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2-Iso-reflexivity of pointed Lipschitz spaces

A. Jiménez-Vargas [a,*], Moisés Villegas-Vallecillos [b]

[a] Departamento de Matemáticas, Universidad de Almería, 04120, Almería, Spain
[b] Departamento de Matemáticas, Universidad de Cádiz, 11510 Puerto Real, Spain

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ABSTRACT

We show that in the case in which X and Y are uniformly concave complete pointed metric spaces, every 2-local isometry Δ from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$ admits a representation as the sum of a weighted composition operator and a homogeneous Lipschitz functional on, at least, a subspace Y_0 of Y which is isometric to Y. Moreover, Δ is both linear and surjective when X is also separable.

1. Introduction

The study of 2-local isometries between Banach spaces has received considerable attention in recent years. This class of maps was introduced by Molnár [15], motivated by the paper [18] of Šemrl who obtained the first results on 2-local automorphisms and 2-local derivations between Banach algebras.

Given two Banach spaces E and F, a mapping $\Delta\colon E \to F$ (no linearity nor surjectivity are assumed) is called a 2-local isometry if for every $x, y \in E$, there exists a surjective linear isometry $T_{x,y}\colon E \to F$, depending possibly on x and y, such that $\Delta(x) = T_{x,y}(x)$ and $\Delta(y) = T_{x,y}(y)$. It is immediate that every 2-local isometry Δ preserves the distance between points. A problem addressed in the literature by different authors is to study when Δ is both linear and surjective.

A Banach space E is said to be 2-iso-reflexive if every 2-local isometry from E into itself is linear and surjective. Molnár [15] proved that the C^*-algebra $B(H)$ of all bounded linear operators on a separable infinite-dimensional Hilbert space H is 2-iso-reflexive. In [16], he raised to study the 2-iso-reflexivity for the space $C(X)$ of all continuous scalar-valued functions on a first countable compact Hausdorff space X, endowed with the supremum norm. This problem was solved by Győry [4], who showed that $C_0(X)$ – the space of all continuous complex-valued functions vanishing at infinity on a first countable σ-compact Hausdorff space X – is 2-iso-reflexive. Al-Halees and Fleming [1] extended Győry's result for 2-local isometries

* Corresponding author.
E-mail addresses: ajimenez@ual.es (A. Jiménez-Vargas), moises.villegas@uca.es (M. Villegas-Vallecillos).

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between spaces of continuous vector-valued functions. Hatori, Miura, Oka and Takagi [7] and Hosseini [8] considered 2-local isometries on uniform algebras and on spaces of continuously differentiable functions, respectively.

The research on 2-local isometries between spaces of Lipschitz functions was initiated in [9]. Given a metric space X, let $\mathrm{Lip}(X)$ be the Banach space of all scalar-valued bounded Lipschitz functions f on X equipped with some of the natural norms: $\max\{\|f\|_\infty, \mathrm{Lip}(f)\}$ or $\|f\|_\infty + \mathrm{Lip}(f)$, where $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f. The isometry group of $\mathrm{Lip}(X)$ is said to be canonical if every surjective linear isometry of $\mathrm{Lip}(X)$ can be expressed as a weighted composition operator of the form $\lambda \cdot (f \circ \phi)$ for all $f \in \mathrm{Lip}(X)$, where λ is an unimodular constant and ϕ is a surjective isometry of X. In [9], we proved that if X is bounded and separable and the isometry group of $\mathrm{Lip}(X)$ is canonical, then $\mathrm{Lip}(X)$ is 2-iso-reflexive.

This study was subsequently extended in several directions. In [10], Li, Peralta, Wang, Wang and the first author studied 2-local isometries between spaces of vector-valued Lipschitz functions. Li, Peralta, Wang and Wang [13] also established some spherical reformulations of the Gleason–Kahane–Zelazko and Kowalski–Słodkowski theorems [3,11,12] that were used to describe 2-weak-local isometries on Lipschitz algebras and uniform algebras. Recently, this spherical variant of the Kowalski–Słodkowski theorem has been extended and applied by Oi [17] to prove that 2-local maps in the set of all surjective isometries (without assuming linearity) on several function spaces are surjective isometries (see also the paper [6] by Hatori and Oi).

We now present the pointed Lipschitz spaces. Let (X, d_X) be a pointed metric space with a basepoint designated by e_X, let $\widetilde{X}$ denote the set

$$\{(x, y) \in X \times X : x \neq y\},$$

and let $\mathbb{K}$ be the field of real or complex numbers. The pointed Lipschitz space $\mathrm{Lip}_0(X)$ is the Banach space of all Lipschitz functions $f : X \to \mathbb{K}$ for which $f(e_X) = 0$, endowed with the norm defined by

$$\mathrm{Lip}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d_X(x, y)} : (x, y) \in \widetilde{X}\right\}.$$

In this paper, we study 2-local isometries between spaces $\mathrm{Lip}_0(X)$. Namely, under the conditions of completeness and uniform concavity on the metric spaces X and Y – which are necessary to have a convenient description of the surjective linear isometries from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$ – we obtain in Section 3 a representation of 2-local isometries from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$ as the sum of a weighted composition operator and a homogeneous Lipschitz functional on, at least, a certain subspace Y_0 of Y which is isometric to Y. Moreover, for a suitable choice of basepoint in Y_0, we show that every 2-local isometry from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$ induces a linear isometry from $\mathrm{Lip}_0(X)$ onto $\mathrm{Lip}_0(Y_0)$. In Section 4, when X is also separable, we prove that Y_0 coincides with Y, and thus every 2-local isometry of $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$ is both linear and surjective. Hence $\mathrm{Lip}_0(X)$ is 2-iso-reflexive.

Our method of proof to obtain the representation of 2-local isometries between spaces $\mathrm{Lip}_0(X)$ follows the strategy of Győry [4] in his study on 2-local isometries of the spaces $C_0(X)$, but also adapts to Lip_0 spaces a technique employed by Győry and Molnár [5] and Cabello Sánchez [2] to describe the form of diameter-preserving linear bijections of $C(X)$.

First, we shall need a representation of the surjective linear isometries between Lip_0 spaces and one difficulty is that such isometries do not admit, in general, a representation as a weighted composition operator. Furthermore, there exists a considerable literature on the study of the isometry group of the spaces Lip under the maximum or sum norms, but to our knowledge only the references [14,19,20] deal with the isometry group of Lip_0 spaces. To this point we devote the following section.

2. Preliminaries

Given $\alpha \in]0,1[$ and two compact pointed metric spaces (X, d_X) and (Y, d_Y), the linear isometries from $\mathrm{Lip}_0(X, d_X^\alpha)$ onto $\mathrm{Lip}_0(Y, d_Y^\alpha)$ were characterized by Mayer-Wolf in [14, Theorem 3.3]. The following extension of this result for linear isometries from $\mathrm{Lip}_0(X)$ onto $\mathrm{Lip}_0(Y)$ is due to Weaver (see [20, Theorem 3.56] for the real-valued case, and [19, Theorem 2.7.3] joint to [20, Theorem 3.39] for the real and complex-valued cases).

