TAIWANESE JOURNAL OF MATHEMATICS
Vol. 23, No. 1, pp. 129–144, February 2019
DOI: 10.11650/tjm/180508

Norm-attaining Composition Operators on Lipschitz Spaces

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Abstract. Every composition operator C_φ on the Lipschitz space $\mathrm{Lip}_0(X)$ attains its norm. This fact is essentially known and we give in this paper a sequential characterization of the extremal functions for the norm of C_φ on $\mathrm{Lip}_0(X)$. We also characterize the norm-attaining composition operators C_φ on the little Lipschitz space $\mathrm{lip}_0(X)$ which separates points uniformly and identify the extremal functions for the norm of C_φ on $\mathrm{lip}_0(X)$. We deduce that compact composition operators on $\mathrm{lip}_0(X)$ are norm-attaining whenever the sphere unit of $\mathrm{lip}_0(X)$ separates points uniformly. In particular, this condition is satisfied by spaces of little Lipschitz functions on Hölder compact metric spaces (X, d^α) with $0 < \alpha < 1$.

1. Introduction

Let (X,d) be a pointed metric space with a basepoint designated by e, let $\widetilde{X}$ denote the set

$$\{(x,y) \in X \times X : x \neq y\},$$

and let $\mathbb{K}$ be the field of real or complex numbers. The *Lipschitz space* $\mathrm{Lip}_0(X)$ is the Banach space of all Lipschitz functions $f\colon X \to \mathbb{K}$ for which $f(e) = 0$, endowed with the *Lipschitz norm* defined by

$$\mathrm{Lip}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : (x,y) \in \widetilde{X} \right\},$$

and the *little Lipschitz space* $\mathrm{lip}_0(X)$ is the closed subspace of $\mathrm{Lip}_0(X)$ of all functions f such that

$$\lim_{t \to 0} \sup \left\{ \frac{|f(x) - f(y)|}{d(x,y)} : (x,y) \in \widetilde{X}, d(x,y) < t \right\} = 0.$$

There exist metric spaces X for which $\mathrm{lip}_0(X) = \{0\}$ as, for instance, $X = [0,1]$ with the usual metric. In contrast, we can consider metric spaces X such that $\mathrm{lip}_0(X)$ *separates*

Received January 23, 2018; Accepted May 28, 2018.
Communicated by Xiang Fang.
2010 *Mathematics Subject Classification.* 47B33, 47B38.
Key words and phrases. composition operator, compact operator, norm-attaining operator, Lipschitz function, little Lipschitz function.
This paper was partially supported by the Spanish Ministry of Economy and Competitiveness project no. MTM2014-58984-P and the European Regional Development Fund (ERDF), and Junta of Andalucía grant FQM-194.

points uniformly in the sense that there exists a constant $a > 1$ such that, for every $x, y \in X$, some $f \in \mathrm{lip}_0(X)$ satisfies $\mathrm{Lip}(f) \leq a$ and $|f(x) - f(y)| = d(x, y)$. For each compact pointed metric space (X, d), the space $\mathrm{lip}_0(X^\alpha)$ enjoys this property being $X^\alpha = (X, d^\alpha)$ and $\alpha \in (0, 1)$. We refer to the Weaver's book [21] for a complete study on those Lipschitz spaces.

Let X be a set and let $\mathcal{F}(X)$ be a linear space of functions from X into $\mathbb{K}$. Given a map $\varphi\colon X \to X$, let us recall that a *composition operator* C_φ on $\mathcal{F}(X)$ is a linear operator from $\mathcal{F}(X)$ into itself defined by $C_\varphi f = f \circ \varphi$. The map φ is called the *symbol* of C_φ. It is said that C_φ *attains its norm* on $\mathcal{F}(X)$ if there exists a function $f \in \mathcal{F}(X)$ with norm one such that $\|C_\varphi\| = \|C_\varphi f\|$. Such a function f is called an *extremal function* for the norm of C_φ. It is an application of a James' theorem (see [4, Chapter One, Theorem 6]) that a Banach space E is reflexive if and only if any compact linear operator on E attains its norm.

Norm-attaining composition operators have been studied for different function spaces by several authors as, for example, the Hardy space and the Dirichlet space by Hammond [8, 9], Bloch spaces by Martín [16] and Montes-Rodríguez [18], and weighted Bloch spaces by Bonet, Lindström and Wolf [1].

We address the question as to when composition operators C_φ acting on the Lipschitz space $\mathrm{Lip}_0(X)$ as well as on the little Lipschitz space $\mathrm{lip}_0(X)$ satisfying the uniform separation property attain their norms and characterize the extremal functions for the norm of C_φ on such spaces.

Composition operators on Lipschitz spaces have been considered by different authors. Assuming that X is a compact metric space and φ is a Lipschitz map of X into X, Kamowitz and Scheinberg [15] proved that a composition operator C_φ is compact on the spaces of bounded Lipschitz functions $\mathrm{Lip}(X)$ and $\mathrm{lip}(X^\alpha)$ with the norm $\|\cdot\|_\infty + \mathrm{Lip}(\cdot)$ if and only if φ is supercontractive. This result was extended in [12] to composition operators on $\mathrm{Lip}_0(X)$ when X is a bounded pointed metric space. Chen, Li, R. Wang and Y.-S. Wang [3] characterized compact weighted composition operators between spaces of scalar-valued Lipschitz functions. Botelho and Jamison [2], Esmaeili and Mahyar [5], and Golbaharan and Mahyar [6, 7] tackled weighted composition operators between spaces of vector-valued Lipschitz functions. When φ is a Lipschitz map from X into X which preserves basepoint (such a map is called a *basepoint-preserving Lipschitz self-map of* X), the proof of Proposition 1.8.2 in [21] reveals that the composition operator C_φ on $\mathrm{Lip}_0(X)$ attains its norm at an explicit extremal function. Apparently, this result of Weaver is one of the few known results concerning norm-attaining composition operators on those Lipschitz spaces.

We now describe the contents of this paper. In Section 2, we characterize the self-maps

φ of X inducing a nonzero bounded composition operator C_φ on the space $\mathrm{Lip}_0(X)$ and the space $\mathrm{lip}_0(X)$ which satisfies the uniform separation property. Specifically, we show that such maps are nonconstant basepoint-preserving Lipschitz.

We recall in Section 3 that every nonzero bounded composition operator C_φ on $\mathrm{Lip}_0(X)$ attains its norm and give a sequential characterization of the extremal functions for the norm $\|C_\varphi\|$.

