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Isometric Composition Operators on Lipschitz Spaces

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Abstract. Given pointed metric spaces X and Y, we characterize the basepoint-preserving Lipschitz maps ϕ from Y to X inducing an isometric composition operator C_ϕ between the Lipschitz spaces $\mathrm{Lip}_0(X)$ and $\mathrm{Lip}_0(Y)$, whenever X enjoys the peak property. This gives an answer to a question posed by Weaver in his book [Lipschitz algebras. Second edition. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018].

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1. Introduction

Let (X, d) be a pointed metric space with a basepoint designated by e_X, and let $\widetilde{X}$ denote the set

$$\{(x, y) \in X \times X : x \neq y\}.$$

The Lipschitz space $\mathrm{Lip}_0(X)$ is the Banach space of all Lipschitz functions $f\colon X \to \mathbb{R}$ with $f(e_X) = 0$, under the Lipschitz norm:

$$\mathrm{Lip}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : (x, y) \in \widetilde{X}\right\}.$$

Throughout the paper, unless specified otherwise, X and Y will denote two pointed metric spaces. Every Lipschitz map ϕ from Y to X which preserves the basepoint produces a bounded composition operator C_ϕ from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$, defined by $C_\phi f = f \circ \phi$ for all $f \in \mathrm{Lip}_0(X)$. The map ϕ is known as the symbol of the operator C_ϕ.

The problem of characterizing those symbols ϕ which induce isometric composition operators C_ϕ (not necessarily surjective) has been raised recently by Weaver in [8, p. 53]. The same question was addressed by some authors for isometric composition operators on Banach spaces of analytic functions (see [5] and the papers that cite it).

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In the surjective case, Weaver proved in [8, Proposition 2.28 (iii)] that those basepoint-preserving Lipschitz symbols ϕ from Y to X that generate surjective isometric composition operators C_ϕ from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$ are precisely the surjective isometries from Y to X, whenever X and Y are complete.

A description of all linear isometries (surjective or not) of $\mathrm{Lip}_0(X)$ does not seem to be known. Given $\alpha \in]0,1[$, we denote by X^α the metric space (X, d^α). If X and Y are compact, the linear isometries from $\mathrm{Lip}_0(X^\alpha)$ onto $\mathrm{Lip}_0(Y^\alpha)$ were characterized by Mayer-Wolf in [6, Theorem 3.3]. He showed that a linear operator $T\colon \mathrm{Lip}_0(X^\alpha) \to \mathrm{Lip}_0(Y^\alpha)$ is a surjective isometry if and only if it is of the form

$$T(f)(y) = \lambda k^{-\alpha} \left(f(\phi(y)) - f(\phi(e_Y)) \right)$$

for all $f \in \mathrm{Lip}_0(X^\alpha)$ and $y \in Y$, where $\lambda \in \mathbb{R}$ with $|\lambda| = 1$ and $\phi\colon Y \to X$ is a bijective k-dilation with $\mathrm{diam}(X) = k \cdot \mathrm{diam}(Y)$. Given $k > 0$, a map $\phi\colon Y \to X$ is a k-dilation if $d_X(\phi(x), \phi(y)) = k \cdot d_Y(x, y)$ for all $x, y \in Y$. Mayer-Wolf's result was extended by Weaver for surjective linear isometries from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$, when X and Y are complete and uniformly concave [8, Theorem 3.56].

According to [8, Definition 3.33], a metric space X is said to be concave if

$$d(x, y) < d(x, z) + d(z, y)$$

for any triple of distinct points $x, y, z \in X$, and uniformly concave if for every distinct points $x, y \in X$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) \leq d(x, z) + d(z, y) - \delta$$

for all $z \in X$ such that $d(x, z), d(y, z) \geq \varepsilon$. The class of uniformly concave metric spaces includes any closed subset of $\mathbb{R}^n$ with the Euclidean norm or any compact subset of a strictly convex Banach space both without colinear triples, the unit sphere of any uniformly convex Banach space and Hölder spaces, among others (see [8, Section 3.5]).