Let us recall that given metric spaces (X, d_X) and (Y, d_Y) and a number $a > 0$, a map $\phi: Y \to X$ is an a-dilation if $d_X(\phi(y_1), \phi(y_2)) = a \cdot d_Y(y_1, y_2)$ for all $y_1, y_2 \in Y$. We denote by $S_{\mathbb{K}}$ the set of all unimodular scalars in $\mathbb{K}$. Moreover, $S_{\mathbb{K}}^+ = \{1\}$ if $\mathbb{K} = \mathbb{R}$ and $S_{\mathbb{K}}^+ = \{e^{it} : t \in [0, \pi[\}$ if $\mathbb{K} = \mathbb{C}$.

Theorem 2.1. *[19,20]. Let X and Y be uniformly concave complete pointed metric spaces. A linear operator $T: \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is a surjective isometry if and only if there exists a number $\lambda \in S_{\mathbb{K}}$ and a surjective a-dilation $\phi: Y \to X$ such that T is of the form*

$$T(f)(y) = \lambda a^{-1} (f(\phi(y)) - f(\phi(e_Y)))$$

for all $f \in \mathrm{Lip}_0(X)$ and $y \in Y$.

According to [20, Definition 3.33], a metric space X is said to be concave if

$$d(x, y) < d(x, z) + d(z, y)$$

for any triple of distinct points $x, y, z \in X$, and uniformly concave if for every distinct points $x, y \in X$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) \leq d(x, z) + d(z, y) - \delta$$

for all $z \in X$ such that $\min\{d(x, z), d(y, z)\} \geq \varepsilon$. In Section 3.5 of [20], the following examples of uniformly concave metric spaces are presented:

(1) Any closed subset of $\mathbb{R}^n$ with the inherited Euclidean norm in which no three points are colinear.
(2) Any compact subset of a strictly convex Banach space in which no three points are colinear.
(3) The unit sphere of any uniformly convex Banach space.
(4) Any metric space $(X, \omega \circ d)$, where $\omega: (0, \infty) \to (0, \infty)$ is a strictly concave distortion function. In particular, any Hölder metric space (X, d^α) with $\alpha \in]0,1[$.

Another important tool in our study is the following peaking functions $h_{(x_1,x_2)}$ borrowed from the proof of [20, Theorem 3.39].

Lemma 2.2. *[20]. Let X be a concave pointed metric space and $(x_1, x_2) \in \widetilde{X}$. Consider the functions $g_{(x_1,x_2)}, h_{(x_1,x_2)}: X \to \mathbb{R}$ defined by*

$$g_{(x_1,x_2)}(z) = \frac{d(x_1, x_2) d(z, x_2)}{d(z, x_1) + d(z, x_2)},$$
$$h_{(x_1,x_2)}(z) = g_{(x_1,x_2)}(z) - g_{(x_1,x_2)}(e_X)$$

for all $z \in X$. Then $h_{(x_1,x_2)}$ belongs to $\mathrm{Lip}_0(X)$, and satisfies that

$$\frac{h_{(x_1,x_2)}(x_1) - h_{(x_1,x_2)}(x_2)}{d(x_1,x_2)} = 1$$

and

$$\frac{\left|h_{(x_1,x_2)}(z) - h_{(x_1,x_2)}(w)\right|}{d(z,w)} < 1$$

for all $(z,w) \in \widetilde{X} \setminus \{(x_1,x_2),(x_2,x_1)\}$.

3. Representation of 2-local isometries

We shall first describe the form of the 2-local isometries between Lip_0 spaces.

Theorem 3.1. *Let X and Y be uniformly concave complete pointed metric spaces and let Δ be a 2-local isometry from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$. Then there exist a subspace Y_0 of Y which is isometric to Y, a number $\lambda \in S_{\mathbb{K}}$ and a surjective a-dilation $\phi\colon Y_0 \to X$ such that*

$$\Delta(f)(y_1) - \Delta(f)(y_2) = \lambda a^{-1}\left(f(\phi(y_1)) - f(\phi(y_2))\right)$$

for all $y_1, y_2 \in Y_0$ and $f \in \mathrm{Lip}_0(X)$.

Proof. The proof will be divided into a sequence of steps. The following one will be frequently applied without any explicit mention.

Step 1. For any $f, g \in \mathrm{Lip}_0(X)$, there are a constant $\lambda_{f,g} \in S_{\mathbb{K}}$ and a surjective $a_{f,g}$-dilation $\phi_{f,g}$ from Y onto X such that

$$\frac{\Delta(f)(y_1) - \Delta(f)(y_2)}{d_Y(y_1,y_2)} = \lambda_{f,g}\frac{f(\phi_{f,g}(y_1)) - f(\phi_{f,g}(y_2))}{d_X(\phi_{f,g}(y_1),\phi_{f,g}(y_2))}$$

and

$$\frac{\Delta(g)(y_1) - \Delta(g)(y_2)}{d_Y(y_1,y_2)} = \lambda_{f,g}\frac{g(\phi_{f,g}(y_1)) - g(\phi_{f,g}(y_2))}{d_X(\phi_{f,g}(y_1),\phi_{f,g}(y_2))}$$

for all $(y_1,y_2) \in \widetilde{Y}$. In the case $f = g$, we shall simply write λ_f, a_f and ϕ_f.

Let $f, g \in \mathrm{Lip}_0(X)$. By hypothesis, there exists a linear isometry $T_{f,g}$ from $\mathrm{Lip}_0(X)$ onto $\mathrm{Lip}_0(Y)$ satisfying $\Delta(f) = T_{f,g}(f)$ and $\Delta(g) = T_{f,g}(g)$. Applying Theorem 2.1, there are a constant $\lambda_{f,g} \in S_{\mathbb{K}}$ and a surjective $a_{f,g}$-dilation $\phi_{f,g}$ from Y onto X such that

$$T_{f,g}(h)(y) = \lambda_{f,g}a_{f,g}^{-1}(h(\phi_{f,g}(y)) - h(\phi_{f,g}(e_Y)))$$

for all $h \in \mathrm{Lip}_0(X)$ and $y \in Y$. Hence

$$\begin{aligned}\frac{\Delta(f)(y_1) - \Delta(f)(y_2)}{d_Y(y_1,y_2)} &= \frac{T_{f,g}(f)(y_1) - T_{f,g}(f)(y_2)}{d_Y(y_1,y_2)} \\ &= \lambda_{f,g}a_{f,g}^{-1}\frac{f(\phi_{f,g}(y_1)) - f(\phi_{f,g}(y_2))}{d_Y(y_1,y_2)} \\ &= \lambda_{f,g}\frac{f(\phi_{f,g}(y_1)) - f(\phi_{f,g}(y_2))}{d_X(\phi_{f,g}(y_1),\phi_{f,g}(y_2))}\end{aligned}$$

for all $(y_1, y_2) \in \widetilde{Y}$. Similarly, we have

$$\frac{\Delta(g)(y_1) - \Delta(g)(y_1)}{d_Y(y_1, y_2)} = \lambda_{f,g} \frac{g(\phi_{f,g}(y_1)) - g(\phi_{f,g}(y_2))}{d_X(\phi_{f,g}(y_1), \phi_{f,g}(y_2))}$$

for all $(y_1, y_2) \in \widetilde{Y}$.