When the space $\mathrm{lip}_0(X)$ separates points uniformly, we will give a complete description of norm-attaining composition operators C_φ on $\mathrm{lip}_0(X)$ in Theorem 4.2. This characterization involves the existence of a point (x_0, y_0) in $\widetilde{X}$ and an extremal function for $\|C_\varphi\|$ that separates the points x_0 and y_0 to their full distance. This fact motivates the following concept. It is said that *the unit sphere of* $\mathrm{lip}_0(X)$ *separates points uniformly* if for every $x, y \in X$, there exists a function $f \in \mathrm{lip}_0(X)$ with $\mathrm{Lip}(f) = 1$ such that $|f(x) - f(y)| = d(x, y)$. We know two different kinds of metric spaces X enjoying this property: when X is uniformly discrete or when X is a Hölder compact metric space. Besides, we will introduce a more constructively defined class of compact metric spaces for which the unit sphere of $\mathrm{lip}_0(X)$ has the uniform separation property. For norm-attaining composition operators on such spaces, we will improve Theorem 4.2 with a sequential characterization which will be now free of extremal functions.

The final part of the paper deals with compact composition operators on spaces $\mathrm{lip}_0(X)$ whose unit spheres separate points uniformly. We will state that every composition operator C_φ on such spaces for which the essential norm of C_φ multiplied by $\sqrt{2}$ is strictly less than the norm of C_φ attains its norm. To prove this fact, we will need a characterization of the weak convergence of sequences in $\mathrm{lip}_0(X)$ and a lower estimate for the essential norm of C_φ on $\mathrm{lip}_0(X)$. As a consequence, we will deduce that compact composition operators on $\mathrm{lip}_0(X)$ are norm-attaining. It is worth noting that infinite dimensional spaces $\mathrm{lip}_0(X)$ and $\mathrm{Lip}_0(X)$ are not reflexive (see [14, Theorem 6.6] and [21, Corollary 2.5.5]).

2. Nonzero bounded composition operators on Lipschitz spaces

In this section, we characterize the class of all functions φ mapping X into itself whose induced composition operator C_φ is a nonzero bounded operator on the Lipschitz space $\mathrm{Lip}_0(X)$ and the little Lipschitz space $\mathrm{lip}_0(X)$ that satisfies the uniform separation property.

Theorem 2.1. *Let X be a pointed metric space and let φ be a self-map of X. Then the composition operator C_φ is a bounded operator from* $\mathrm{Lip}_0(X)$ *into* $\mathrm{Lip}_0(X)$ *if and only if φ is Lipschitz and preserves basepoint. Besides,* $C_\varphi\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(X)$ *is nonzero if and only if φ is nonconstant.*

Proof. Assume that $C_\varphi\colon \operatorname{Lip}_0(X) \to \operatorname{Lip}_0(X)$ is bounded. If $\varphi\colon X \to X$ were not Lipschitz, there would exist a sequence $\{(x_n, y_n)\}$ in $\widetilde{X}$ satisfying $d(\varphi(x_n), \varphi(y_n))/d(x_n, y_n) \geq n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define the functions $g_n, f_n\colon X \to \mathbb{R}$ by

$$g_n(x) = \frac{d(x, \varphi(x_n)) - d(x, \varphi(y_n))}{2},$$
$$f_n(x) = g_n(x) - g_n(e).$$

Clearly, f_n belongs to $\operatorname{Lip}_0(X)$ with $\operatorname{Lip}(f_n) = 1$ and satisfies $|f_n(\varphi(x_n)) - f_n(\varphi(y_n))| = d(\varphi(x_n), \varphi(y_n))$. Hence we have

$$\operatorname{Lip}(f_n \circ \varphi) \geq \frac{|f_n(\varphi(x_n)) - f_n(\varphi(y_n))|}{d(x_n, y_n)} = \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} \geq n$$

for all $n \in \mathbb{N}$, and thus the sequence $\{C_\varphi f_n\}$ is not bounded in $\operatorname{Lip}_0(X)$. This contradicts that $C_\varphi\colon \operatorname{Lip}_0(X) \to \operatorname{Lip}_0(X)$ is bounded, and proves that φ is Lipschitz. On the other hand, since C_φ maps $\operatorname{Lip}_0(X)$ into itself, we have that $C_\varphi f(e) = 0$ for all $f \in \operatorname{Lip}_0(X)$, that is, $f(\varphi(e)) = f(e)$ for all $f \in \operatorname{Lip}_0(X)$ which implies that $\varphi(e) = e$ because $\operatorname{Lip}_0(X)$ separates the points of X.

Conversely, suppose that φ is Lipschitz and $\varphi(e) = e$. For every $f \in \operatorname{Lip}_0(X)$, we have $f(\varphi(e)) = f(e) = 0$ and $f \circ \varphi$ is Lipschitz with $\operatorname{Lip}(f \circ \varphi) \leq \operatorname{Lip}(f)\operatorname{Lip}(\varphi)$. Hence C_φ maps $\operatorname{Lip}_0(X)$ into $\operatorname{Lip}_0(X)$. In order to see that C_φ is bounded, we use the closed graph theorem. Let $\{f_n\}$ be a sequence in $\operatorname{Lip}_0(X)$ such that $\operatorname{Lip}(f_n) \to 0$ as $n \to \infty$, and assume that $\operatorname{Lip}((f_n \circ \varphi) - g) \to 0$ as $n \to \infty$ for some function $g \in \operatorname{Lip}_0(X)$. Observe that, for any function $f \in \operatorname{Lip}_0(X)$, it holds that $|f(x)| \leq \operatorname{Lip}(f)d(x, e)$ for all $x \in X$. Using this inequality, we can deduce that, for each $x \in X$, the sequence $\{f_n(\varphi(x))\}$ converges to 0 and also to $g(x)$ as $n \to \infty$, and so $g(x) = 0$. This gives $g = 0$. Hence $C_\varphi\colon \operatorname{Lip}_0(X) \to \operatorname{Lip}_0(X)$ is bounded.

We now prove the second assertion. Assume that φ is constant. Then $\varphi(x) = \varphi(e) = e$ for all $x \in X$. Hence $C_\varphi f(x) = f(\varphi(x)) = f(e) = 0$ for each $f \in \operatorname{Lip}_0(X)$ and all $x \in X$, and therefore $C_\varphi = 0$. Conversely, suppose that φ is not constant. This implies that $X \setminus \{e\} \neq \emptyset$ and we can take a point $x \in X \setminus \{e\}$ such that $\varphi(x) \neq \varphi(e) = e$. Since $\operatorname{Lip}_0(X)$ separates the points of X, some $f \in \operatorname{Lip}_0(X)$ satisfies that $f(\varphi(x)) \neq f(e) = 0$ and thus C_φ is nonzero. □

As we have commented above, $\operatorname{lip}_0(X)$ has especial interest when it separates points uniformly. So we avoid the cases in which $\operatorname{lip}_0(X)$ reduces to the zero function.