Uniform concavity is closely related to the extremal structure of the closed unit ball $B_{\mathcal{F}(X)}$ of the Lipschitz free Banach space

$$\mathcal{F}(X) := \overline{\mathrm{lin}}\, \{\delta_x \colon x \in X\} \subset \mathrm{Lip}_0(X)^*,$$

where $\delta_x(f) := f(x)$ for every $x \in X$ and $f \in \mathrm{Lip}_0(X)$. Let us recall that $\mathcal{F}(X)$ is the canonical predual of $\mathrm{Lip}_0(X)$. By Theorems 3.39 in [8] and 4.1 in [1], X is uniformly concave if and only if every molecule $(\delta_x - \delta_y)/d(x, y)$ is a preserved extreme point of $B_{\mathcal{F}(X)}$.

The notion of peaking function has been a very important tool in the study of the isometric theory of Lipschitz spaces. According to [7, Definition 2.4.1], a function $f \in \mathrm{Lip}_0(X)$ with $\mathrm{Lip}(f) \leq 1$ is said to peak at $(x, y) \in \widetilde{X}$ if

$$\frac{f(x) - f(y)}{d(x, y)} = 1,$$

and for each open set $U \subset \widetilde{X}$ containing (x, y) and (y, x), there exists $\delta > 0$ such that

$$\frac{|f(z) - f(w)|}{d(z, w)} \leq 1 - \delta$$

for all $(z, w) \in \widetilde{X} \backslash U$. Colloquially, if f peaks at $(x, y) \in \widetilde{X}$, we have that $|f(z) - f(w)|/d(z, w)$ is uniformly less than 1 when (z, w) is away from (x, y) and (y, x).

We say that a pointed metric space X has the peak property if for every $(x, y) \in \widetilde{X}$, there is a function $f \in \mathrm{Lip}_0(X)$ with $\mathrm{Lip}(f) \leq 1$ that peaks at (x, y). Hölder spaces constitute a class of metric spaces with the peak property (see the proof of Proposition 2.4.5 in [7]). By [3, Theorem 5.4], X has the peak property if and only if every molecule $(\delta_x - \delta_y)/d(x, y)$ is a strongly exposed point of $B_{\mathcal{F}(X)}$.

In this note, we characterize all basepoint-preserving Lipschitz maps ϕ from Y to X whose induced composition operators C_ϕ from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$ are isometries, whenever X has the peak property. We also give a condition for ϕ to induce an isometric composition operator C_ϕ without any restriction on X.

2. Results

Let us recall that a map $\phi \colon Y \to X$ is nonexpansive if $d_X(\phi(x), \phi(y)) \leq d_Y(x, y)$ for all $x, y \in Y$. A nonexpansive map $\phi \colon Y \to X$ which preserves the basepoint can induce or not an isometric composition operator $C_\phi \colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$. For example, each k-dilation $\phi \colon Y \to X$ with $k \in]0, 1]$ is nonexpansive and if, in addition, $\phi \colon Y \to X$ is nonconstant, preserves the basepoint and has dense range, then $C_\phi \colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is an isometry if and only if $k = 1$ (that is, if ϕ is an isometry).

We first give a sufficient condition for a basepoint-preserving Lipschitz map ϕ from Y to X to be the symbol of an isometric composition operator from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$.

Theorem 2.1. *Let X and Y be pointed metric spaces and let $\phi \colon Y \to X$ be a Lipschitz map which preserves the basepoint. Assume that ϕ is nonexpansive and satisfies the property (M): for every point $(x, y) \in \widetilde{X}$, there exists a sequence $\{(x_n, y_n)\}$ in $\widetilde{Y}$ such that $\{\phi(x_n)\} \to x$, $\{\phi(y_n)\} \to y$ and*

$$\left\{ \frac{d_X(\phi(x_n), \phi(y_n))}{d_Y(x_n, y_n)} \right\} \to 1.$$

Then $C_\phi \colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is an isometry.

Proof. Since ϕ is nonexpansive, we have

$$\mathrm{Lip}(f \circ \phi) \leq \mathrm{Lip}(f)\mathrm{Lip}(\phi) \leq \mathrm{Lip}(f)$$

for every $f \in \mathrm{Lip}_0(X)$. In order to check the converse inequality, take $f \in \mathrm{Lip}_0(X)$. Hence there exists a sequence $\{(a_m, b_m)\}$ in $\widetilde{X}$ such that

$$\lim_{m\to\infty} \frac{|f(a_m) - f(b_m)|}{d_X(a_m, b_m)} = \mathrm{Lip}(f).$$

Fix $m \in \mathbb{N}$. By assumption we can find a sequence $\{(x_n^{(m)}, y_n^{(m)})\}$ in $\widetilde{Y}$ satisfying that

$$\lim_{n\to\infty} d_X(\phi(x_n^{(m)}), a_m) = 0, \quad \lim_{n\to\infty} d_X(\phi(y_n^{(m)}), b_m) = 0$$

and

$$\lim_{n\to\infty} \frac{d_X(\phi(x_n^{(m)}), \phi(y_n^{(m)}))}{d_Y(x_n^{(m)}, y_n^{(m)})} = 1.$$

