.*

**Proof.** Let $\Delta \in 2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(C^{(n)}(X)))$. With a similar proof to that of Theorem 3, we can prove that $\Delta$ is linear. Hence $\Delta \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(C^{(n)}(X)))$. Then $\Delta \in \mathcal{G}^2(C^{(n)}(X))$ by Corollary 4. □

Let us recall that a generalized bi-circular projection of a Banach space $E$ is a linear projection $P\colon E \to E$ such that $P + \lambda(\mathrm{id}_E - P)$ is a linear surjective isometry for some $\lambda \in \mathbb{T} \setminus \{1\}$. We denote by $\mathcal{GBP}(E)$ the set of all generalized bi-circular projections of $E$.

A complete description of such projections on $C^{(1)}([0,1], E)$ with $E$ a finite-dimensional complex Hilbert space was given by Botelho and Jamison [1]. Here, the next theorem describes the class of generalized bi-circular projections of $C^{(n)}(X)$.

**Theorem 5.** *A map $P\colon C^{(n)}(X) \to C^{(n)}(X)$ is a generalized bi-circular projection if and only if there exist a function $\theta \in A_n(X)$ with $\theta(x) \in \{\pm 1\}$ for all $x \in X$, and a function $\sigma \in B_n(X,X)$ with $\sigma^2 = \mathrm{id}_X$ such that*

$$P(f)(x) = \frac{1}{2}\left[f(x) + \theta(x)f(\sigma(x))\right] \qquad (x \in X,\ f \in C^{(n)}(X)).$$

**Proof.** Assume that $P \in \mathcal{GBP}(C^{(n)}(X))$. Then $P + \lambda(\mathrm{id}_{C^{(n)}(X)} - P) \in \mathcal{G}(C^{(n)}(X))$ for some $\lambda \in \mathbb{T}$ with $\lambda \neq 1$. By Theorem 1, we can find two functions $\theta \in A_n(X)$ and $\sigma \in B_n(X,X)$ such that

$$[P + \lambda(\mathrm{id}_{C^{(n)}(X)} - P)](f)(x) = \theta(x)f(\sigma(x)) \qquad (f \in C^{(n)}(X),\ x \in X),$$

which gives the following formula for $P$:

$$P(f)(x) = (1-\lambda)^{-1}[-\lambda f(x) + \theta(x)f(\sigma(x))] \qquad (f \in C^{(n)}(X),\ x \in X).$$

Since $P^2 = P$, we obtain the following equation:

$$\lambda f(x) - (\lambda+1)\theta(x)f(\sigma(x)) + [\theta(x)]^2 f(\sigma^2(x)) = 0 \qquad (f \in C^{(n)}(X),\ x \in X).$$

Suppose that there exists $x_0 \in X$ such that $x_0 \neq \sigma(x_0)$ and $x_0 \neq \sigma^2(x_0)$. Take a function $h \in C^{(n)}(X)$ such that $h(x_0) = 1$ and $h(\sigma(x_0)) = 0 = h(\sigma^2(x_0))$. Taking $f = h$ and $x = x_0$ in the equation above, we obtain $\lambda = 0$, a contradiction. Hence $\sigma(x) = x$ or $\sigma^2(x) = x$ for all $x \in X$. In any case we conclude that $\sigma^2 = \mathrm{id}_X$.

We now distinguish two cases. If $\sigma \neq \mathrm{id}_X$, we can take a point $x_0 \in X$ with $x_0 \neq \sigma(x_0)$ and a function $g \in C^{(n)}(X)$ such that $g(x_0) = 1$ and $g(\sigma(x_0)) = 0$. Substituting now in the equation, first $f = g$ and $x = x_0$, and after $f = 1_X$ and any $x$, we infer that $\lambda + [\theta(x_0)]^2 = 0$ and $\lambda - (\lambda+1)\theta(x) + [\theta(x)]^2 = 0$ for all $x \in X$, respectively. Hence $\lambda = -1$ and $[\theta(x)]^2 = 1$ for all $x \in X$. Hence $\theta(x) \in \{-1, 1\}$ for all $x \in X$ and the formula of $P$ yields

$$P(f)(x) = \frac{1}{2}\left[f(x) + \theta(x)f(\sigma(x))\right] \qquad (f \in C^{(n)}(X),\ x \in X).$$

