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On 2-local diameter-preserving maps between $C(X)$-spaces

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Abstract
The 2-locality problem of diameter-preserving maps between $C(X)$-spaces is addressed in this paper. For any compact Hausdorff space X with at least three points, we give an example of a 2-local diameter-preserving map on $C(X)$ which is not linear. However, we show that for first countable compact Hausdorff spaces X and Y, every 2-local diameter-preserving map from $C(X)$ to $C(Y)$ is linear and surjective up to constants in some sense. This fact yields the 2-algebraic reflexivity of isometries with respect to the diameter norms on the quotient spaces.

Keywords 2-local map · Diameter-preserving map · Function space · Weighted composition operator

Mathematics Subject Classification 46B04 · 47B38

1 Introduction and results

Let E and F be Banach spaces and let $\mathcal{S}$ be a subset of $\mathcal{L}(E, F)$, the space of linear operators from E to F. Let us recall that a linear map $T: E \to F$ is a local $\mathcal{S}$-map if for every $e \in E$, there exists a $T_e \in \mathcal{S}$, depending possibly on e, such that $T_e(e) = T(e)$. On the other hand, a map $\Delta: E \to F$ (which is not assumed to be linear) is called a 2-local $\mathcal{S}$-map if for any $e, u \in E$, there exists a $T_{e,u} \in \mathcal{S}$, depending in general on e and u, such that $T_{e,u}(e) = \Delta(e)$ and $T_{e,u}(u) = \Delta(u)$.

Most of the published works on local and 2-local $\mathcal{S}$-maps concern the set $\mathcal{S} = \mathcal{G}(E)$, the group of surjective linear isometries of E. In this case, the local and 2-

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local $\mathcal{G}(E)$-maps are known as local and 2-local isometries of E, respectively. The main question which one raises is for which Banach spaces, every local isometry is a surjective isometry or, equivalently, which Banach spaces have an algebraically reflexive isometry group. In the 2-local setting, the basic problem is to show that every 2-local isometry is a surjective linear isometry.

In [21], Molnár initiated the study of 2-local isometries on operator algebras and proposed to investigate the 2-locality of isometries on function algebras. In this line, Győry [11] dealt with 2-local isometries on spaces of continuous functions. In [18], Villegas and the first author adapted the Győry's technique to analyze the 2-local isometries on Lipschitz algebras. Hatori, Miura, Oka, and Takagi [13] considered 2-local isometries on uniform algebras including certain algebras of holomorphic functions. More recently, Hosseini [15], Hatori and Oi [14] and Li, Peralta, L. Wang and Y.-S. Wang [20] have investigated 2-local isometries of different function algebras such as uniform algebras, Lipschitz algebras, and algebras of continuously differentiable functions.

Our aim in this paper is to study the 2-locality problem for isometries between certain quotient Banach spaces which appear in a natural form when one treats with maps between $C(X)$-spaces which preserve the diameter of the range.

Let $C(X)$ be the Banach space of all continuous complex-valued functions on a compact Hausdorff space X, with the usual supremum norm. A map $\Delta\colon C(X) \to C(Y)$ (not necessarily linear) is diameter-preserving if

$$\rho(\Delta(f) - \Delta(g)) = \rho(f - g) \quad (f, g \in C(X)),$$

where for each $f \in C(X)$,

$$\rho(f) = \sup\{|f(x) - f(z)| : x, z \in X\}.$$

Győry and Molnár [12] introduced such maps and gave a complete description of diameter-preserving linear bijections of $C(X)$, when X is a first countable compact Hausdorff space. Cabello Sánchez [5] and González and Uspenskij [10] established the same characterization without the first countability assumption. As usual, $\mathbb{T}$ denotes the unit circle of $\mathbb{C}$. We also put

$$\mathbb{T}^+ = \{e^{it} : t \in [0, \pi)\}.$$

Moreover, 1_X and 0_X stand for the constant functions 1 and 0 on X, respectively.

Theorem 1 [5,10,12]. *Let X and Y be compact Hausdorff spaces. A linear bijection $T\colon C(X) \to C(Y)$ is diameter-preserving if and only if there exist a homeomorphism $\phi\colon Y \to X$, a linear functional $\mu\colon C(X) \to \mathbb{C}$ and a scalar $\lambda \in \mathbb{T}$ with $\lambda \neq -\mu(1_X)$ such that*

$$T(f) = \lambda f \circ \phi + \mu(f)1_Y \quad (f \in C(X)).$$

The main problem addressed in the study of diameter-preserving maps between function algebras is establishing a representation of such a map as the sum of a weighted composition operator and a functional as in Theorem 1. We have a precise description of diameter-preserving maps for most of the classical function spaces (see for example [1–3,5,7,9,23] for diameter-preserving linear maps and [4,8,16] for the non-linear case).

In the case that $\mathcal{S}$ is the set of all diameter-preserving linear bijections from $C(X)$ to $C(Y)$, we studied in a recent paper [17] the local $\mathcal{S}$-maps, called local diameter-preserving maps. Namely, we proved that for first countable compact Hausdorff spaces X and Y, every local diameter-preserving map from $C(X)$ to $C(Y)$ is a diameter-preserving bijection. The first countability assumption on the topological spaces is a mild and appropriate condition when one deals with these problems. For example, the isometry group and the automorphism group of $C(X)$ are algebraically reflexive whenever X is first countable [22], while these results are no longer true when X does not enjoy this property (see section 7 in [6]).