By Theorem 2.1 and the definition of 2-local isometry, there exists a bijection from Y onto X. Hence $|Y| = |X|$, where $|S|$ denotes the cardinality of the set S. Since Theorem 3.1 is easy to verify when $|Y| = 1$, we shall suppose $|Y| \geq 2$ from now on.

Step 2. For each $(x_1, x_2) \in \widetilde{X}$, define the sets:

$$\mathcal{B}_{(x_1,x_2),f} = \left\{ ((y_1, y_2), \lambda) \in \widetilde{Y} \times S_{\mathbb{K}} \colon \frac{\Delta(f)(y_1) - \Delta(f)(y_2)}{d_Y(y_1, y_2)} = \lambda \frac{f(x_1) - f(x_2)}{d_X(x_1, x_2)} \right\} \quad (f \in \mathrm{Lip}_0(X)),$$

$$\mathcal{B}_{(x_1,x_2)} = \bigcap_{f \in \mathrm{Lip}_0(X)} \mathcal{B}_{(x_1,x_2),f}.$$

Then $\{\mathcal{B}_{(x_1,x_2)} \colon (x_1, x_2) \in \widetilde{X}\}$ is a family of nonempty subsets of $\widetilde{Y} \times S_{\mathbb{K}}$.

Let $(x_1, x_2) \in \widetilde{X}$, $f \in \mathrm{Lip}_0(X)$ and consider the function $h_{(x_1,x_2)} \in \mathrm{Lip}_0(X)$ defined in Lemma 2.2. We shall first show that $\mathcal{B}_{(x_1,x_2),h_{(x_1,x_2)}}$ is a nonempty subset of $\widetilde{Y} \times S_{\mathbb{K}}$. By Step 1, there exist a number $\lambda_{h_{(x_1,x_2)},f} \in S_{\mathbb{K}}$ and a surjective $a_{h_{(x_1,x_2)},f}$-dilation $\phi_{h_{(x_1,x_2)},f} \colon Y \to X$ such that

$$\frac{\Delta(h_{(x_1,x_2)})(y_1) - \Delta(h_{(x_1,x_2)})(y_2)}{d_Y(y_1, y_2)} = \lambda_{h_{(x_1,x_2)},f} \frac{h_{(x_1,x_2)}(\phi_{h_{(x_1,x_2)},f}(y_1)) - h_{(x_1,x_2)}(\phi_{h_{(x_1,x_2)},f}(y_2))}{d_X(\phi_{h_{(x_1,x_2)},f}(y_1), \phi_{h_{(x_1,x_2)},f}(y_2))}$$

and

$$\frac{\Delta(f)(y_1) - \Delta(f)(y_2)}{d_Y(y_1, y_2)} = \lambda_{h_{(x_1,x_2)},f} \frac{f(\phi_{h_{(x_1,x_2)},f}(y_1)) - f(\phi_{h_{(x_1,x_2)},f}(y_2))}{d_X(\phi_{h_{(x_1,x_2)},f}(y_1), \phi_{h_{(x_1,x_2)},f}(y_2))}$$

for all $(y_1, y_2) \in \widetilde{Y}$. From the first equality, we deduce that

$$\left(\left(\phi^{-1}_{h_{(x_1,x_2)},f}(x_1), \phi^{-1}_{h_{(x_1,x_2)},f}(x_2) \right), \lambda_{h_{(x_1,x_2)},f} \right), \left(\left(\phi^{-1}_{h_{(x_1,x_2)},f}(x_2), \phi^{-1}_{h_{(x_1,x_2)},f}(x_1) \right), -\lambda_{h_{(x_1,x_2)},f} \right)$$

belong to $\mathcal{B}_{(x_1,x_2),h_{(x_1,x_2)}}$, and hence $\mathcal{B}_{(x_1,x_2),h_{(x_1,x_2)}}$ is nonempty, as desired.

We shall next prove that $\mathcal{B}_{(x_1,x_2),h_{(x_1,x_2)}}$ is contained in $\mathcal{B}_{(x_1,x_2),f}$. Let $((y_1, y_2), \lambda) \in \mathcal{B}_{(x_1,x_2),h_{(x_1,x_2)}}$ be arbitrary. We have

$$\begin{aligned} \lambda &= \lambda \frac{h_{(x_1,x_2)}(x_1) - h_{(x_1,x_2)}(x_2)}{d_X(x_1, x_2)} \\ &= \frac{\Delta(h_{(x_1,x_2)})(y_1) - \Delta(h_{(x_1,x_2)})(y_2)}{d_Y(y_1, y_2)} \\ &= \lambda_{h_{(x_1,x_2)},f} \frac{h_{(x_1,x_2)}(\phi_{h_{(x_1,x_2)},f}(y_1)) - h_{(x_1,x_2)}(\phi_{h_{(x_1,x_2)},f}(y_2))}{d_X(\phi_{h_{(x_1,x_2)},f}(y_1), \phi_{h_{(x_1,x_2)},f}(y_2))} \end{aligned}$$

and therefore

$$\frac{\left|h_{(x_1,x_2)}(\phi_{h_{(x_1,x_2)},f}(y_1)) - h_{(x_1,x_2)}(\phi_{h_{(x_1,x_2)},f}(y_2))\right|}{d_X(\phi_{h_{(x_1,x_2)},f}(y_1), \phi_{h_{(x_1,x_2)},f}(y_2))} = 1.$$

Now Lemma 2.2 implies either

$$(\phi_{h_{(x_1,x_2)},f}(y_1), \phi_{h_{(x_1,x_2)},f}(y_2)) = (x_1, x_2),$$

or

$$(\phi_{h_{(x_1,x_2)},f}(y_1), \phi_{h_{(x_1,x_2)},f}(y_2)) = (x_2, x_1).$$

Hence $\lambda_{h_{(x_1,x_2)},f} = \lambda$ in the first case, or $\lambda_{h_{(x_1,x_2)},f} = -\lambda$ in the second one. It follows that

$$\begin{aligned}\frac{\Delta(f)(y_1) - \Delta(f)(y_2)}{d_Y(y_1, y_2)} &= \lambda_{h_{(x_1,x_2)},f}\frac{f(\phi_{h_{(x_1,x_2)},f}(y_1)) - f(\phi_{h_{(x_1,x_2)},f}(y_2))}{d_X(\phi_{h_{(x_1,x_2)},f}(y_1), \phi_{h_{(x_1,x_2)},f}(y_2))} \\ &= \lambda\frac{f(x_1) - f(x_2)}{d_X(x_1, x_2)},\end{aligned}$$

hence $((y_1, y_2), \lambda) \in \mathcal{B}_{(x_1,x_2),f}$ and this proves that

$$\mathcal{B}_{(x_1,x_2),h_{(x_1,x_2)}} \subseteq \mathcal{B}_{(x_1,x_2),f}.$$

As a consequence, we obtain that $\mathcal{B}_{(x_1,x_2)} = \mathcal{B}_{(x_1,x_2),h_{(x_1,x_2)}}$.