It follows that $\lim_{n\to\infty} d_X(\phi(x_n^{(m)}), \phi(y_n^{(m)})) = d_X(a_m, b_m) > 0$ and, therefore, there exists $p \in \mathbb{N}$ such that $d_X(\phi(x_n^{(m)}), \phi(y_n^{(m)})) > 0$ for all $n \geq p$. We have

$$\begin{aligned}
\mathrm{Lip}(f \circ \phi) &= \sup_{x\neq y} \frac{|f(\phi(x)) - f(\phi(y))|}{d_Y(x, y)} \\
&\geq \frac{\left|f(\phi(x_n^{(m)})) - f(\phi(y_n^{(m)}))\right|}{d_Y(x_n^{(m)}, y_n^{(m)})} \\
&= \frac{\left|f(\phi(x_n^{(m)})) - f(\phi(y_n^{(m)}))\right|}{d_X(\phi(x_n^{(m)}), \phi(y_n^{(m)}))} \frac{d_X(\phi(x_n^{(m)}), \phi(y_n^{(m)}))}{d_Y(x_n^{(m)}, y_n^{(m)})} \\
&\geq \frac{d_X(\phi(x_n^{(m)}), \phi(y_n^{(m)}))}{d_Y(x_n^{(m)}, y_n^{(m)})} \left(\frac{|f(a_m) - f(b_m)|}{d_X(\phi(x_n^{(m)}), \phi(y_n^{(m)}))} \right. \\
&\quad \left. - \mathrm{Lip}(f) \frac{d_X(a_m, \phi(x_n^{(m)}))}{d_X(\phi(x_n^{(m)}), \phi(y_n^{(m)}))} - \mathrm{Lip}(f) \frac{d_X(b_m, \phi(y_n^{(m)}))}{d_X(\phi(x_n^{(m)}), \phi(y_n^{(m)}))} \right)
\end{aligned}$$

for all $n \geq p$, and taking limits as $n \to \infty$, we obtain

$$\mathrm{Lip}(f \circ \phi) \geq \frac{|f(a_m) - f(b_m)|}{d_X(a_m, b_m)}.$$

Since m was arbitrary, we conclude that

$$\mathrm{Lip}(f \circ \phi) \geq \lim_{m\to\infty} \frac{|f(a_m) - f(b_m)|}{d_X(a_m, b_m)} = \mathrm{Lip}(f).$$

□

Note that there are symbols ϕ satisfying the conditions in Theorem 2.1; for example, every basepoint-preserving isometry with dense range $\phi\colon Y \to X$.

In general, we can establish a kind of reciprocal result of Theorem 2.1.

Proposition 2.2. *Let X and Y be pointed metric spaces and let $\phi\colon Y \to X$ be a Lipschitz map which preserves the basepoint. Assume that $C_\phi\colon \mathrm{Lip}_0(X) \to$*

$\mathrm{Lip}_0(Y)$ *is an isometry. Then* ϕ *is nonexpansive and has the following additional property: for every point* $(x, y) \in \widetilde{X}$*, there exists a sequence* $\{(x_n, y_n)\}$ *in* $\widetilde{Y}$ *such that* $\{\phi(x_n)\} \to x$*,* $\{\phi(y_n)\} \to y$ *and*