It is natural to arise the corresponding question in the 2-local context, that is, is every 2-local diameter-preserving map a diameter-preserving linear bijection? Unfortunately or not, the answer is negative as we shall see in a counterexample.

Let us recall that a map $\Delta\colon C(X) \to C(Y)$ (not assumed to be linear) is a 2-local diameter-preserving map if for any $f, g \in C(X)$, there exists a diameter-preserving linear bijection $T_{f,g}$ from $C(X)$ to $C(Y)$ such that $T_{f,g}(f) = \Delta(f)$ and $T_{f,g}(g) = \Delta(g)$.

Example (*A 2-local diameter-preserving non-linear map between $C(X)$-spaces*) Let X be a compact Hausdorff space with at least three points. Let $\mu\colon C(X) \to \mathbb{C}$ be a homogeneous non-additive functional such that $\mu(1_X) \neq -1$ and $\mu(1_X - f) = \mu(1_X) - \mu(f)$ for all $f \in C(X)$. To give an example of such a functional μ, fix three distinct points $x_1, x_2, x_3 \in X$ and define $\mu\colon C(X) \to \mathbb{C}$ by

$$\mu(f) = \begin{cases} f(x_1) & \text{if } f(x_1) = f(x_2) \text{ and } f(x_1) \neq f(x_3), \\ f(x_3) & \text{otherwise.} \end{cases}$$

It is easy to see that μ is homogeneous and $\mu(1_X - f) = \mu(1_X) - \mu(f)$ for all $f \in C(X)$. Meanwhile, μ is not additive, since we can take $f, g \in C(X)$ such that $f(x_1) = f(x_2) = 1$ and $f(x_3) = 0$ and also $g(x_1) = g(x_3) = 1$ and $g(x_2) = 0$, and then $\mu(f + g) = 1 \neq 2 = \mu(f) + \mu(g)$.

Define now the map $\Delta\colon C(X) \to C(X)$ by

$$\Delta(f) = f + \mu(f)1_X \quad (f \in C(X)).$$

For each pair $f, g \in C(X)$, consider a linear functional $\mu_{f,g}\colon C(X) \to \mathbb{C}$ satisfying

$$\mu_{f,g}(f) = \mu(f), \quad \mu_{f,g}(g) = \mu(g), \quad \mu_{f,g}(1_X) = \mu(1_X).$$

Notice that such a functional $\mu_{f,g}$ exists. Indeed, if $\{f, g, 1_X\}$ is linearly independent, the existence of $\mu_{f,g}$ can be established by extending linearly to $C(X)$ a convenient linear functional defined on $\operatorname{span}\{f, g, 1_X\}$. If $\{f, g, 1_X\}$ is linearly dependent and $1_X \in \operatorname{span}\{f, g\}$, then we can find a linear functional $\mu_{f,g}$ on $C(X)$

such that $\mu_{f,g}(f) = \mu(f)$ and $\mu_{f,g}(g) = \mu(g)$ (note that μ is homogeneous). Since $1_X = af + bg$ for some $a, b \in \mathbb{C}$, the hypotheses on μ easily imply that $\mu_{f,g}(1_X) = \mu(1_X)$, as desired. In the case where $\{f, g, 1_X\}$ is linearly dependent and $1_X \notin \operatorname{span}\{f, g\}$ we conclude that f and g are linearly dependent and we may assume that $f = cg$ for some scalar c. In this case, there exists a linear functional $\mu_{f,g}$ on $C(X)$ such that $\mu_{f,g}(1_X) = \mu(1_X)$ and $\mu_{f,g}(g) = \mu(g)$. Hence $\mu_{f,g}(f) = \mu(f)$ since μ is homogeneous. Thus in each case we can find a linear functional $\mu_{f,g} \colon C(X) \to \mathbb{C}$ with the desired properties.

Finally, for any $f, g \in C(X)$, define $T_{f,g} \colon C(X) \to C(X)$ by

$$T_{f,g}(h) = h + \mu_{f,g}(h)1_X \quad (h \in C(X)).$$

Then $T_{f,g}$ is a diameter-preserving linear bijection by Theorem 1. Clearly, for any $f, g \in C(X)$, we have $T_{f,g}(f) = \Delta(f)$ and $T_{f,g}(g) = \Delta(g)$. Hence Δ is a 2-local diameter-preserving map which is homogeneous but not additive.

However, we will show that, in the case where X and Y are first countable, every 2-local diameter-preserving map (which is immediately diameter-preserving) is linear and surjective up to constants in some sense. Our approach consists of analyzing the 2-local isometries of the following quotient Banach spaces which appear closely related to diameter-preserving maps.

Given a compact Hausdorff space X, let $C_\rho(X)$ denote the quotient space $C(X)/\ker(\rho)$. Clearly, $C_\rho(X)$ is a Banach space with the norm

$$\|\pi_X(f)\|_\rho = \rho(f) \quad (f \in C(X)),$$

where $\pi_X \colon C(X) \to C_\rho(X)$ is the canonical quotient surjection. Let us recall that a mapping $T \colon C_\rho(X) \to C_\rho(Y)$ (which is not assumed to be linear or surjective) is an isometry whenever

$$\|T(\pi_X(f)) - T(\pi_X(g))\|_\rho = \|\pi_X(f) - \pi_X(g)\|_\rho \quad (f, g \in C(X)).$$

Our main result is the following theorem on 2-local isometries between $C_\rho(X)$-spaces.