In the another case, if $\sigma = \mathrm{id}_X$, taking $f = 1_X$ in the equation we obtain

$$[\theta(x) - \lambda][\theta(x) - 1] = [\theta(x)]^2 - (\lambda+1)\theta(x) + \lambda = 0 \qquad (x \in X).$$

Since $\lambda \neq 1$, we deduce that either $\theta(x) = \lambda$ for all $x \in X$ or $\theta(x) = 1$ for all $x \in X$. Using the formula, we conclude that

$$P(f)(x) = 0 = \frac{1}{2}\left[f(x) - f(\sigma(x))\right] \qquad (f \in C^{(n)}(X),\ x \in X)$$

or

$$P(f)(x) = f(x) = \frac{1}{2}\left[f(x) + f(\sigma(x))\right] \qquad (f \in C^{(n)}(X),\ x \in X).$$

Conversely, if $P$ is given as the average of the identity operator with an isometric reflection on $C^{(n)}(X)$, then it is immediate to check that $P \in \mathcal{GBP}(C^{(n)}(X))$. $\square$

We now show that every approximate local generalized bi-circular projection of $C^{(n)}(X)$ is a generalized bi-circular projection.

**Corollary 6.** *The set $\mathcal{GBP}(C^{(n)}(X))$ is topologically reflexive.*

**Proof.** Let $P \in \mathrm{ref}_{\mathrm{top}}(\mathcal{GBP}(C^{(n)}(X)))$. By Theorem 5, for each $f \in C^{(n)}(X)$ there exist two sequences $\{\theta_{f,i}\}_{i\in\mathbb{N}}$ in $A_n(X)$ with $\theta_{f,i}(x) \in \{\pm 1\}$ for all $x \in X$, and $\{\sigma_{f,i}\}_{i\in\mathbb{N}}$ in $B_n(X,X)$ with $\sigma_{f,i}^2 = \mathrm{id}_X$ satisfying

$$\lim_{i\to\infty} \frac{1}{2}\left[f + \theta_{f,i}(f \circ \sigma_{f,i})\right] = P(f).$$

Hence

$$\lim_{i\to\infty} \theta_{f,i}(f \circ \sigma_{f,i}) = 2P(f) - f,$$

and so $2P - \mathrm{id}_{C^{(n)}(X)} \in \mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(C^{(n)}(X)))$. Hence $2P - \mathrm{id}_{C^{(n)}(X)} \in \mathcal{G}^2(C^{(n)}(X))$ by Corollary 4 and therefore $P \in \mathcal{GBP}(C^{(n)}(X))$. $\square$

We close the paper with the study of approximate 2-local generalized bi-circular projection of $C^{(n)}(X)$.

**Theorem 6.** *The set $\mathcal{GBP}(C^{(n)}(X))$ is 2-topologically reflexive.*

**Proof.** Let $\Delta \in 2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathcal{GBP}(C^{(n)}(X)))$. For any $f, g \in C^{(n)}(X)$, there exist sequences $\{\theta_{f,i}\}_{i\in\mathbb{N}}$ in $A_n(X)$ with $\theta_{f,i}(x) \in \{\pm 1\}$ for all $x \in X$ and $\{\sigma_{f,i}\}_{i\in\mathbb{N}}$ in $B_n(X,X)$ with $\sigma_{f,i}^2(x) = x$ for all $x \in X$ satisfying

$$\lim_{i\to\infty} \frac{1}{2}\left[f + \theta_{f,g,i}(f \circ \sigma_{f,g,i})\right] = \Delta(f),$$
$$\lim_{i\to\infty} \frac{1}{2}\left[g + \theta_{f,g,i}(g \circ \sigma_{f,g,i})\right] = \Delta(g).$$

Hence, for every $f, g \in C^{(n)}(X)$, we have

$$\lim_{i\to\infty} \theta_{f,g,i}(f \circ \sigma_{f,g,i}) = 2\Delta(f) - f,$$
$$\lim_{i\to\infty} \theta_{f,g,i}(g \circ \sigma_{f,g,i}) = 2\Delta(g) - g,$$

and this says that $2\Delta - \mathrm{id}_{C^{(n)}(X)} \in 2\text{-}\mathrm{ref}_{\mathrm{top}}(\mathcal{G}^2(C^{(n)}(X)))$. Hence $2\Delta - \mathrm{id}_{C^{(n)}(X)} \in \mathcal{G}^2(C^{(n)}(X))$ by Corollary 5. Therefore, from Theorem 5 we conclude that $\Delta \in \mathcal{GBP}(C^{(n)}(X))$. $\square$

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