Step 3. For every $(x_1, x_2) \in \widetilde{X}$, there exist $(y_1, y_2) \in \widetilde{Y}$ and $\lambda \in S_{\mathbb{K}}^{+}$ such that

$$\mathcal{B}_{(x_1,x_2)} = \{((y_1, y_2), \lambda), ((y_2, y_1), -\lambda)\}.$$

Let $(x_1, x_2) \in \widetilde{X}$. By Step 2, we can take some $((y_1, y_2), \lambda) \in \mathcal{B}_{(x_1,x_2)}$. Observe that $((y_2, y_1), -\lambda) \in \mathcal{B}_{(x_1,x_2)}$. Let $((y_3, y_4), \beta) \in \mathcal{B}_{(x_1,x_2)}$ be arbitrary. We have

$$\begin{aligned}\frac{\Delta(f)(y_1) - \Delta(f)(y_2)}{d_Y(y_1, y_2)} &= \lambda\frac{f(x_1) - f(x_2)}{d_X(x_1, x_2)}, \\ \frac{\Delta(f)(y_3) - \Delta(f)(y_4)}{d_Y(y_3, y_4)} &= \beta\frac{f(x_1) - f(x_2)}{d_X(x_1, x_2)},\end{aligned}$$

for all $f \in \mathrm{Lip}_0(X)$. Taking $f = h_{(x_1,x_2)}$ and applying Step 1, we deduce

$$\begin{aligned}\lambda_f\frac{f(\phi_f(y_1)) - f(\phi_f(y_2))}{d_X(\phi_f(y_1), \phi_f(y_2))} &= \lambda, \\ \lambda_f\frac{f(\phi_f(y_3)) - f(\phi_f(y_4))}{d_X(\phi_f(y_3), \phi_f(y_4))} &= \beta.\end{aligned}$$

It follows that

$$\frac{|f(\phi_f(y_1)) - f(\phi_f(y_2))|}{d_X(\phi_f(y_1), \phi_f(y_2))} = \frac{|f(\phi_f(y_3)) - f(\phi_f(y_4))|}{d_X(\phi_f(y_3), \phi_f(y_4))} = 1,$$

and in the light of Lemma 2.2 these equalities yield

$$\{\phi_f(y_1), \phi_f(y_2)\} = \{\phi_f(y_3), \phi_f(y_4))\} = \{x_1, x_2\}.$$

We have four possibilities:

(1) $x_1 = \phi_f(y_1),\ x_2 = \phi_f(y_2),\ x_1 = \phi_f(y_3),\ x_2 = \phi_f(y_4)$.
(2) $x_1 = \phi_f(y_1),\ x_2 = \phi_f(y_2),\ x_1 = \phi_f(y_4),\ x_2 = \phi_f(y_3)$.
(3) $x_1 = \phi_f(y_2),\ x_2 = \phi_f(y_1),\ x_1 = \phi_f(y_4),\ x_2 = \phi_f(y_3)$.
(4) $x_1 = \phi_f(y_2),\ x_2 = \phi_f(y_1),\ x_1 = \phi_f(y_3),\ x_2 = \phi_f(y_4)$.

Using the injectivity of ϕ_f, we infer that

$$((y_3, y_4), \beta) \in \{((y_1, y_2), \lambda), ((y_2, y_1), -\lambda)\}.$$

Therefore

$$\mathcal{B}_{(x_1,x_2)} = \{((y_1, y_2), \lambda), ((y_2, y_1), -\lambda)\}.$$

Finally, notice that either $\lambda \in S^+_{\mathbb{K}}$ or $-\lambda \in S^+_{\mathbb{K}}$.

Step 4. For every $(x_1, x_2) \in \widetilde{X}$, the set

$$\mathcal{A}_{(x_1,x_2)} = \left\{(y_1, y_2) \in \widetilde{Y} \mid \exists \lambda \in S^+_{\mathbb{K}} \colon ((y_1, y_2), \lambda) \in \mathcal{B}_{(x_1,x_2)}\right\}$$

is a singleton by Step 3. Let $\Gamma\colon \widetilde{X} \to \widetilde{Y}$ be the map given by

$$\{\Gamma(x_1, x_2)\} = \mathcal{A}_{(x_1,x_2)}.$$

We have $(y_2, y_1) = \Gamma(x_2, x_1)$ if $(y_1, y_2) = \Gamma(x_1, x_2)$.

Let $(x_1, x_2) \in \widetilde{X}$ and assume that $(y_1, y_2) = \Gamma(x_1, x_2)$. Then $(y_1, y_2) \in \mathcal{A}_{(x_1,x_2)}$. Therefore $((y_1, y_2), \lambda) \in \mathcal{B}_{(x_1,x_2)}$ for some $\lambda \in S^+_{\mathbb{K}}$. It follows that $((y_2, y_1), \lambda) \in \mathcal{B}_{(x_2,x_1)}$, hence $(y_2, y_1) \in \mathcal{A}_{(x_2,x_1)}$ and thus $(y_2, y_1) = \Gamma(x_2, x_1)$, as required.

Step 5. For every $(x_1, x_2) \in \widetilde{X}$, there exists a number $\lambda(x_1, x_2) \in S^+_{\mathbb{K}}$ such that

$$\frac{\Delta(f)(y_1) - \Delta(f)(y_2)}{d_Y(y_1, y_2)} = \lambda(x_1, x_2)\frac{f(x_1) - f(x_2)}{d_X(x_1, x_2)}$$

for all $f \in \mathrm{Lip}_0(X)$, where $(y_1, y_2) = \Gamma(x_1, x_2)$. Furthermore, $\lambda(x_1, x_2) = \lambda(x_2, x_1)$.