Theorem 2 *Let X and Y be first countable compact Hausdorff spaces and let $T \colon C_\rho(X) \to C_\rho(Y)$ be a 2-local isometry. Then T is a surjective linear isometry.*

2 Proofs

The first key tool to prove Theorem 2 is the fact that every isometry T between $C_\rho(X)$-spaces induces a convenient (injective) diameter-preserving map Δ between the corresponding $C(X)$-spaces, which is linear or surjective if so is T. Towards this end, fix two points $u_0 \in X$ and $w_0 \in Y$ and consider the linear bijections

$$\Psi_X \colon C(X) \to C_\rho(X) \oplus \mathbb{C}, \quad \Psi_X(f) = (\pi_X(f), f(u_0)) \quad (f \in C(X))$$

and

$$\Psi_Y : C(Y) \to C_\rho(Y) \oplus \mathbb{C}, \quad \Psi_Y(g) = (\pi_Y(g), g(w_0)) \quad (g \in C(Y)).$$

Lemma 1 *Let X and Y be compact Hausdorff spaces and let $T : C_\rho(X) \to C_\rho(Y)$ be an isometry. Then $\Delta : C(X) \to C(Y)$ defined by*

$$\Delta(f) = \Psi_Y^{-1}(T(\pi_X(f)), f(u_0)) \quad (f \in C(X))$$

is an injective diameter-preserving map. Moreover, T is linear (respectively, surjective) if and only if so is Δ.

Proof Given $f, g \in C(X)$, we put $h = \Delta(f) - \Delta(g)$. Then

$$\begin{aligned} h &= \Psi_Y^{-1}(T(\pi_X(f)), f(u_0)) - \Psi_Y^{-1}(T(\pi_X(g)), g(u_0)) \\ &= \Psi_Y^{-1}(T(\pi_X(f)) - T(\pi_X(g)), f(u_0) - g(u_0)). \end{aligned}$$

Hence

$$(\pi_Y(h), h(w_0)) = \Psi_Y(h) = (T(\pi_X(f)) - T(\pi_X(g)), f(u_0) - g(u_0)),$$

and, consequently, $\pi_Y(h) = T(\pi_X(f)) - T(\pi_X(g))$. This implies that

$$\begin{aligned} \rho(\Delta(f) - \Delta(g)) &= \|\pi_Y(h)\|_\rho = \|T(\pi_X(f)) - T(\pi_X(g))\|_\rho \\ &= \|\pi_X(f) - \pi_X(g)\|_\rho \\ &= \rho(f - g), \end{aligned}$$

that is, Δ is diameter-preserving. Clearly, Δ is injective. It is also easy to see that Δ is linear if so is T. Assume now that T is surjective. Then, given $g \in C(Y)$ there exists $f \in C(X)$ such that $T(\pi_X(f)) = \pi_Y(g)$. Replacing g by $g + \lambda$ for some $\lambda \in \mathbb{C}$, we can assume that $g(w_0) = f(u_0)$. Hence

$$\Delta(f) = \Psi_Y^{-1}(T(\pi_X(f)), f(u_0)) = \Psi_Y^{-1}(\pi_Y(g), g(w_0)) = g,$$

which shows that Δ is surjective, as well. A similar reasoning shows that if Δ is linear (respectively, surjective), then so is T. □

We now prove our main theorem.

Proof of Theorem 2 Let $T : C_\rho(X) \to C_\rho(Y)$ be a 2-local isometry. We prove the theorem through a series of claims. We note that some claims have similar proofs to those of the corresponding steps in the proof of [17, Theorem 2]. For this reason, we will only include here the proof of those claims whose arguments differ essentially from similar steps in [17].

Claim 1 *The map* $\Delta\colon C(X) \to C(Y)$ *defined by*

$$\Delta(f) = \Psi_Y^{-1}(T(\pi_X(f)), f(u_0)) \quad (f \in C(X)),$$

is a 2-local diameter-preserving map.

Let $f, g \in C(X)$. By hypotheses, there exists a surjective linear isometry $T_{f,g}\colon C_\rho(X) \to C_\rho(Y)$ such that $T_{f,g}(\pi_X(f)) = T(\pi_X(f))$ and $T_{f,g}(\pi_X(g)) = T(\pi_X(g))$. Define $\Delta_{f,g}\colon C(X) \to C(Y)$ by

$$\Delta_{f,g}(h) = \Psi_Y^{-1}(T_{f,g}(\pi_X(h)), h(u_0)) \quad (h \in C(X)).$$

By Lemma 1, $\Delta_{f,g}$ is a diameter-preserving linear bijection from $C(X)$ to $C(Y)$ satisfying $\Delta_{f,g}(f) = \Delta(f)$ and $\Delta_{f,g}(g) = \Delta(g)$.

The following fact will be used repeatedly without any explicit mention in our proof.

Claim 2 *For any* $f, g \in C(X)$*, there exists a diameter-preserving linear bijection* $\Delta_{f,g}$ *of* $C(X)$ *to* $C(Y)$ *such that* $\Delta_{f,g}(f) = \Delta(f)$ *and* $\Delta_{f,g}(g) = \Delta(g)$*. Moreover, there exist a homeomorphism* $\phi_{f,g}\colon Y \to X$*, a linear functional* $\mu_{f,g}$ *on* $C(X)$ *and a scalar* $\lambda_{f,g} \in \mathbb{T}$ *with* $\lambda_{f,g} \neq -\mu_{f,g}(1_X)$ *such that*

$$\Delta(f)(y) = \lambda_{f,g} f(\phi_{f,g}(y)) + \mu_{f,g}(f) \quad (y \in Y)$$

and

$$\Delta(g)(y) = \lambda_{f,g} g(\phi_{f,g}(y)) + \mu_{f,g}(g) \quad (y \in Y).$$

It follows from Claim 1 and Theorem 1.