Theorem 2.2. *Let X be a compact pointed metric space and let φ be a self-map of X. Assume that $\operatorname{lip}_0(X)$ separates points uniformly. Then C_φ is a bounded operator from $\operatorname{lip}_0(X)$ into $\operatorname{lip}_0(X)$ if and only if φ is Lipschitz and preserves basepoint. Besides, $C_\varphi\colon \operatorname{lip}_0(X) \to \operatorname{lip}_0(X)$ is nonzero if and only if φ is nonconstant.*

Proof. Suppose that $C_\varphi\colon \operatorname{lip}_0(X) \to \operatorname{lip}_0(X)$ is bounded. Using that $\operatorname{lip}_0(X)$ separates points uniformly, similar arguments to those above in Theorem 2.1 show that φ is Lipschitz and preserves basepoint.

For the converse implication, assume that φ is Lipschitz and $\varphi(e) = e$. We first show that $f \circ \varphi \in \operatorname{lip}_0(X)$ for all $f \in \operatorname{lip}_0(X)$. Note that $f \circ \varphi \in \operatorname{Lip}_0(X)$ as in the proof of Theorem 2.1. Besides, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$(x, y) \in \widetilde{X},\ d(x, y) < \delta \quad \Longrightarrow \quad \frac{|f(x) - f(y)|}{d(x, y)} < \frac{\varepsilon}{1 + \operatorname{Lip}(\varphi)}.$$

Let $(x, y) \in \widetilde{X}$ be with $d(x, y) < \delta/(1+\operatorname{Lip}(\varphi))$. If $\varphi(x) \neq \varphi(y)$, we have $0 < d(\varphi(x), \varphi(y)) < \delta$ and hence

$$\frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x, y)} = \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(\varphi(x), \varphi(y))} \frac{d(\varphi(x), \varphi(y))}{d(x, y)} < \frac{\varepsilon}{1 + \operatorname{Lip}(\varphi)} \operatorname{Lip}(\varphi) < \varepsilon.$$

Thus $f \circ \varphi \in \operatorname{lip}_0(X)$. Hence C_φ maps $\operatorname{lip}_0(X)$ into $\operatorname{lip}_0(X)$ and, using the closed graph theorem as in the proof of Theorem 2.1, we show that $C_\varphi\colon \operatorname{lip}_0(X) \to \operatorname{lip}_0(X)$ is bounded. The second equivalence is proved similarly as in Theorem 2.1. □

3. Norm-attaining composition operators on $\operatorname{Lip}_0(X)$

We recall in this section that every nonzero bounded composition operator C_φ on $\operatorname{Lip}_0(X)$ attains its norm and give a sequential characterization of the extremal functions for $\|C_\varphi\|$.

Theorem 3.1. [21, Proposition 1.8.2] *Let X be a pointed metric space and let $\varphi\colon X \to X$ be a nonconstant basepoint-preserving Lipschitz map. Then the norm of the composition operator $C_\varphi\colon \operatorname{Lip}_0(X) \to \operatorname{Lip}_0(X)$ is given by the formula*

$$\|C_\varphi\| = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))}{d(x, y)}.$$

Furthermore, the operator C_φ on $\operatorname{Lip}_0(X)$ is norm-attaining and, for each $y \in X$, an extremal function for $\|C_\varphi\|$ is the function $f_y\colon X \to \mathbb{R}$, defined by $f_y(z) = d(z, \varphi(y)) - d(e, \varphi(y))$ for all $z \in X$.

Proof. For any $f \in \operatorname{Lip}_0(X)$ with $\operatorname{Lip}(f) = 1$, we have

$$\operatorname{Lip}(C_\varphi f) = \operatorname{Lip}(f \circ \varphi) \leq \operatorname{Lip}(f) \operatorname{Lip}(\varphi) = \operatorname{Lip}(\varphi),$$

and therefore $\|C_\varphi\| \leq \operatorname{Lip}(\varphi)$. Now, for each point $y \in X$, define $f_y\colon X \to \mathbb{R}$ as in the statement. It is easy to see that $f_y \in \operatorname{Lip}_0(X)$ with $\operatorname{Lip}(f_y) = 1$. We obtain

$$\operatorname{Lip}(\varphi) = \sup_{x \neq y} \frac{d(\varphi(x), \varphi(y))}{d(x, y)} = \sup_{x \neq y} \frac{|f_y(\varphi(x)) - f_y(\varphi(y))|}{d(x, y)} = \operatorname{Lip}(C_\varphi f_y) \leq \|C_\varphi\|,$$

and this completes the proof. □

Theorem 3.2. *Let X be a pointed metric space and let $\varphi\colon X \to X$ be a nonconstant basepoint-preserving Lipschitz map. Then a function f in $\mathrm{Lip}_0(X)$ with $\mathrm{Lip}(f) = 1$ is extremal for the norm of the operator C_φ on $\mathrm{Lip}_0(X)$ if and only if there exists a sequence $\{(\varphi(x_n), \varphi(y_n))\}$ in $\widetilde{X}$ such that*

$$\lim_{n\to\infty} \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} = \|C_\varphi\| \quad \textit{and} \quad \lim_{n\to\infty} \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(\varphi(x_n), \varphi(y_n))} = 1.$$

Proof. Suppose that $f \in \mathrm{Lip}_0(X)$ with $\mathrm{Lip}(f) = 1$ is extremal for $\|C_\varphi\|$. Then

$$\|C_\varphi\| = \mathrm{Lip}(f \circ \varphi) = \sup_{x \neq y} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x, y)}.$$

Hence, for each $n \in \mathbb{N}$, we can take a point $(x_n, y_n) \in \widetilde{X}$ such that

$$\left(1 - \frac{1}{n}\right) \|C_\varphi\| < \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(x_n, y_n)} \leq \|C_\varphi\|.$$

By using Theorem 3.1, it follows that

$$\left(1 - \frac{1}{n}\right) \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} < \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(x_n, y_n)}.$$