Let $(x_1, x_2) \in \widetilde{X}$ and $(y_1, y_2) = \Gamma(x_1, x_2)$. By Step 4, there is a number $\lambda(x_1, x_2) \in S^+_{\mathbb{K}}$ such that

$$\frac{\Delta(f)(y_1) - \Delta(f)(y_2)}{d_Y(y_1, y_2)} = \lambda(x_1, x_2)\frac{f(x_1) - f(x_2)}{d_X(x_1, x_2)}$$

for all $f \in \mathrm{Lip}_0(X)$. Since $(y_2, y_1) = \Gamma(x_2, x_1)$ by Step 4, we also have

$$\frac{\Delta(f)(y_2) - \Delta(f)(y_1)}{d_Y(y_2, y_1)} = \lambda(x_2, x_1)\frac{f(x_2) - f(x_1)}{d_X(x_2, x_1)}$$

for all $f \in \mathrm{Lip}_0(X)$. Combining the equations obtained above, we infer that

$$\begin{aligned}\lambda(x_1,x_2)\frac{f(x_1)-f(x_2)}{d_X(x_1,x_2)} &= \frac{\Delta(f)(y_1)-\Delta(f)(y_2)}{d_Y(y_1,y_2)}\\ &= -\frac{\Delta(f)(y_2)-\Delta(f)(y_1)}{d_Y(y_2,y_1)}\\ &= -\lambda(x_2,x_1)\frac{f(x_2)-f(x_1)}{d_X(x_2,x_1)}\\ &= \lambda(x_2,x_1)\frac{f(x_1)-f(x_2)}{d_X(x_1,x_2)}\end{aligned}$$

for all $f \in \mathrm{Lip}_0(X)$, and taking $f = h_{(x_1,x_2)}$ yields $\lambda(x_1,x_2) = \lambda(x_2,x_1)$.

Step 6. The map Γ is a bijection from $\widetilde{X}$ to $\cup_{(x_1,x_2)\in\widetilde{X}}\mathcal{A}_{(x_1,x_2)}$.

Let $(y_1,y_2) \in \cup_{(x_1,x_2)\in\widetilde{X}}\mathcal{A}_{(x_1,x_2)}$. Then $(y_1,y_2) \in \mathcal{A}_{(x_1,x_2)}$ for some $(x_1,x_2) \in \widetilde{X}$. By Step 4, $\mathcal{A}_{(x_1,x_2)} = \{(y_1,y_2)\}$, and thus $\Gamma(x_1,x_2) = (y_1,y_2)$. Hence Γ is surjective.

In order to prove that it is injective, let $(x_1,x_2),(x_3,x_4) \in \widetilde{X}$ be such that

$$(y_1,y_2) = \Gamma(x_1,x_2) = \Gamma(x_3,x_4),$$

where $(y_1,y_2) \in \cup_{(x_1,x_2)\in\widetilde{X}}\mathcal{A}_{(x_1,x_2)}$. By Step 5, we have

$$\lambda(x_1,x_2)\frac{f(x_1)-f(x_2)}{d_X(x_1,x_2)} = \frac{\Delta(f)(y_1)-\Delta(f)(y_2)}{d_Y(y_1,y_2)} = \lambda(x_3,x_4)\frac{f(x_3)-f(x_4)}{d_X(x_3,x_4)}$$

for all $f \in \mathrm{Lip}_0(X)$, with $\lambda(x_1,x_2),\lambda(x_3,x_4) \in S^+_{\mathbb{K}}$. Taking $f = h_{(x_1,x_2)}$, we deduce that either $(x_1,x_2) = (x_4,x_3)$ or $(x_1,x_2) = (x_3,x_4)$. In the former case, we would have $\lambda(x_1,x_2) = -\lambda(x_3,x_4)$, which is impossible. Therefore $(x_1,x_2) = (x_3,x_4)$.

Step 7. Let $(x_1,x_2),(x_3,x_4) \in \widetilde{X}$, $(y_1,y_2) = \Gamma(x_1,x_2)$ and $(y_3,y_4) = \Gamma(x_3,x_4)$. Then

$$|\{x_1,x_2\}\cap\{x_3,x_4\}| = |\{y_1,y_2\}\cap\{y_3,y_4\}|.$$

By Step 5, we have the equalities:

$$\begin{aligned}\frac{\Delta(f)(y_1)-\Delta(f)(y_2)}{d_Y(y_1,y_2)} &= \lambda(x_1,x_2)\frac{f(x_1)-f(x_2)}{d_X(x_1,x_2)},\\ \frac{\Delta(f)(y_3)-\Delta(f)(y_4)}{d_Y(y_3,y_4)} &= \lambda(x_3,x_4)\frac{f(x_3)-f(x_4)}{d_X(x_3,x_4)},\end{aligned}$$

for all $f \in \mathrm{Lip}_0(X)$, where $\lambda(x_1,x_2),\lambda(x_3,x_4) \in S^+_{\mathbb{K}}$. To simplify the writing, we denote $g = h_{(x_1,x_2)}$ and $h = h_{(x_3,x_4)}$. Taking $f = g$ in the former equality and $f = h$ in the latter one, we obtain

$$\begin{aligned}\frac{\Delta(g)(y_1)-\Delta(g)(y_2)}{d_Y(y_1,y_2)} &= \lambda(x_1,x_2) = \lambda_{g,h}\frac{g(\phi_{g,h}(y_1))-g(\phi_{g,h}(y_2))}{d_X(\phi_{g,h}(y_1),\phi_{g,h}(y_2))},\\ \frac{\Delta(h)(y_3)-\Delta(h)(y_4)}{d_Y(y_3,y_4)} &= \lambda(x_3,x_4) = \lambda_{g,h}\frac{h(\phi_{g,h}(y_3))-h(\phi_{g,h}(y_4))}{d_X(\phi_{g,h}(y_3),\phi_{g,h}(y_4))}.\end{aligned}$$

By Lemma 2.2, it follows that $\{\phi_{g,h}(y_1),\phi_{g,h}(y_2)\} = \{x_1,x_2\}$ and $\{\phi_{g,h}(y_3),\phi_{g,h}(y_4)\} = \{x_3,x_4\}$, respectively, and the step holds.

Let X_2 stand for the family of all subsets of X having exactly two elements. Step 7 can be reformulated as follows: if $\Lambda_X\colon \widetilde{X} \to X_2$ and $\Lambda_Y\colon \widetilde{Y} \to Y_2$ are the maps defined by $\Lambda_X(x_1,x_2) = \{x_1,x_2\}$ and $\Lambda_Y(y_1,y_2) = \{y_1,y_2\}$, respectively, we have

$$|\Lambda_X(x_1,x_2) \cap \Lambda_X(x_3,x_4)| = |\Lambda_Y(\Gamma(x_1,x_2)) \cap \Lambda_Y(\Gamma(x_3,x_4))|$$

for all $(x_1,x_2),(x_3,x_4) \in \widetilde{X}$.