Claim 3 Δ *is injective, diameter-preserving and homogeneous.*

Let $f, g \in C(X)$. If $\Delta(f) = \Delta(g)$, then $f = g$ by the injectivity of $\Delta_{f,g}$ and therefore Δ is injective. Clearly, Δ is diameter-preserving because

$$\rho(\Delta(f) - \Delta(g)) = \rho(\Delta_{f,g}(f) - \Delta_{f,g}(g)) = \rho(f - g).$$

Finally, given $\lambda \in \mathbb{C}$, we have

$$\Delta(\lambda f) = \Delta_{f,\lambda f}(\lambda f) = \lambda \Delta_{f,\lambda f}(f) = \lambda \Delta(f),$$

and thus Δ is homogeneous.

By Claim 2, there exists a homeomorphism from Y onto X. Hence Y and X have the same cardinality. Since Theorem 2 is quite easy to verify when Y is a singleton, we suppose from now on that X and Y have at least two points.

Given a set X with cardinal number $|X| \geq 2$, we set

$$\widetilde{X} = \{(x_1, x_2) \in X \times X : x_1 \neq x_2\},$$

$$X_2 = \left\{\{x_1, x_2\} : (x_1, x_2) \in \widetilde{X}\right\},$$

and we define the natural correspondence $\Lambda_X : \widetilde{X} \to X_2$ by

$$\Lambda_X\left((x_1, x_2)\right) = \{x_1, x_2\} \quad \left((x_1, x_2) \in \widetilde{X}\right).$$

Given a compact Hausdorff space X and a point $(x_1, x_2) \in \widetilde{X}$, Urysohn's lemma guarantees the existence of a continuous function $h_{(x_1,x_2)} : X \to [0, 1]$ such that

$$h_{(x_1,x_2)}(x_1) - h_{(x_1,x_2)}(x_2) = \rho(h_{(x_1,x_2)}).$$

In fact, $h_{(x_1,x_2)}(x_1) = 1$ and $h_{(x_1,x_2)}(x_2) = 0$. Furthermore, since X is also first countable, we can take $h_{(x_1,x_2)}$ such that $h^{-1}_{(x_1,x_2)}(\{1\}) = \{x_1\}$ and $h^{-1}_{(x_1,x_2)}(\{0\}) = \{x_2\}$. In particular,

$$\left|h_{(x_1,x_2)}(z) - h_{(x_1,x_2)}(w)\right| < \rho(h_{(x_1,x_2)})$$

for all $(z, w) \in \widetilde{X} \setminus \{(x_1, x_2), (x_2, x_1)\}$.

Claim 4 *For any* $(x_1, x_2) \in \widetilde{X}$*, the set*

$$\mathcal{B}_{(x_1,x_2)} = \bigcap_{f \in C(X)} \mathcal{B}_{(x_1,x_2),f}$$

is non-empty, where

$$\mathcal{B}_{(x_1,x_2),f} = \left\{((y_1, y_2), \lambda) \in \widetilde{Y} \times \mathbb{T} : \Delta(f)(y_1) - \Delta(f)(y_2) = \lambda\left(f(x_1) - f(x_2)\right)\right\} \quad (f \in C(X)).$$

Let $(x_1, x_2) \in \widetilde{X}$ and $f \in C(X)$. Observe first that $\mathcal{B}_{(x_1,x_2),f}$ is non-empty. Indeed, it suffices to choose $y_1, y_2 \in Y$ such that $\phi_{f,f}(y_i) = x_i$ for $i = 1, 2$. Then

$$\Delta(f)(y_1) - \Delta(f)(y_2) = \Delta_{f,f}(f)(y_1) - \Delta_{f,f}(f)(y_2) = \lambda_{f,f}\left(f(x_1) - f(x_2)\right),$$

and therefore $((y_1, y_2), \lambda_{f,f}) \in \mathcal{B}_{(x_1,x_2),f}$.

We next prove that $\mathcal{B}_{(x_1,x_2),g} \subseteq \mathcal{B}_{(x_1,x_2),f}$ where $g = h_{(x_1,x_2)}$. By Claim 2, there exists a diameter-preserving linear bijection $\Delta_{g,f}$ from $C(X)$ to $C(Y)$ such that $\Delta_{g,f}(g) = \Delta(g)$ and $\Delta_{g,f}(f) = \Delta(f)$. Furthermore, we have a homeomorphism $\phi_{g,f}$ from Y onto X, a linear functional $\mu_{g,f}$ on $C(X)$ and a scalar $\lambda_{g,f} \in \mathbb{T}$ with $\lambda_{g,f} \neq -\mu_{g,f}(1_X)$ such that

$$\Delta_{g,f}(h)(y) = \lambda_{g,f} h(\phi_{g,f}(y)) + \mu_{g,f}(h) \quad (h \in C(X),\ y \in Y).$$