This implies that $\{(\varphi(x_n), \varphi(y_n))\}$ is a sequence in $\widetilde{X}$, and

$$1 - \frac{1}{n} < \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(\varphi(x_n), \varphi(y_n))} \leq \mathrm{Lip}(f) = 1$$

for all $n \in \mathbb{N}$, and therefore

$$\lim_{n\to\infty} \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(\varphi(x_n), \varphi(y_n))} = 1,$$

as required. By the Bolzano-Weierstrass theorem, taking a subsequence if necessary, we can suppose that the sequence $\{d(\varphi(x_n), \varphi(y_n))/d(x_n, y_n)\}$ converges. By the inequality above for $\|C_\varphi\|$, we get that

$$\|C_\varphi\| = \lim_{n\to\infty} \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(x_n, y_n)},$$

and from this we conclude that

$$\|C_\varphi\| = \lim_{n\to\infty} \left[\frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(\varphi(x_n), \varphi(y_n))} \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} \right] = \lim_{n\to\infty} \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)}.$$

Conversely, let f be a function in $\mathrm{Lip}_0(X)$ with $\mathrm{Lip}(f) = 1$ and suppose that there exists a sequence $\{(\varphi(x_n), \varphi(y_n))\}$ in $\widetilde{X}$ such that both conditions

$$\lim_{n\to\infty} \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} = \|C_\varphi\| \quad \text{and} \quad \lim_{n\to\infty} \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(\varphi(x_n), \varphi(y_n))} = 1$$

are satisfied. This last limit shows that $f(\varphi(x_n)) \neq f(\varphi(y_n))$ for all $n \geq m$ and some $m \in \mathbb{N}$, and thus

$$\lim_{n\ to\infty} \frac{d(\varphi(x_n), \varphi(y_n))}{|f(\varphi(x_n)) - f(\varphi(y_n))|} = 1.$$

By the Bolzano-Weierstrass theorem, we can assume that the sequence

$$\left\{ \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(x_n, y_n)} \right\}$$

converges by taking a subsequence if necessary. We now can obtain

$$\begin{aligned}
\|C_\varphi\| &= \lim_{n\to\infty} \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} \\
&= \lim_{n\to\infty} \left[\frac{d(\varphi(x_n), \varphi(y_n))}{|f(\varphi(x_n)) - f(\varphi(y_n))|} \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(x_n, y_n)} \right] \\
&= \lim_{n\to\infty} \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(x_n, y_n)} \\
&\leq \sup_{x\neq y} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x, y)} = \operatorname{Lip}(f \circ \varphi) \leq \|C_\varphi\|,
\end{aligned}$$

and this says us that f is an extremal function for $\|C_\varphi\|$, as desired. □

4. Norm-attaining composition operators on $\operatorname{lip}_0(X)$

Our first aim in this section is to characterize norm-attaining composition operators on $\operatorname{lip}_0(X)$ whenever these spaces separate points uniformly.

We will need a formula for the norm of the operator C_φ on $\operatorname{lip}_0(X)$, similar to that of Theorem 3.1 when C_φ is defined on $\operatorname{Lip}_0(X)$.

Theorem 4.1. *Let X be a compact pointed metric space and let $\varphi\colon X \to X$ be a non-constant basepoint-preserving Lipschitz map. Assume* $\operatorname{lip}_0(X)$ *separates points uniformly. Then the norm of the composition operator $C_\varphi\colon$* $\operatorname{lip}_0(X) \to \operatorname{lip}_0(X)$ *is given by*

$$\|C_\varphi\| = \sup_{x\neq y} \frac{d(\varphi(x), \varphi(y))}{d(x, y)}.$$

Proof. We obtain that

$$\|C_\varphi\| \leq \sup_{x\neq y} \frac{d(\varphi(x), \varphi(y))}{d(x, y)},$$

as in the proof of Theorem 3.1. Conversely, according to [21, Corollary 3.3.5], for every $a > 1$ and every $(x, y) \in \widetilde{X}$, some $f \in \operatorname{lip}_0(X)$ satisfies $\operatorname{Lip}(f) \leq a$ and $|f(\varphi(x)) - f(\varphi(y))| = d(\varphi(x), \varphi(y))$. We have

$$\frac{d(\varphi(x), \varphi(y))}{d(x, y)} = \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x, y)} \leq \operatorname{Lip}(C_\varphi f) \leq \|C_\varphi\| \operatorname{Lip}(f) \leq \|C_\varphi\| a.$$

Taking supremum over x and y, it follows that

$$\sup_{x\neq y}\frac{d(\varphi(x),\varphi(y))}{d(x,y)}\leq \|C_\varphi\|a.$$

Since $a>1$ was arbitrary, we conclude that

$$\sup_{x\neq y}\frac{d(\varphi(x),\varphi(y))}{d(x,y)}\leq \|C_\varphi\|. \qquad \square$$

We now characterize norm-attaining composition operators C_φ on $\mathrm{lip}_0(X)$.

Theorem 4.2. *Let X be a compact pointed metric space and let $\varphi\colon X\to X$ be a nonconstant basepoint-preserving Lipschitz map. Assume that $\mathrm{lip}_0(X)$ separates points uniformly. Then a composition operator $C_\varphi\colon \mathrm{lip}_0(X)\to\mathrm{lip}_0(X)$ is norm-attaining if and only if there exist a point $(x_0,y_0)\in\widetilde{X}$, a sequence $\{(\varphi(x_n),\varphi(y_n))\}$ in $\widetilde{X}$ with $\lim_{n\to\infty}\varphi(x_n)=x_0$ and $\lim_{n\to\infty}\varphi(y_n)=y_0$ such that*

$$\|C_\varphi\|=\lim_{n\to\infty}\frac{d(\varphi(x_n),\varphi(y_n))}{d(x_n,y_n)},$$

and a function f in $\mathrm{lip}_0(X)$ with $\mathrm{Lip}(f)=1$ such that $|f(x_0)-f(y_0)|=d(x_0,y_0)$. In this case, f is an extremal function for $\|C_\varphi\|$.