Step 8. Assume $|X| \geq 3$. For each $x \in X$ and any $(x_1,x_2) \in \widetilde{X}$ with $x_1 \neq x \neq x_2$, there exists a unique point, depending only on x and denoted by $\varphi(x)$, in the intersection

$$\Lambda_Y(\Gamma(x,x_1)) \cap \Lambda_Y(\Gamma(x,x_2)).$$

The map $\varphi\colon X \to Y$ is injective and $\{\varphi(x_1),\varphi(x_2)\} = \Lambda_Y(\Gamma(x_1,x_2))$ for every $(x_1,x_2) \in \widetilde{X}$.

Let $x \in X$ and let $x_1,x_2 \in X$ be with $x_1 \neq x_2$ and $x_1 \neq x \neq x_2$. By Step 7, there exists a unique point, denoted here by y, in the intersection $\Lambda_Y(\Gamma(x,x_1)) \cap \Lambda_Y(\Gamma(x,x_2))$.

We claim that $y \in \Lambda_Y(\Gamma(x,x_3))$ for every $x_3 \in X$ with $x_3 \neq x$, what shows that y does not depend on x_1 and x_2 and thus it depends only on x. Indeed, if $|X| = 3$, this is obvious. Assume $|X| \geq 4$. Pick $x_3 \in X \setminus \{x,x_1,x_2\}$ and suppose on the contrary that $y \notin \Lambda_Y(\Gamma(x,x_3))$. We can write $\Lambda_Y(\Gamma(x,x_1)) = \{y,y_1\}$ and $\Lambda_Y(\Gamma(x,x_2)) = \{y,y_2\}$ for some $y_1,y_2 \in Y$ with $y_1 \neq y \neq y_2$. In the light of Step 7, we obtain $y_1 \neq y_2$. Since the cardinality of both sets $\Lambda_Y(\Gamma(x,x_3)) \cap \Lambda_Y(\Gamma(x,x_1))$ and $\Lambda_Y(\Gamma(x,x_3)) \cap \Lambda_Y(\Gamma(x,x_2))$ is 1, we deduce that $\Lambda_Y(\Gamma(x,x_3)) = \{y_1,y_2\}$. This implies that $\Gamma(x,x_3) = (y_1,y_2)$ or $\Gamma(x,x_3) = (y_2,y_1)$. We shall only prove the first case and the other is similarly proven. Since $\lambda(x,x_3),\lambda(x,x_1),\lambda(x,x_2) \in S_{\mathbb{K}}^{+}$, an easy argument yields the equation:

$$\begin{aligned}
\lambda(x,x_3)\frac{f(x)-f(x_3)}{d_X(x,x_3)} &= \frac{\Delta(f)(y_1)-\Delta(f)(y_2)}{d_Y(y_1,y_2)} \\
&= \frac{d_Y(y_1,y)}{d_Y(y_1,y_2)}\frac{\Delta(f)(y_1)-\Delta(f)(y)}{d_Y(y_1,y)} + \frac{d_Y(y,y_2)}{d_Y(y_1,y_2)}\frac{\Delta(f)(y)-\Delta(f)(y_2)}{d_Y(y,y_2)} \\
&= \lambda(x,x_1)\frac{d_Y(y_1,y)}{d_Y(y_1,y_2)}\frac{f(x)-f(x_1)}{d_X(x,x_1)} + \lambda(x,x_2)\frac{d_Y(y,y_2)}{d_Y(y_1,y_2)}\frac{f(x)-f(x_2)}{d_X(x,x_2)}
\end{aligned}$$

for all $f \in \mathrm{Lip}_0(X)$. Taking first a function $f \in \mathrm{Lip}_0(X)$ satisfying $f(x) = f(x_2) = 1$ and $f(x_1) = f(x_3) = 0$, and after another $f \in \mathrm{Lip}_0(X)$ for which $f(x) = f(x_1) = 1$ and $f(x_2) = f(x_3) = 0$, we deduce that

$$\lambda(x,x_3)\frac{d_Y(y_1,y_2)}{d_X(x,x_3)} = \lambda(x,x_1)\frac{d_Y(y_1,y)}{d_X(x,x_1)} = \lambda(x,x_2)\frac{d_Y(y,y_2)}{d_X(x,x_2)}.$$

Using this we can simplify the cited equation by obtaining $f(x) = f(x_1)+f(x_2)-f(x_3)$ for all $f \in \mathrm{Lip}_0(X)$, a contradiction. This proves our claim.

We shall next prove the injectivity of φ. Suppose first $|X| = 3$, say $X = \{x_1,x_2,x_3\}$ (one of them is e_X). If $\varphi(x_1) = \varphi(x_2) = y_1$, then $y_1 \in \Lambda_Y(\Gamma(x_1,x_2)) \cap \Lambda_Y(\Gamma(x_1,x_3)) \cap \Lambda_Y(\Gamma(x_2,x_3))$. As the cardinality of each one of the three sets in this intersection is 2, there are $y_2,y_3,y_4 \in Y \setminus \{y_1\}$ such that $\Lambda_Y(\Gamma(x_1,x_2)) = \{y_1,y_2\}$, $\Lambda_Y(\Gamma(x_1,x_3)) = \{y_1,y_3\}$ and $\Lambda_Y(\Gamma(x_2,x_3)) = \{y_1,y_4\}$. Applying Step 7 yields $y_2 \neq y_3 \neq y_4 \neq y_2$, and thus $|Y| \geq 4$ which contradicts that $|X| = |Y|$.

Assume now $|X| \geq 4$. Let $x_1,x_2 \in X$ be with $x_1 \neq x_2$ and suppose $\varphi(x_1) = \varphi(x_2) = y_2$. Take $\{z_1,z_2\} \in X_2$ such that $\{z_1,z_2\} \cap \{x_1,x_2\} = \emptyset$. We have $y_2 \in \Lambda_Y(\Gamma(x_1,z_1)) \cap \Lambda_Y(\Gamma(x_2,z_2))$; but since $|\Lambda_X(x_1,z_1) \cap \Lambda_X(x_2,z_2)| = 0$, we have $|\Lambda_Y(\Gamma(x_1,z_1)) \cap \Lambda_Y(\Gamma(x_2,z_2))| = 0$ by Step 7, a contradiction. This completes the proof that φ is injective.

For the second assertion, note that if $(x_1,x_2)\in\widetilde{X}$, then $\varphi(x_1)$ and $\varphi(x_2)$ are distinct and belong to $\Lambda_Y(\Gamma(x_1,x_2))$ (see Step 4). Hence $\{\varphi(x_1),\varphi(x_2)\}=\Lambda_Y(\Gamma(x_1,x_2))$.

Step 9. There exist a nonempty subset $Y_0\subseteq Y$ and a bijection $\phi\colon Y_0\to X$ such that $\{y_1,y_2\}=\Lambda_Y(\Gamma(\phi(y_1),\phi(y_2)))$ for all $y_1,y_2\in Y_0$ with $y_1\neq y_2$.