Let $((y_1, y_2), \lambda) \in \mathcal{B}_{(x_1,x_2),g}$ be arbitrary. We have

$$\lambda\left(g(x_1) - g(x_2)\right) = \Delta(g)(y_1) - \Delta(g)(y_2)$$

$$\begin{aligned} &= \Delta_{g,f}(g)(y_1) - \Delta_{g,f}(g)(y_2) \\ &= \lambda_{g,f}\left(g(\phi_{g,f}(y_1)) - g(\phi_{g,f}(y_2))\right) \end{aligned}$$

and therefore

$$\left|g(\phi_{g,f}(y_1)) - g(\phi_{g,f}(y_2))\right| = 1.$$

This implies that either

$$(\phi_{g,f}(y_1), \phi_{g,f}(y_2)) = (x_1, x_2),$$

or

$$(\phi_{g,f}(y_1), \phi_{g,f}(y_2)) = (x_2, x_1).$$

Hence $\lambda_{g,f} = \lambda$ in the first case, or $\lambda_{g,f} = -\lambda$ in the second one. It follows that

$$\begin{aligned} \Delta(f)(y_1) - \Delta(f)(y_2) &= \Delta_{g,f}(f)(y_1) - \Delta_{g,f}(f)(y_2) \\ &= \lambda_{g,f}\left(f(\phi_{g,f}(y_1)) - f(\phi_{g,f}(y_2))\right) \\ &= \lambda\left(f(x_1) - f(x_2)\right), \end{aligned}$$

whence $((y_1, y_2), \lambda) \in \mathcal{B}_{(x_1,x_2),f}$ and this proves that $\mathcal{B}_{(x_1,x_2),g} \subseteq \mathcal{B}_{(x_1,x_2),f}$. Consequently, we obtain that $\mathcal{B}_{(x_1,x_2)} = \mathcal{B}_{(x_1,x_2),g}$.

The proof of Claim 5 is similar to that of Step 4 in [17].

Claim 5 *For every* $(x_1, x_2) \in \widetilde{X}$, *there exist* $(y_1, y_2) \in \widetilde{Y}$ *and* $\lambda \in \mathbb{T}$ *such that*

$$\mathcal{B}_{(x_1,x_2)} = \{((y_1, y_2), \lambda), ((y_2, y_1), -\lambda)\}.$$

It is immediate from Claim 5 that for every $(x_1, x_2) \in \widetilde{X}$, the set

$$\mathcal{A}_{(x_1,x_2)} = \left\{(y_1, y_2) \in \widetilde{Y} \mid \exists\lambda \in \mathbb{T}^+ : ((y_1, y_2), \lambda) \in \mathcal{B}_{(x_1,x_2)}\right\}$$

is a singleton. Let $\Gamma\colon \widetilde{X} \to \widetilde{Y}$ be the map given by $\Gamma((x_1, x_2)) = (y_1, y_2)$ where for each $(x_1, x_2) \in \widetilde{X}$, the element $(y_1, y_2) \in \widetilde{Y}$ is the unique point of $A_{(x_1,x_2)}$. We note that if $\mathcal{A}_{(x_1,x_2)} = \{(y_1, y_2)\}$, then the definition of $\mathcal{A}_{(x_1,x_2)}$ shows that there exists a (unique) scalar $\beta(x_1, x_2) \in \mathbb{T}^+$, depending on the pair (x_1, x_2), such that

$$\Delta(f)(y_1) - \Delta(f)(y_2) = \beta(x_1, x_2)\,(f(x_1) - f(x_2)) \quad (f \in C(X)).$$

This concludes that

$$\Delta(f)(y_2) - \Delta(f)(y_1) = \beta(x_1, x_2)\,(f(x_2) - f(x_1)) \quad (f \in C(X)),$$

that is, $\beta(x_2, x_1) = \beta(x_1, x_2)$ and $\Gamma((x_2, x_1)) = (y_2, y_1)$.

Claim 6 *The map* Γ *is a bijection from* $\widetilde{X}$ *to* $\cup_{(x_1,x_2)\in\widetilde{X}}\mathcal{A}_{(x_1,x_2)}$.

The surjectivity of Γ is immediate, since $(y_1, y_2) = \Gamma((x_1, x_2))$ if and only if $(y_1, y_2) \in \mathcal{A}_{(x_1,x_2)}$. To prove the injectivity of Γ, let $(x_1, x_2), (x_3, x_4) \in \widetilde{X}$ be such that

$$(y_1, y_2) = \Gamma((x_1, x_2)) = \Gamma((x_3, x_4)).$$

Then we have

$$\beta(x_1, x_2)(f(x_1) - f(x_2)) = \Delta(f)(y_1) - \Delta(f)(y_2) = \beta(x_3, x_4)(f(x_3) - f(x_4))$$

for all $f \in C(X)$, where $\beta(x_1, x_2), \beta(x_3, x_4) \in \mathbb{T}^+$. Substituting f by $h_{(x_1,x_2)}$, we deduce that $\{x_3, x_4\} = \{x_1, x_2\}$. Now since both scalars $\beta(x_1, x_2)$ and $\beta(x_3, x_4)$ are in $\mathbb{T}^+$, we get $(x_3, x_4) = (x_1, x_2)$, as desired.