Proof. Assume that C_φ is a norm-attaining composition operator on $\mathrm{lip}_0(X)$. Then there exists a function $f\in\mathrm{lip}_0(X)$ with $\mathrm{Lip}(f)=1$ such that $\|C_\varphi\|=\mathrm{Lip}(f\circ\varphi)$, that is, f is an extremal function for $\|C_\varphi\|$. Since $f\in\mathrm{lip}_0(X)$, there exists $\delta>0$ such that

$$\frac{|f(x)-f(y)|}{d(x,y)}<\frac{1}{2}$$

whenever $0<d(x,y)<\delta$. If $(x,y)\in X\times X$ with $0<d(\varphi(x),\varphi(y))<\delta$, then

$$\begin{aligned}\frac{|f(\varphi(x))-f(\varphi(y))|}{d(x,y)}&=\frac{|f(\varphi(x))-f(\varphi(y))|}{d(\varphi(x),\varphi(y))}\frac{d(\varphi(x),\varphi(y))}{d(x,y)}\\&<\frac{1}{2}\frac{d(\varphi(x),\varphi(y))}{d(x,y)}\\&\leq\frac{1}{2}\,\mathrm{Lip}(\varphi)=\frac{1}{2}\|C_\varphi\|,\end{aligned}$$

where we have used Theorem 4.1. Let $\widetilde{X}_\delta=\{(x,y)\in X\times X:\delta\leq d(\varphi(x),\varphi(y))\}$. We have

$$\|C_\varphi\|=\mathrm{Lip}(f\circ\varphi)=\sup_{(x,y)\in\widetilde{X}_\delta}\frac{|f(\varphi(x))-f(\varphi(y))|}{d(x,y)}.$$

By that condition of supremum of $\|C_\varphi\|$, for each $n\in\mathbb{N}$ we can find a point $(x_n,y_n)\in\widetilde{X}_\delta$ such that

$$\left(1-\frac{1}{n}\right)\|C_\varphi\|<\frac{|f(\varphi(x_n))-f(\varphi(y_n))|}{d(x_n,y_n)}.$$

Since X is compact, taking subsequences if necessary, we can suppose that $\{x_n\}$ and $\{y_n\}$ converge to points a and b in X, respectively. Put $x_0 = \varphi(a)$ and $y_0 = \varphi(b)$. Clearly, $(x_0, y_0) \in \widetilde{X}_\delta$ and thus $(x_0, y_0) \in \widetilde{X}$. Besides, $\{\varphi(x_n)\}$ and $\{\varphi(y_n)\}$ converge to x_0 and y_0 in X, respectively. Since

$$\begin{aligned}\left(1 - \frac{1}{n}\right)\|C_\varphi\| &< \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(x_n, y_n)} \\ &= \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(\varphi(x_n), \varphi(y_n))}\frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} \\ &\leq \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} \leq \|C_\varphi\|\end{aligned}$$

for all $n \in \mathbb{N}$, taking limits as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} = \|C_\varphi\|,$$

and since we can assume, taking a subsequence if necessary, that the sequence

$$\left\{\frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(\varphi(x_n), \varphi(y_n))}\right\}$$

converges by the Bolzano-Weierstrass theorem, we infer that

$$\frac{|f(x_0) - f(y_0)|}{d(x_0, y_0)} = \lim_{n\to\infty} \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(\varphi(x_n), \varphi(y_n))} = 1.$$

This completes the proof of an implication.

Conversely, suppose that there exist a point (x_0, y_0) in $\widetilde{X}$, a sequence $\{(\varphi(x_n), \varphi(y_n))\}$ in $\widetilde{X}$ and a function f in $\mathrm{lip}_0(X)$ satisfying the hypotheses of the theorem. By the Bolzano-Weierstrass theorem, we can assume that the sequence

$$\left\{\frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(x_n, y_n)}\right\}$$

converges by taking a subsequence if necessary. Note that

$$\lim_{n\to\infty} \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(\varphi(x_n), \varphi(y_n))} = \frac{|f(x_0) - f(y_0)|}{d(x_0, y_0)} = 1$$

and therefore $f(\varphi(x_n)) \neq f(\varphi(y_n))$ for all $n \geq m$ and some $m \in \mathbb{N}$. We have

$$\begin{aligned}\|C_\varphi\| &= \lim_{n\to\infty} \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)} \\ &= \lim_{n\to\infty} \left[\frac{d(\varphi(x_n), \varphi(y_n))}{d(x_0, y_0)}\frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(x_n, y_n)}\frac{|f(x_0) - f(y_0)|}{|f(\varphi(x_n)) - f(\varphi(y_n))|}\right] \\ &= \lim_{n\to\infty} \frac{|f(\varphi(x_n)) - f(\varphi(y_n))|}{d(x_n, y_n)} \\ &\leq \sup_{x\neq y} \frac{|f(\varphi(x)) - f(\varphi(y))|}{d(x, y)} = \mathrm{Lip}(f \circ \varphi) \leq \|C_\varphi\|.\end{aligned}$$

Hence f is an extremal function for $\|C_\varphi\|$ and this completes the proof. □

A condition in Theorem 4.2 justifies the introduction of the following property.

Definition 4.3. Let X be a pointed metric space. It is said that the unit sphere of $\mathrm{lip}_0(X)$ separates points uniformly if for every $x, y \in X$, there exists a function $f \in \mathrm{lip}_0(X)$ with $\mathrm{Lip}(f) = 1$ such that $|f(x) - f(y)| = d(x, y)$.

We now discuss some examples of spaces $\mathrm{lip}_0(X)$ whose unit spheres separate points uniformly. Note that $\mathrm{lip}_0(X)$ enjoys that property when X is uniformly discrete meaning that $\inf\{d(x, y) : x \neq y\} > 0$ because, in this case, we have $\mathrm{lip}_0(X) = \mathrm{Lip}_0(X)$ by [13, Lemma 2.5] and, for each $y \in X$, the function $z \mapsto d(z, y) - d(e, y)$ from X into $\mathbb{R}$ satisfies the required conditions in Definition 4.3. On the other hand, if (X, d) is a compact pointed metric space and α is a scalar in $(0, 1)$, then $\mathrm{lip}_0(X^\alpha)$ has the aforementioned property (see, for example, [17, p. 62]).

In order to provide more examples, we appeal to [10, Definition 2] and denote by Ω the set of increasing functions $\omega \colon [0, \infty) \to [0, \infty)$ such that $\omega(0) = 0$, $\lim_{t\to 0} \omega(t) = 0$, $\lim_{t\to 0} \omega(t)/t = +\infty$ and the function $\omega(t)/t$ is decreasing for $t > 0$. Some important elements of Ω are $\omega(t) = t^\alpha$ with $\alpha \in (0, 1)$. Each element in Ω permits to replace the metric d on X with a new metric $\omega \circ d$ and we can consider so the space $\mathrm{Lip}_0(X, \omega \circ d)$. In the case $\omega(t) = t^\alpha$, we would obtain the space $\mathrm{Lip}_0(X^\alpha)$.