$$\lim_{n\to\infty} \frac{d_X(\phi(x_n), \phi(y_n))}{d_Y(x_n, y_n)} \leq 1.$$

Proof. Clearly, $\|C_\phi\| \leq 1$, and since $\|C_\phi\| = \mathrm{Lip}(\phi)$ by [8, Proposition 2.23] (completeness of X and Y is not needed to prove this formula in [8]), it follows that ϕ is nonexpansive. In order to show that ϕ has the above-cited property, let $(x, y) \in \widetilde{X}$. Note that C_ϕ is injective and, therefore, $\phi(Y)$ is dense in X by [8, Proposition 2.25 (ii)] (completeness of X and Y is not necessary to prove this fact). Hence we can take sequences $\{x_n\}$ and $\{y_n\}$ in Y such that $\{\phi(x_n)\} \to x$ and $\{\phi(y_n)\} \to y$. It follows that $\lim_{n\to\infty} d_X(\phi(x_n), \phi(y_n)) = d_X(x, y) > 0$; hence there exists $p \in \mathbb{N}$ such that $d_X(\phi(x_n), \phi(y_n)) > 0$ for all $n \geq p$ and thus $d_Y(x_n, y_n) > 0$ for all $n \geq p$. Since $d_X(\phi(x_n), \phi(y_n))/d_Y(x_n, y_n) \leq 1$ for all $n \geq p$, taking subsequences if necessary, we obtain that

$$\lim_{n\to\infty} \frac{d_X(\phi(x_n), \phi(y_n))}{d_Y(x_n, y_n)} \leq 1.$$

□

We shall next prove that the basepoint-preserving Lipschitz maps $\phi\colon Y \to X$ for which C_ϕ is an isometry from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$, are precisely the nonexpansive maps satisfying the property (M), whenever X has the peak property.

We shall make use of the following sequential characterization of peaking functions. It appears without proof in [3] and we prove it here for completeness.

Lemma 2.3 [3]. *Let* X *be a pointed metric space,* $(x, y) \in \widetilde{X}$ *and* $f \in \mathrm{Lip}_0(X)$ *with* $\mathrm{Lip}(f) \leq 1$*. Then* f *peaks at* (x, y) *if and only if*

$$\frac{f(x) - f(y)}{d(x, y)} = 1,$$

and the following property (P) holds: if $\{(x_n, y_n)\}$ *is a sequence in* $\widetilde{X}$ *such that*

$$\left\{\frac{f(x_n) - f(y_n)}{d(x_n, y_n)}\right\} \to 1,$$

then $\{x_n\} \to x$ *and* $\{y_n\} \to y$*.*

Proof. Assume that f peaks at (x, y). Then

$$\frac{f(x) - f(y)}{d(x, y)} = 1.$$

In order to prove that f satisfies the property (P), let $\{(x_n, y_n)\}$ be a sequence in $\widetilde{X}$ such that

$$\left\{\frac{f(x_n) - f(y_n)}{d(x_n, y_n)}\right\} \to 1.$$

If the conclusion of the property (P) is not satisfied, we could find a real number $\varepsilon > 0$ and subsequences $\{x_{\sigma(n)}\}$ and $\{y_{\sigma(n)}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, satisfying that $d(x_{\sigma(n)}, x) \geq \varepsilon$ for all $n \in \mathbb{N}$ or $d(y_{\sigma(n)}, y) \geq \varepsilon$ for all $n \in \mathbb{N}$. Clearly, the set

$$\left\{n \in \mathbb{N} \colon d(x_{\sigma(n)}, y) \geq \varepsilon\right\} \cup \left\{n \in \mathbb{N} \colon d(y_{\sigma(n)}, x) \geq \varepsilon\right\}$$

is nonempty. Taking subsequences of $\{x_{\sigma(n)}\}$ and $\{y_{\sigma(n)}\}$, we can suppose that $d(x_{\sigma(n)}, y) \geq \varepsilon$ for all $n \in \mathbb{N}$ or $d(y_{\sigma(n)}, x) \geq \varepsilon$ for all $n \in \mathbb{N}$. Since f peaks at (x, y), there exists $\delta > 0$ such that

$$\frac{\left|f(x_{\sigma(n)}) - f(y_{\sigma(n)})\right|}{d(x_{\sigma(n)}, y_{\sigma(n)})} \leq 1 - \delta$$

for all $n \in \mathbb{N}$, and since

$$\left\{\frac{f(x_n) - f(y_n)}{d(x_n, y_n)}\right\} \to 1,$$

we would arrive at a contradiction. This proves that $\{x_n\} \to x$ and $\{y_n\} \to y$.