Claim 7 *For any* $\{x_1, x_2\}, \{x_3, x_4\} \in X_2$, *we have*

$$|\{x_1, x_2\} \cap \{x_3, x_4\}| = |\Lambda_Y(\Gamma((x_1, x_2))) \cap \Lambda_Y(\Gamma((x_3, x_4)))|.$$

Let $\{x_1, x_2\}, \{x_3, x_4\} \in X_2$. If $\{x_1, x_2\} = \{x_3, x_4\}$, then either $\Gamma((x_1, x_2)) = \Gamma((x_3, x_4))$ or $\Gamma((x_1, x_2)) = \Gamma((x_4, x_3))$ and thus the equality holds. Assume that $\{x_1, x_2\} \neq \{x_3, x_4\}$. Then $(x_1, x_2) \neq (x_3, x_4)$ and $(x_1, x_2) \neq (x_4, x_3)$. Hence $\Gamma((x_1, x_2)) = (y_1, y_2)$ and $\Gamma((x_3, x_4)) = (y_3, y_4)$ for some $\{y_1, y_2\}, \{y_3, y_4\} \in Y_2$ with $\{y_1, y_2\} \neq \{y_3, y_4\}$ by the injectivity of Γ and the fact that $\Lambda_Y(\Gamma((x_1, x_2))) = \Lambda_Y(\Gamma((x_2, x_1)))$. We have two equations:

$$\Delta(f)(y_1) - \Delta(f)(y_2) = \beta(x_1, x_2)(f(x_1) - f(x_2)),$$
$$\Delta(f)(y_3) - \Delta(f)(y_4) = \beta(x_3, x_4)(f(x_3) - f(x_4)),$$

for all $f \in C(X)$, where $\beta(x_1, x_2), \beta(x_3, x_4) \in \mathbb{T}^+$. Put $g = h_{(x_1,x_2)}$ and $h = h_{(x_3,x_4)}$. Then using the first equality for g and the second one for h, we obtain

$$\Delta(g)(y_1) - \Delta(g)(y_2) = \beta(x_1, x_2)(g(x_1) - g(x_2)),$$
$$\Delta(h)(y_3) - \Delta(h)(y_4) = \beta(x_3, x_4)(h(x_3) - h(x_4)).$$

By Claim 2, there exist a homeomorphism $\phi_{g,h}$ from Y onto X, a linear functional $\mu_{g,h}$ on $C(X)$ and a scalar $\lambda_{g,h} \in \mathbb{T}$ with $\lambda_{g,h} \neq -\mu_{g,h}(1_X)$ such that

$$\Delta(g)(y) = \lambda_{g,h} g(\phi_{g,h}(y)) + \mu_{g,h}(g)$$

and

$$\Delta(h)(y) = \lambda_{g,h} h(\phi_{g,h}(y)) + \mu_{g,h}(h)$$

for all $y \in Y$. Therefore,

$$\Delta(g)(y_1) - \Delta(g)(y_2) = \lambda_{g,h}\left(g(\phi_{g,h}(y_1)) - g(\phi_{g,h}(y_2))\right),$$
$$\Delta(h)(y_3) - \Delta(h)(y_4) = \lambda_{g,h}\left(h(\phi_{g,h}(y_3)) - h(\phi_{g,h}(y_4))\right),$$

and it follows that

$$\lambda_{g,h}\left(g(\phi_{g,h}(y_1)) - g(\phi_{g,h}(y_2))\right) = \beta(x_1, x_2)\left(g(x_1) - g(x_2)\right),$$
$$\lambda_{g,h}\left(h(\phi_{g,h}(y_3)) - h(\phi_{g,h}(y_4))\right) = \beta(x_3, x_4)\left(h(x_3) - h(x_4)\right).$$

These equalities imply that

$$(\phi_{g,h}(y_1), \phi_{g,h}(y_2))) \in \{(x_1, x_2), (x_2, x_1)\}$$

and

$$(\phi_{g,h}(y_3), \phi_{g,h}(y_4)) \in \{(x_3, x_4), (x_4, x_3)\}.$$

Then we have four possibilities:

(1) $x_1 = \phi_{g,h}(y_1),\ x_2 = \phi_{g,h}(y_2),\ x_3 = \phi_{g,h}(y_3),\ x_4 = \phi_{g,h}(y_4)$.
(2) $x_1 = \phi_{g,h}(y_1),\ x_2 = \phi_{g,h}(y_2),\ x_3 = \phi_{g,h}(y_4),\ x_4 = \phi_{g,h}(y_3)$.
(3) $x_1 = \phi_{g,h}(y_2),\ x_2 = \phi_{g,h}(y_1),\ x_3 = \phi_{g,h}(y_3),\ x_4 = \phi_{g,h}(y_4)$.
(4) $x_1 = \phi_{g,h}(y_2),\ x_2 = \phi_{g,h}(y_1),\ x_3 = \phi_{g,h}(y_4),\ x_4 = \phi_{g,h}(y_3)$.

If $|\{x_1, x_2\} \cap \{x_3, x_4\}| = 1$, then we infer from the injectivity of $\phi_{g,h}$ that

$$|\Lambda_Y(\Gamma((x_1, x_2))) \cap \Lambda_Y(\Gamma((x_3, x_4)))| = |\{y_1, y_2\} \cap \{y_3, y_4\}| = 1,$$

while if $|\{x_1, x_2\} \cap \{x_3, x_4\}| = 0$, then

$$|\Lambda_Y(\Gamma((x_1, x_2))) \cap \Gamma((x_3, x_4))| = |\{y_1, y_2\} \cap \{y_3, y_4\}| = 0.$$

The proof of Claim 8 is the same as that of Step 10 of [17].