Proposition 4.4. *If (X, d) is a compact pointed metric space and $\omega \in \Omega$, then the unit sphere of $\mathrm{lip}_0(X, \omega \circ d)$ separates points uniformly.*

Proof. Fix two points $x, y \in X$ with $x \neq y$ and define the functions $h_{x,y}, g_{x,y}, f_{x,y} \colon X \to \mathbb{R}$ by

$$\begin{aligned} h_{x,y}(z) &= \max\{d(x, y) - d(x, z), 0\}, \\ g_{x,y}(z) &= \frac{\omega(d(x, y))}{d(x, y)} h_{x,y}(z), \\ f_{x,y}(z) &= g_{x,y}(z) - g_{x,y}(e) \end{aligned}$$

for all $z \in X$. An easy computation gives

$$\frac{|f_{x,y}(z) - f_{x,y}(u)|}{\omega(d(z, u))} \leq \frac{\omega(d(x, y))}{d(x, y)} \frac{\min\{d(z, u), d(x, y)\}}{\omega(d(z, u))}$$

for all $z, u \in X$ with $z \neq u$. Hence $f_{x,y} \in \mathrm{Lip}_0(X)$ with $|f_{x,y}(x) - f_{x,y}(y)| = \omega(d(x, y))$. If $d(z, u) \leq d(x, y)$, we have

$$\frac{|f_{x,y}(z) - f_{x,y}(u)|}{\omega(d(z, u))} \leq \frac{\omega(d(x, y))}{d(x, y)} \frac{d(z, u)}{\omega(d(z, u))} \leq 1$$

because $t \mapsto \omega(t)/t$ $(t > 0)$ is decreasing; and if $d(z, u) > d(x, y)$, we also have

$$\frac{|f_{x,y}(z) - f_{x,y}(u)|}{\omega(d(z, u))} \leq \frac{\omega(d(x, y))}{d(x, y)} \frac{d(x, y)}{\omega(d(z, u))} \leq 1$$

because ω is increasing. So we have proved that $f_{x,y} \in \mathrm{Lip}_0(X, \omega \circ d)$ with

$$\mathrm{Lip}(f_{x,y}, \omega \circ d) = \frac{|f_{x,y}(x) - f_{x,y}(y)|}{\omega(d(x,y))} = 1.$$

We next show that $\mathrm{Lip}_0(X)$ is contained in $\mathrm{lip}_0(X, \omega \circ d)$. Indeed, let $f \in \mathrm{Lip}_0(X)$. Then $f \in \mathrm{Lip}_0(X, \omega \circ d)$ also because

$$\frac{|f(z) - f(u)|}{\omega(d(z,u))} = \frac{|f(z) - f(u)|}{d(z,u)} \frac{d(z,u)}{\omega(d(z,u))} \leq \mathrm{Lip}(f) \frac{1 + \mathrm{diam}(X)}{\omega(1 + \mathrm{diam}(X))}$$

for all $z, u \in X$ with $z \neq u$. Moreover, given $\varepsilon > 0$, we can find $\delta > 0$ such that $t/\omega(t) < \varepsilon/(1 + \mathrm{Lip}(f))$ whenever $0 < t < \delta$. Then $0 < d(z,u) < \delta$ implies

$$\frac{|f(z) - f(u)|}{\omega(d(z,u))} \leq \mathrm{Lip}(f) \frac{d(z,u)}{\omega(d(z,u))} < \varepsilon,$$

and thus $f \in \mathrm{lip}_0(X, \omega \circ d)$. Therefore $f_{x,y}$ satisfies the conditions of Definition 4.3 and this proves the proposition. □

For spaces $\mathrm{lip}_0(X)$ whose unit spheres separate points uniformly, we next derive from Theorem 4.2 a characterization for norm-attaining composition operators on $\mathrm{lip}_0(X)$ which is now free of extremal functions.

Corollary 4.5. *Let X be a compact pointed metric space and let $\varphi\colon X \to X$ be a nonconstant basepoint-preserving Lipschitz map. Assume that the unit sphere of* $\mathrm{lip}_0(X)$ *separates points uniformly. Then a composition operator $C_\varphi\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X)$ is norm-attaining if and only if there exist a point $(x_0, y_0) \in \widetilde{X}$, a sequence $\{(\varphi(x_n), \varphi(y_n))\}$ in $\widetilde{X}$ with* $\lim_{n\to\infty} \varphi(x_n) = x_0$ *and* $\lim_{n\to\infty} \varphi(y_n) = y_0$ *such that*

$$\|C_\varphi\| = \lim_{n\to\infty} \frac{d(\varphi(x_n), \varphi(y_n))}{d(x_n, y_n)}.$$

Furthermore, if $C_\varphi\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X)$ is norm-attaining, then any function f in $\mathrm{lip}_0(X)$ *with* $\mathrm{Lip}(f) = 1$ *satisfying that $|f(x_0) - f(y_0)| = d(x_0, y_0)$, is extremal for $\|C_\varphi\|$.*

Our following goal is to show that compact composition operators on spaces $\mathrm{lip}_0(X)$ whose unit spheres separate points uniformly are norm-attaining. Really, we will deduce this fact from a much more general result that involves the concept of *essential norm* $\|T\|_e$ of a bounded operator $T\colon X \to Y$ between Banach spaces defined by

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is a compact operator from } X \text{ to } Y\}.$$

We prepare its proof with two lemmas whose proofs use the same methods applied in the proofs of Lemma 3.2 and Theorem 3.1 in [11] for the special case of spaces $\mathrm{lip}_0(X^\alpha)$ with $0 < \alpha < 1$.

The first one is a characterization of the weak convergence of sequences in $\mathrm{lip}_0(X)$ which is an easy consequence of the uniform boundedness principle and Rainwater's theorem [19, p. 33]. For the application of this last theorem, we use [21, Corollary 3.3.6] which describes the extreme points of the unit ball of the dual space $\mathrm{lip}_0(X)^*$ when $\mathrm{lip}_0(X)$ separates points uniformly.

Lemma 4.6. *Let X be a compact pointed metric space and let $\{f_n\}$ be a sequence in $\mathrm{lip}_0(X)$. Assume $\mathrm{lip}_0(X)$ separates points uniformly. Then $\{f_n\}$ converges to 0 weakly in $\mathrm{lip}_0(X)$ if and only if $\{f_n\}$ is bounded in $\mathrm{lip}_0(X)$ and converges to 0 pointwise on X.*

We will follow the proof of [11, Theorem 3.1] to prove the next lemma, but we include it because that adaptation is not immediate.