Assume first $|X|=2$. Then $|Y|=2$ by Step 1. Hence $X=\{x,e_X\}$ and $Y=\{y,e_Y\}$ for certain $x\in X\setminus\{e_X\}$ and $y\in Y\setminus\{e_Y\}$. Clearly, $\widetilde{X}=\{(x,e_X),(e_X,x)\}$ and $\widetilde{Y}=\{(y,e_Y),(e_Y,y)\}$. Since Γ is a map from $\widetilde{X}$ to $\widetilde{Y}$, we have $\Lambda_Y(\Gamma(x,e_X))=\{y,e_Y\}$. Take $Y_0=Y$ and the bijection $\phi\colon Y_0\to X$ defined by $\phi(y)=x$ and $\phi(e_Y)=e_X$, and the proof is finished if $|X|=2$.

Assume now $|X|\geq 3$. Let $\varphi\colon X\to Y$ be the injective map defined in Step 8. Then $Y_0=\varphi(X)$ and $\phi=\varphi^{-1}\colon Y_0\to X$ satisfy the required conditions.

Step 10. There exist numbers $a\in\mathbb{R}^+$ and $\lambda\in S_{\mathbb{K}}$ such that $\phi\colon Y_0\to X$ is an a-dilation and

$$\Delta(f)(y_1)-\Delta(f)(y_2)=\lambda a^{-1}\left(f(\phi(y_1))-f(\phi(y_2))\right)$$

for all $y_1,y_2\in Y_0$ and $f\in\mathrm{Lip}_0(X)$.

Let $Y_0\subseteq Y$ and $\phi\colon Y_0\to X$ be the set and the bijection given in Step 9. Let $y_1,y_2\in Y_0$ with $y_1\neq y_2$. By Step 9, $\{y_1,y_2\}=\Lambda_Y(\Gamma(\phi(y_1),\phi(y_2)))$. Hence either $\Gamma(\phi(y_1),\phi(y_2))=(y_1,y_2)$ or $\Gamma(\phi(y_1),\phi(y_2))=(y_2,y_1)$. By Step 5, we have

$$\frac{\Delta(f)(y_1)-\Delta(f)(y_2)}{d_Y(y_1,y_2)}=\beta(\phi(y_1),\phi(y_2))\frac{f(\phi(y_1))-f(\phi(y_2))}{d_X(\phi(y_1),\phi(y_2))}$$

for all $f\in\mathrm{Lip}_0(X)$, where $\beta(\phi(y_1),\phi(y_2))\in\{\pm\lambda(\phi(y_1),\phi(y_2))\}$ and $\lambda(\phi(y_1),\phi(y_2))\in S_{\mathbb{K}}^+$.

We now claim that $\beta(\phi(y_1),\phi(y_2))$ and $d_Y(y_1,y_2)/d_X(\phi(y_1),\phi(y_2))$ do not depend on their variables y_1,y_2. It is clear when $|Y_0|=2$ because $\beta(\phi(y_1),\phi(y_2))=\beta(\phi(y_2),\phi(y_1))$ by Step 5. Otherwise, let $y_3\in Y_0$ be with $y_3\notin\{y_1,y_2\}$. We have the equation:

$$\begin{aligned}\beta(\phi(y_1),\phi(y_2))\frac{d_Y(y_1,y_2)}{d_X(\phi(y_1),\phi(y_2))}&\left(f(\phi(y_1))-f(\phi(y_2))\right)\\&=\Delta(f)(y_1)-\Delta(f)(y_2)\\&=(\Delta(f)(y_1)-\Delta(f)(y_3))+(\Delta(f)(y_3)-\Delta(f)(y_2))\\&=\beta(\phi(y_1),\phi(y_3))\frac{d_Y(y_1,y_3)}{d_X(\phi(y_1),\phi(y_3))}\left(f(\phi(y_1))-f(\phi(y_3))\right)\\&+\beta(\phi(y_3),\phi(y_2))\frac{d_Y(y_3,y_2)}{d_X(\phi(y_3),\phi(y_2))}\left(f(\phi(y_3))-f(\phi(y_2))\right)\end{aligned}$$

for all $f\in\mathrm{Lip}_0(X)$. For each $i\in\{1,2\}$, consider the set

$$F_i=\{\phi(y_1),\phi(y_2),\phi(y_3)\}\setminus\{\phi(y_i)\}$$

and the functions $g_i,f_i\colon X\to\mathbb{R}$ defined, respectively, by

$$\begin{aligned}g_i(z)&=\frac{d_X(z,F_i)}{d_X(z,\phi(y_i))+d_X(z,F_i)},\\f_i(z)&=g_i(z)-g_i(e_X).\end{aligned}$$

Clearly, $f_i \in \mathrm{Lip}_0(X)$, and taking $f = f_i$ for $i = 1, 2$ in the equation above, it follows that

$$\begin{aligned}\beta(\phi(y_1),\phi(y_3))\frac{d_Y(y_1,y_3)}{d_X(\phi(y_1),\phi(y_3))} &= \beta(\phi(y_1),\phi(y_2))\frac{d_Y(y_1,y_2)}{d_X(\phi(y_1),\phi(y_2))}\\ &= \beta(\phi(y_3),\phi(y_2))\frac{d_Y(y_3,y_2)}{d_X(\phi(y_3),\phi(y_2))},\end{aligned}$$

as claimed. Since $\beta(\phi(\cdot),\phi(\cdot))$ has the unit modulus, we deduce that

$$\frac{d_Y(y_1,y_3)}{d_X(\phi(y_1),\phi(y_3))} = \frac{d_Y(y_1,y_2)}{d_X(\phi(y_1),\phi(y_2))} = \frac{d_Y(y_3,y_2)}{d_X(\phi(y_3),\phi(y_2))}$$

and therefore

$$\beta(\phi(y_1),\phi(y_3)) = \beta(\phi(y_1),\phi(y_2)) = \beta(\phi(y_3),\phi(y_2)).$$

By the arbitrariness of y_1, y_2 and y_3, the first equality in the two preceding equations means that the two functions $d_Y(\cdot,\cdot)/d_X(\phi(\cdot),\phi(\cdot))$ and $\beta(\phi(\cdot),\phi(\cdot))$ does not depend on the second variable, while the second equality in both equation says us that the same occurs with the first one. Hence there exist two constants $a \in \mathbb{R}^+$ and $\lambda \in S_{\mathbb{K}}$ such that

$$d_X(\phi(y_1),\phi(y_2)) = a \cdot d_Y(y_1,y_2),$$

and

$$\beta(\phi(y_1),\phi(y_2)) = \lambda,$$

for all $y_1, y_2 \in Y_0$ with $y_1 \neq y_2$. Therefore ϕ is an a-dilation from Y_0 onto X. Since X is complete, so also is Y_0.