Conversely, suppose that

$$\frac{f(x) - f(y)}{d(x, y)} = 1$$

and the property (P) is satisfied, but f does not peak at (x, y). Hence there exist $\varepsilon > 0$ and a sequence $\{(x_n, y_n)\}$ in $\widetilde{X}$ satisfying that $\max\{d(x_n, x), d(y_n, y)\} \geq \varepsilon$ for all $n \in \mathbb{N}$ and $\max\{d(x_n, y), d(y_n, x)\} \geq \varepsilon$ for all $n \in \mathbb{N}$ such that

$$\frac{f(x_n) - f(y_n)}{d(x_n, y_n)} > 1 - \frac{1}{n}$$

for all $n \in \mathbb{N}$. Since $\mathrm{Lip}(f) \leq 1$, it follows that

$$\left\{\frac{f(x_n) - f(y_n)}{d(x_n, y_n)}\right\} \to 1,$$

but the sequence $\{(x_n, y_n)\}$ does not satisfy the conclusion of the property (P), a contradiction. □

We are now ready to prove our main result.

Theorem 2.4. *Let X and Y be pointed metric spaces and let $\phi \colon Y \to X$ be a basepoint-preserving Lipschitz map. Assume that X enjoys the peak property. Then $C_\phi \colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is an isometry if and only if ϕ is nonexpansive and satisfies the property (M): for every point $(x, y) \in \widetilde{X}$, there exists a sequence $\{(x_n, y_n)\}$ in $\widetilde{Y}$ such that $\{\phi(x_n)\} \to x$, $\{\phi(y_n)\} \to y$ and*

$$\left\{\frac{d_X(\phi(x_n), \phi(y_n))}{d_Y(x_n, y_n)}\right\} \to 1.$$

Proof. The sufficiency follows from Theorem 2.1. To prove the necessity, assume that $C_\phi \colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is an isometry. Then ϕ is nonexpansive by Proposition 2.2. We now show that ϕ enjoys the property (M). Let $(x, y) \in \widetilde{X}$. Since X has the peak property, there exists a function $f_{(x,y)} \in \mathrm{Lip}_0(X)$ with

$\mathrm{Lip}(f_{(x,y)}) = 1$ that peaks at (x, y). Note that $\mathrm{Lip}(f \circ \phi) = \mathrm{Lip}(f)$ for all $f \in \mathrm{Lip}_0(X)$. Since $\mathrm{Lip}(f_{(x,y)} \circ \phi) = 1$, we can take a sequence $\{(x_n, y_n)\}$ in $\widetilde{Y}$ such that

$$\left\{ \frac{\left|f_{(x,y)}(\phi(x_n)) - f_{(x,y)}(\phi(y_n))\right|}{d_Y(x_n, y_n)} \right\} \to 1.$$

Using that $\mathrm{Lip}(f_{(x,y)}) = 1$ and $\mathrm{Lip}(\phi) \leq 1$, we obtain

$$\left\{ \frac{d_X(\phi(x_n), \phi(y_n))}{d_Y(x_n, y_n)} \right\} \to 1.$$

An easy argument yields that

$$\left\{ \frac{\left|f_{(x,y)}(\phi(x_n)) - f_{(x,y)}(\phi(y_n))\right|}{d_X(\phi(x_n), \phi(y_n))} \right\} \to 1.$$

Taking subsequences, we have that

$$\left\{ \frac{f_{(x,y)}(\phi(x_{\sigma(n)}) - f_{(x,y)}(\phi(y_{\sigma(n)}))}{d_X(\phi(x_{\sigma(n)}), \phi(y_{\sigma(n)}))} \right\} \to 1,$$

or

$$\left\{ \frac{f_{(x,y)}(\phi(x_{\sigma(n)})) - f_{(x,y)}(\phi(y_{\sigma(n)}))}{d_X(\phi(x_{\sigma(n)}), \phi(y_{\sigma(n)}))} \right\} \to -1.$$

Applying Lemma 2.3, it follows that $\{\phi(x_{\sigma(n)})\} \to x$ and $\{\phi(y_{\sigma(n)})\} \to y$ in the first case, or $\{\phi(y_{\sigma(n)})\} \to x$ and $\{\phi(x_{\sigma(n)})\} \to y$ in the second one. This proves the theorem. □

Remark 2.5. Our description of isometric composition operators on $\mathrm{Lip}_0(X)$ is a Lipschitz version of a characterization of isometric composition operators on the Bloch space $\mathcal{B}$, obtained by Martín and Vukotić [5].

In view of another characterization of isometric composition operators on $\mathcal{B}$ stated by Colonna in [2, Theorem 5], it would be interesting to study under which conditions the basepoint-preserving Lipschitz self-maps ϕ of X inducing an isometric composition operator C_ϕ on $\mathrm{Lip}_0(X)$ are precisely those having Lipschitz constant equal to one.

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