Lemma 4.7. *Let X be a compact pointed metric space and let $\varphi\colon X \to X$ be a nonzero basepoint-preserving Lipschitz map. Assume that $\mathrm{lip}_0(X)$ separates points uniformly. Then the essential norm of the operator $C_\varphi\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X)$ satisfies the lower estimate*

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))}{d(x,y)} \leq \sqrt{2}\|C_\varphi\|_e.$$

Proof. According to the proof of [11, Theorem 3.1], we first note that

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))}{d(x,y)} = \inf_{t>0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))}{d(x,y)} \tag{4.1}$$

and obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying that

$$0 < d(x_n, y_n) < \frac{1}{n(1+\mathrm{Lip}(\varphi))} \tag{4.2}$$

for all $n \in \mathbb{N}$, and

$$\inf_{t>0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))}{d(x,y)} = \lim_{n\to\infty} \frac{d(\varphi(x_n),\varphi(y_n))}{d(x_n,y_n)}. \tag{4.3}$$

Passing to a subsequence if necessary, we can assume that $\{x_n\}$ and $\{y_n\}$ converge respectively to points x_0 and y_0 in X. By (4.2), note that $x_0 = y_0$. Consider the closed subset of X given by

$$X_0 = \{\varphi(x_n) : n \in \mathbb{N}_0\} \cup \{\varphi(y_n) : n \in \mathbb{N}_0\} \cup \{e\}.$$

Since $\mathrm{lip}_0(X)$ separates points uniformly, for every $a > 1$ and every $n \in \mathbb{N}_0$, we have $g_n(\varphi(y_n)) = 0$ and $g_n(\varphi(x_n)) = d(\varphi(x_n), \varphi(y_n))$ for some $g_n \in \mathrm{lip}_0(X_0)$ with $\mathrm{Lip}(g_n) \leq a$ (see [21, Corollary 3.3.5] and [20, Theorem 1]). Since $[n/(a(n+1))]g_n \in \mathrm{lip}_0(X_0)$ with $\mathrm{Lip}([n/(a(n+1))]g_n) < 1$, applying [21, Theorem 3.2.6], for every $n \in \mathbb{N}_0$ there exists

$f_n \in \mathrm{lip}_0(X)$ with $f_n(x) = [n/(a(n+1))]g_n(x)$ for all $x \in X_0$, $\mathrm{Lip}(f_n) < \sqrt{2}$ and $\|f_n\|_\infty = \|[n/(a(n+1))]g_n\|_\infty$. Since

$$\begin{aligned}\frac{d(\varphi(x_n),\varphi(y_n))}{d(x_n,y_n)} &= \frac{|g_n(\varphi(x_n)) - g_n(\varphi(y_n))|}{d(x_n,y_n)}\\ &= a\left(\frac{n+1}{n}\right)\frac{|f_n(\varphi(x_n)) - f_n(\varphi(y_n))|}{d(x_n,y_n)}\\ &\leq a\left(\frac{n+1}{n}\right)\mathrm{Lip}(C_\varphi f_n)\end{aligned}$$

for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty}\frac{d(\varphi(x_n),\varphi(y_n))}{d(x_n,y_n)} \leq a \limsup_{n\to\infty}\mathrm{Lip}(C_\varphi f_n). \tag{4.4}$$

Note that $\{f_n\}$ is bounded in $\mathrm{lip}_0(X)$ and $\|f_n\|_\infty \leq 1/a(n+1)$ for all $n \in \mathbb{N}$. Thus it converges weakly to zero in $\mathrm{lip}_0(X)$ by Lemma 4.6. Now, if K is any compact operator from $\mathrm{lip}_0(X)$ into $\mathrm{lip}_0(X)$, we have $\lim_{n\to\infty}\mathrm{Lip}(Kf_n) = 0$ because compact operators are completely continuous. Hence

$$\begin{aligned}\limsup_{n\to\infty}\mathrm{Lip}(C_\varphi f_n) &= \limsup_{n\to\infty}(\mathrm{Lip}(C_\varphi f_n) - \mathrm{Lip}(Kf_n))\\ &\leq \limsup_{n\to\infty}\mathrm{Lip}((C_\varphi - K)f_n)\\ &\leq \sqrt{2}\|C_\varphi - K\|.\end{aligned} \tag{4.5}$$

Connecting (4.1), (4.3), (4.4) and (4.5), we deduce that

$$\lim_{t\to 0}\sup_{0<d(x,y)<t}\frac{d(\varphi(x),\varphi(y))}{d(x,y)} \leq a\sqrt{2}\|C_\varphi - K\|.$$

Taking infimum over all compact operators K on $\mathrm{lip}_0(X)$, we obtain

$$\lim_{t\to 0}\sup_{0<d(x,y)<t}\frac{d(\varphi(x),\varphi(y))}{d(x,y)} \leq a\sqrt{2}\|C_\varphi\|_e.$$

Since $a > 1$ was arbitrary, we derive the lower estimate

$$\lim_{t\to 0}\sup_{0<d(x,y)<t}\frac{d(\varphi(x),\varphi(y))}{d(x,y)} \leq \sqrt{2}\|C_\varphi\|_e. \qquad \square$$

We now are ready to establish one of our announced results.

Corollary 4.8. *Let X be a compact pointed metric space and let $\varphi\colon X \to X$ be a non-constant basepoint-preserving Lipschitz map. Assume that the unit sphere of* $\mathrm{lip}_0(X)$ *separates points uniformly. If the composition operator $C_\varphi\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X)$ satisfies that $\sqrt{2}\|C_\varphi\|_e < \|C_\varphi\|$, then C_φ is norm-attaining.*

Proof. As $\sqrt{2}\|C_\varphi\|_e < \|C_\varphi\|$, by Lemma 4.7 and Theorem 4.1 we have

$$\lim_{d(x,y)\to 0} \frac{d(\varphi(x),\varphi(y))}{d(x,y)} < \sup_{x\neq y} \frac{d(\varphi(x),\varphi(y))}{d(x,y)} = \|C_\varphi\|. \tag{4.6}$$

We can take a sequence $\{(x_n, y_n)\}$ in $\widetilde{X}$ such that

$$\lim_{n\to\infty} \frac{d(\varphi(x_n),\varphi(y_n))}{d(x_n,y_n)} = \sup_{x\neq y} \frac{d(\varphi(x),\varphi(y))}{d(x,y)}. \tag{4.7}$$

By the compactness of X, taking subsequences if necessary, we can suppose that $\{x_n\}$ and $\{y_n\}$ converge to a and b in X, respectively. Put $\varphi(a) = x_0$ and $\varphi(b) = y_0$. By the continuity of φ, $\{\varphi(x_n)\}$ and $\{\varphi(y_n)\}$ converge to x_0 and y_0, respectively. By (4.7) and (4.6), we have $a \neq b$ and $x_0 \neq y_0$. It follows that

$$\lim_{n\to\infty} \frac{d(\varphi(x_n),\varphi(y_n))}{d(x_n,y_n)} = \frac{d(x_0,y_0)}{d(a,b)} > 0,$$

and thus we can take a subsequence of $\{(\varphi(x_n),\varphi(y_n))\}$ in $\widetilde{X}$ which satisfies the hypotheses of Corollary 4.5. Therefore C_φ attains its norm. □