Claim 8 *Assume $|X| \geq 3$. For each $x \in X$ and any $\{x_1, x_2\} \in X_2$ with $x_1 \neq x \neq x_2$, there exists a unique point, depending only on x and denoted by $\varphi(x)$, in the intersection $\Gamma(\{x, x_1\}) \cap \Gamma(\{x, x_2\})$. Then the map $\varphi\colon X \to Y$ is injective and $\{\varphi(x_1), \varphi(x_2)\} = \Lambda_Y(\Gamma((x_1, x_2)))$ for all $\{x_1, x_2\} \in X_2$.*

Let $Y_0 = \varphi(X)$. Since the map $\varphi\colon X \to Y$ is injective, its inverse $\phi_0\colon Y_0 \to X$ is a bijection which satisfies

$$\{y_1, y_2\} = \Lambda_Y(\Gamma((\phi_0(y_1), \phi_0(y_2))) \quad (\{y_1, y_2\} \in (Y_0)_2).$$

Now the same argument as in Step 12 of [17] yields the next claim.

Claim 9 *There exists a scalar* $\lambda \in \mathbb{T}$ *such that*

$$\Delta(f)(y_1) - \Delta(f)(y_2) = \lambda\,(f(\phi_0(y_1)) - f(\phi_0(y_2))) \quad (f \in C(X),\ y_1, y_2 \in Y_0)\,.$$

Using the above claim we can define a functional $\mu\colon C(X) \to \mathbb{C}$ by

$$\mu(f) = \Delta(f)(y_0) - \lambda f(\phi_0(y_0)) \quad (f \in C(X)),$$

where y_0 is an arbitrary point of Y_0. Then it is obvious that μ is well-defined and homogeneous and, moreover,

$$\Delta(f)(y) = \lambda f(\phi_0(y)) + \mu(f) \quad (f \in C(X),\ y \in Y_0). \tag{1}$$

Note that $\Delta(1_X)$ is a non-zero constant function by Claim 3. Hence it follows from (1) that $\mu(1_X) \neq -\lambda$.

The proof of Step 15 of [17] can be applied to get the next claim.

Claim 10 $\phi_0\colon Y_0 \to X$ *is a homeomorphism.*

In the next claims we will show that the homeomorphism $\phi_0\colon Y_0 \to X$ can be extended to a homeomorphism $\phi\colon Y \to X$ satisfying $\Delta(f)(y) = \lambda f(\phi(y)) + \mu(f)$ for all $f \in C(X)$ and $y \in Y$. To do this we first prove the next claim.

Claim 11 *The map* $S\colon C(X) \to C(Y)$ *defined by*

$$S(f)(y) = \lambda^{-1}(\Delta(f)(y) - \mu(f)) \quad (f \in C(X),\ y \in Y)$$

is a unital algebra homomorphism.

Fix a point $y \in Y$ and define the functional $S_y\colon C(X) \to \mathbb{C}$ by

$$S_y(f) = \lambda^{-1}(\Delta(f)(y) - \mu(f)) \quad (f \in C(X)).$$

Since $\Delta(1_X)$ is a constant function, it follows from the equality (1) that $S_y(1_X) = 1$. We next prove that S_y is linear and multiplicative. Since $S_y(0_X) = 0$, by the Kowalski–Słodkowski theorem [19] it suffices to show that $S_y(f) - S_y(g) \in (f-g)(X)$ for every $f, g \in C(X)$. Let $f, g \in C(X)$. Since $\phi_0\colon Y_0 \to X$ is a bijective map, there exists $y_0 \in Y_0$ such that $\phi_0(y_0) = \phi_{f,g}(y)$. Construct the sequence $\{y_i\}_{i=0}^{\infty}$ in Y_0 such that

$$\phi_0(y_{i+1}) = \phi_{f,g}(y_i) \quad (i \in \mathbb{N} \cup \{0\}).$$

Since Y_0 is a first countable compact Hausdorff space, passing through a subsequence we may assume that $\{y_i\}_i \to z_0$ for some $z_0 \in Y_0$. Hence, tending $i \to \infty$ in the above equality, we get $\phi_0(z_0) = \phi_{f,g}(z_0)$. Since $z_0, y_i \in Y_0$, Claim 9 provides the equations:

$$\Delta(f)(z_0) - \Delta(f)(y_i) = \lambda(f(\phi_0(z_0)) - f(\phi_0(y_i)))$$

and

$$\Delta(g)(z_0) - \Delta(g)(y_i) = \lambda(g(\phi_0(z_0)) - g(\phi_0(y_i))).$$

On the other hand, since $\phi_{f,g}(z_0) = \phi_0(z_0)$ and $\phi_{f,g}(y_i) = \phi_0(y_{i+1})$, we have

$$\begin{aligned}\Delta(f)(z_0) - \Delta(f)(y_i) &= \lambda_{f,g}(f(\phi_{f,g}(z_0)) - f(\phi_{f,g}(y_i))) \\ &= \lambda_{f,g}(f(\phi_0(z_0)) - f(\phi_0(y_{i+1})))\end{aligned}$$

and

$$\begin{aligned}\Delta(g)(z_0) - \Delta(g)(y_i) &= \lambda_{f,g}(g(\phi_{f,g}(z_0)) - g(\phi_{f,g}(y_i))) \\ &= \lambda_{f,g}(g(\phi_0(z_0)) - g(\phi_0(y_{i+1}))).\end{aligned}$$

Hence, using the cited equations above, for each $i \in \mathbb{N} \cup \{0\}$ we have

$$f(\phi_0(z_0)) - f(\phi_0(y_i)) = \lambda^{-1}\lambda_{f,g}(f(\phi_0(z_0)) - f(\phi_0(y_{i+1})))$$

and

$$g(\phi_0(z_0)) - g(\phi_0(y_i)) = \lambda^{-1}\lambda_{f,g}(g(\phi_0(z_0)) - g(\phi_0(y_{i+1}))).$$