Remark 4.9. Taking into account Theorem 3.2.6 in [21], note that the proof of Lemma 4.7 shows that if C_φ is a nonzero bounded composition operator from $\mathrm{lip}_0(X,\mathbb{R})$ into $\mathrm{lip}_0(X,\mathbb{R})$, then

$$\lim_{t\to 0} \sup_{0<d(x,y)<t} \frac{d(\varphi(x),\varphi(y))}{d(x,y)} \leq \|C_\varphi\|_e.$$

Unfortunately, we have not been able to prove this estimate for operators C_φ from $\mathrm{lip}_0(X,\mathbb{C})$ into $\mathrm{lip}_0(X,\mathbb{C})$. The reason is that the complex version of Theorem 3.2.6 in [21] follows from the real version separating into real and imaginary parts, and this introduces a factor of $\sqrt{2}$ in the extension of a complex-valued little Lipschitz function. Therefore, we obtain the same conclusion in Corollary 4.8 when $C_\varphi\colon \mathrm{lip}_0(X,\mathbb{R}) \to \mathrm{lip}_0(X,\mathbb{R})$ satisfies $\|C_\varphi\|_e < \|C_\varphi\|$.

Since a bounded operator is compact if and only if its essential norm equals 0, an application of Corollary 4.8 yields the desired result:

Corollary 4.10. *Let X be a compact pointed metric space and let $\varphi\colon X \to X$ be a nonconstant basepoint-preserving Lipschitz map. Assume that the unit sphere of* $\mathrm{lip}_0(X)$ *separates points uniformly. Then every compact composition operator C_φ on* $\mathrm{lip}_0(X)$ *is norm-attaining.*

It is known (see [12, 15]) that if X is a compact pointed metric space and $\varphi\colon X \to X$ is a basepoint-preserving Lipschitz map, then a composition operator C_φ on $\mathrm{lip}_0(X)$ is compact if and only if φ is supercontractive, that is,

$$\lim_{d(x,y)\to 0} \frac{d(\varphi(x),\varphi(y))}{d(x,y)} = 0.$$

As a consequence of Corollary 4.10, we obtain the following.

Corollary 4.11. *Let X be a compact pointed metric space and let $\varphi\colon X \to X$ be a nonconstant basepoint-preserving supercontractive Lipschitz map. Assume that the unit sphere of* $\mathrm{lip}_0(X)$ *separates points uniformly. Then every composition operator C_φ on* $\mathrm{lip}_0(X)$ *is norm-attaining.*

References

[1] J. Bonet, M. Lindström and E. Wolf, *Norm-attaining weighted composition operators on weighted Banach spaces of analytic functions*, Arch. Math. (Basel) **99** (2012), no. 6, 537–546.

[2] F. Botelho and J. Jamison, *Composition operators on spaces of Lipschitz functions*, Acta Sci. Math. (Szeged) **77** (2011), no. 3-4, 621–632.

[3] D. Chen, L. Li, R. Wang and Y.-S. Wang, *Nonvanishing preservers and compact weighted composition operators between spaces of Lipschitz functions*, Abstr. Appl. Anal. **2013**, Art. ID 741050, 8 pp.

[4] J. Diestel, *Geometry of Banach Spaces–Selected topics*, Lecture Notes in Mathematics **485**, Springer-Verlag, Berlin-New York, 1975

[5] K. Esmaeili and H. Mahyar, *Weighted composition operators between vector-valued Lipschitz function spaces*, Banach J. Math. Anal. **7** (2013), no. 1, 59–72.

[6] A. Golbaharan and H. Mahyar, *Linear operators of Banach spaces with range in Lipschitz algebras*, Bull. Iranian Math. Soc. **42** (2016), no. 1, 69–78.

[7] ______, *Weighted composition operators on Lipschitz algebras*, Houston J. Math. **42** (2016), no. 3, 905–917.

[8] C. Hammond, *On the norm of a composition operator with linear fractional symbol*, Acta Sci. Math. (Szeged) **69** (2003), no. 3-4, 813–829.

[9] ______, *The norm of a composition operator with linear symbol acting on the Dirichlet space*, J. Math. Anal. Appl. **303** (2005), no. 2, 499–508.

[10] L. G. Hanin, *Kantorovich-Rubinstein norm and its application in the theory of Lipschitz spaces*, Proc. Amer. Math. Soc. **115** (1992), no. 2, 345–352.

[11] A. Jiménez-Vargas, M. Lacruz and M. Villegas-Vallecillos, *Essential norm of composition operators on Banach spaces of Hölder functions*, Abstr. Appl. Anal. **2011**, Article ID 590853, 13 pp.

[12] A. Jiménez-Vargas and M. Villegas-Vallecillos, *Compact composition operators on noncompact Lipschitz spaces*, J. Math. Anal. Appl. **398** (2013), no. 1, 221–229.

[13] J. A. Johnson, *Banach spaces of Lipschitz functions and vector-valued Lipschitz functions*, Trans. Amer. Math. Soc. **148** (1970), 147–169.

[14] N. J. Kalton, *Spaces of Lipschitz and Hölder functions and their applications*, Collect. Math. **55** (2004), no. 2, 171–217.

[15] H. Kamowitz and S. Scheinberg, *Some properties of endomorphisms of Lipschitz algebras*, Studia Math. **96** (1990), no. 3, 255–261.

[16] M. J. Martín, *Norm-attaining composition operators on the Bloch spaces*, J. Math. Anal. Appl. **369** (2010), no. 1, 15–21.

[17] E. Mayer-Wolf, *Isometries between Banach spaces of Lipschitz functions*, Israel J. Math. **38** (1981), no. 1-3, 58–74.

[18] A. Montes-Rodríguez, *The Pick-Schwarz lemma and composition operators on Bloch spaces*, International Workshop on Operator Theory (Cefalù, 1997), Rend. Circ. Mat. Palermo (2) Suppl. **1998** (1998), no. 56, 167–170.

[19] R. R. Phelps, *Lectures on Choquet's Theorem*, D. Van Nostrand, Princeton, N.J.-Toronto, 1966.

[20] N. Weaver, *Quotients of little Lipschitz algebras*, Proc. Amer. Math. Soc. **125** (1997), no. 9, 2643–2648.

[21] ______, *Lipschitz algebras*, World Scientific, River Edge, NJ, 1999.

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