Now, it follows by induction that for each $i \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, we have

$$f(\phi_0(z_0)) - f(\phi_0(y_i)) = (\lambda^{-1}\lambda_{f,g})^n(f(\phi_0(z_0)) - f(\phi_0(y_{i+n}))),$$

and

$$g(\phi_0(z_0)) - g(\phi_0(y_i)) = (\lambda^{-1}\lambda_{f,g})^n(g(\phi_0(z_0)) - g(\phi_0(y_{i+n}))),$$

Thus letting $n \to \infty$, we get

$$f(\phi_0(z_0)) = f(\phi_0(y_i)) \quad (i \in \mathbb{N} \cup \{0\}).$$

and

$$g(\phi_0(z_0)) = g(\phi_0(y_i)) \quad (i \in \mathbb{N} \cup \{0\}).$$

Therefore, for each $i \in \mathbb{N} \cup \{0\}$, we infer from the above-mentioned equations that

$$\Delta(f)(z_0) - \Delta(f)(y_i) = \lambda(f(\phi_0(z_0)) - f(\phi_0(y_i))) = 0,$$

and

$$\Delta(g)(z_0) - \Delta(g)(y_i) = \lambda(g(\phi_0(z_0)) - g(\phi_0(y_i))) = 0,$$

that is,

$$\Delta(f)(z_0) = \Delta(f)(y_i), \quad \Delta(g)(z_0) = \Delta(g)(y_i) \quad (i \in \mathbb{N} \cup \{0\}).$$

Taking limits when $i \to \infty$, we deduce that

$$\Delta(f)(z_0) = \Delta(f)(y_0), \quad \Delta(g)(z_0) = \Delta(g)(y_0).$$

On the other hand, notice that $f(\phi_{f,g}(y)) = f(\phi_0(y_0)) = f(\phi_0(z_0))$, and consequently

$$\begin{aligned}\Delta(f)(y) &= \lambda_{f,g} f(\phi_{f,g}(y)) + \mu_{f,g}(f)\\ &= \lambda_{f,g} f(\phi_0(z_0)) + \mu_{f,g}(f)\\ &= \lambda_{f,g} f(\phi_{f,g}(z_0)) + \mu_{f,g}(f)\\ &= \Delta(f)(z_0).\end{aligned}$$

Therefore we have

$$\Delta(f)(y) = \Delta(f)(z_0) = \Delta(f)(y_0),$$

and, similarly, we can obtain

$$\Delta(g)(y) = \Delta(g)(z_0) = \Delta(g)(y_0).$$

Now, using the equality (1) and the definition of S_y, we can write

$$\begin{aligned}\Delta(f)(y_0) &= \lambda f(\phi_0(y_0)) + \mu(f) = \lambda f(\phi_0(y_0)) + \Delta(f)(y) - \lambda S_y(f),\\ \Delta(g)(y_0) &= \lambda g(\phi_0(y_0)) + \mu(g) = \lambda g(\phi_0(y_0)) + \Delta(g)(y) - \lambda S_y(g),\end{aligned}$$

which imply

$$\begin{aligned}S_y(f) &= f(\phi_0(y_0)) + \lambda^{-1}(\Delta(f)(y) - \Delta(f)(y_0)) = f(\phi_0(y_0)),\\ S_y(g) &= g(\phi_0(y_0)) + \lambda^{-1}(\Delta(g)(y) - \Delta(g)(y_0)) = g(\phi_0(y_0)).\end{aligned}$$

Finally, we deduce the required condition:

$$S_y(f) - S_y(g) = f(\phi_0(y_0)) - g(\phi_0(y_0)) \in (f-g)(X).$$

Hence S_y is a unital multiplicative linear functional on $C(X)$. Since y was arbitrary, we conclude that $S\colon C(X) \to C(Y)$ is a unital algebra homomorphism.

Claim 12 *There exists a homeomorphism $\phi\colon Y \to X$ such that*

$$\Delta(f) = \lambda f \circ \phi + \mu(f) 1_Y \quad (f \in C(X)).$$

Let $S\colon C(X) \to C(Y)$ be the unital algebra homomorphism given in Claim 11. By Gelfand theory, S induces a continuous map $\phi\colon Y \to X$ such that $S(f) = f \circ \phi$ for all $f \in C(X)$, and thus $\Delta(f) = \lambda f \circ \phi + \mu(f)1_Y$ for all $f \in C(X)$. Now, a similar proof to that of Step 17 in [17] shows that ϕ is a homeomorphism from Y onto X.

We note that $\phi(y) = \phi_0(y)$ for all $y \in Y_0$, since by Claim 12 and the equation (1) we have $f(\phi(y)) = f(\phi_0(y))$ for all $f \in C(X)$ and $y \in Y_0$.

Claim 13 *For each $f \in C(X)$, we have $T(\pi_X(f)) = \pi_Y(\lambda f \circ \phi)$. In particular, T is linear and surjective.*

Let $f \in C(X)$. By Claim 12 and the definition of Δ, we have

$$\lambda f \circ \phi + \mu(f)1_Y = \Delta(f) = \Psi_Y^{-1}(T(\pi_X(f)), f(u_0)).$$

Hence $\Psi_Y(\lambda f \circ \phi + \mu(f)1_Y) = (T(\pi_X(f)), f(u_0))$ which implies

$$\pi_Y(\lambda f \circ \phi) = \pi_Y(\lambda f \circ \phi + \mu(f)1_Y) = T(\pi_X(f)).$$

This completes the proof of Theorem 2. □

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