Finally, we have

$$\begin{aligned}\frac{\Delta(f)(y_1)-\Delta(f)(y_2)}{d_Y(y_1,y_2)} &= \beta(\phi(y_1),\phi(y_2))\frac{f(\phi(y_1))-f(\phi(y_2))}{d_X(\phi(y_1),\phi(y_2))}\\ &= \lambda a^{-1}\frac{f(\phi(y_1))-f(\phi(y_2))}{d_Y(y_1,y_2)}\end{aligned}$$

for all $y_1, y_2 \in Y_0$ with $y_1 \neq y_2$ and $f \in \mathrm{Lip}_0(X)$ and therefore

$$\Delta(f)(y_1)-\Delta(f)(y_2) = \lambda a^{-1}\left(f(\phi(y_1))-f(\phi(y_2))\right)$$

for all $y_1, y_2 \in Y_0$ and $f \in \mathrm{Lip}_0(X)$.

Step 11. There exists a surjective isometry $\psi\colon Y \to Y_0$.

For all $y_1, y_2 \in Y_0$ and $f \in \mathrm{Lip}_0(X)$, we have

$$\Delta(f)(y_1)-\Delta(f)(y_2) = \lambda a^{-1}\left(f(\phi(y_1))-f(\phi(y_2))\right),$$

with Y_0, λ, a, ϕ being as in the statement of Step 10.

Pick $y_1, y_2 \in Y_0$ with $y_1 \neq y_2$, and denote $x_1 = \phi(y_1)$ and $x_2 = \phi(y_2)$. By Step 1, we have

$$\Delta(f)(y) = \lambda_{h(x_1,x_2)} a^{-1}_{h(x_1,x_2)} \left(f\left(\phi_{h(x_1,x_2)}(y)\right) - f\left(\phi_{h(x_1,x_2)}(e_Y)\right)\right)$$

for all $y \in Y$ and $f \in \mathrm{Lip}_0(X)$. Hence

$$\begin{aligned}\lambda &= \lambda \frac{h_{(x_1,x_2)}(x_1) - h_{(x_1,x_2)}(x_2)}{d_X(x_1, x_2)} \\ &= \frac{\Delta(h_{(x_1,x_2)})(y_1) - \Delta(h_{(x_1,x_2)})(y_2)}{d_Y(y_1, y_2)} \\ &= \lambda_{h(x_1,x_2)} \frac{h_{(x_1,x_2)}(\phi_{h(x_1,x_2)}(y_1)) - h_{(x_1,x_2)}(\phi_{h(x_1,x_2)}(y_2))}{d_X(\phi_{h(x_1,x_2)}(y_1), \phi_{h(x_1,x_2)}(y_2))}.\end{aligned}$$

By Lemma 2.2, it follows that

$$(\phi_{h(x_1,x_2)}(y_1), \phi_{h(x_1,x_2)}(y_2)) \in \{(x_1, x_2), (x_2, x_1)\}.$$

Define the mapping $\psi = \phi^{-1} \circ \phi_{h(x_1,x_2)} \colon Y \to Y$. Clearly, $Y_0 = \psi(Y)$ and

$$d_Y(\psi(z_1), \psi(z_2)) = a^{-1} d_X(\phi_{h(x_1,x_2)}(z_1), \phi_{h(x_1,x_2)}(z_2)) = a^{-1} a_{h(x_1,x_2)} d_Y(z_1, z_2)$$

for all $z_1, z_2 \in Y$. In particular, we have

$$d_Y(\psi(y_1), \psi(y_2)) = a^{-1} a_{h(x_1,x_2)} d_Y(y_1, y_2),$$

but

$$\begin{aligned}d_Y(\psi(y_1), \psi(y_2)) &= d_Y(\phi^{-1}(\phi_{h(x_1,x_2)}(y_1), \phi^{-1}(\phi_{h(x_1,x_2)}(y_2))) \\ &= d_Y(\phi^{-1}(x_1), \phi^{-1}(x_2)) = d_Y(y_1, y_2).\end{aligned}$$

Therefore $a^{-1} a_{h(x_1,x_2)} = 1$, and thus $\psi \colon Y \to Y_0$ is an isometry. This completes the proof of Theorem 3.1. □

This theorem can be reformulated as follows.

Corollary 3.2. *Let X and Y be uniformly concave complete pointed metric spaces and let Δ be a 2-local isometry from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$. Then there exist a subspace Y_0 of Y which is isometric to Y, a surjective a-dilation $\phi\colon Y_0 \to X$, a number $\lambda \in S_{\mathbb{K}}$ and a homogeneous Lipschitz function $\mu\colon \mathrm{Lip}_0(X) \to \mathbb{K}$ such that*

$$\Delta(f)(y) = \lambda a^{-1} f(\phi(y)) + \mu(f)$$

for all $y \in Y_0$ and $f \in \mathrm{Lip}_0(X)$.

Proof. For every $y_1, y_2 \in Y_0$ and $f \in \mathrm{Lip}_0(X)$, we can write

$$\Delta(f)(y_1) - \Delta(f)(y_2) = \lambda a^{-1} \left(f(\phi(y_1)) - f(\phi(y_2))\right),$$

with Y_0, λ, a, ϕ being as in the statement of Theorem 3.1. Define $\mu\colon \mathrm{Lip}_0(X) \to \mathbb{K}$ by

$$\mu(f) = \Delta(f)(y) - \lambda a^{-1} f(\phi(y))$$

for all $f \in \text{Lip}_0(X)$, where y is an arbitrary point in Y_0. Note that $\mu(f)$ does not depend on y, and μ is well-defined.

Given $\lambda \in \mathbb{K}$ and $f \in \text{Lip}_0(X)$, by hypothesis there is a linear isometry $T_{f,\lambda f}$ from $\text{Lip}_0(X)$ onto $\text{Lip}_0(Y)$ such that $\Delta(f) = T_{f,\lambda f}(f)$ and $\Delta(\lambda f) = T_{f,\lambda f}(\lambda f)$. We have

$$\Delta(\lambda f) = T_{f,\lambda f}(\lambda f) = \lambda T_{f,\lambda f}(f) = \lambda\Delta(f),$$

and thus Δ is homogeneous. Hence so is μ.

In order to prove that μ is Lipschitz, let us recall first that Δ is an isometry. Observe also that for any $x \in X$, the evaluation functional $\delta_x \colon \text{Lip}_0(X) \to \mathbb{K}$, given by $\delta_x(f) = f(x)$ for all $f \in \text{Lip}_0(X)$, is linear and continuous with $\|\delta_x\| = d(x, e_X)$.

Finally, given $f, g \in \text{Lip}_0(X)$, we have

$$\begin{aligned} |\mu(f) - \mu(g)| &= \left|(\Delta(f)(y) - \Delta(g)(y)) + \lambda a^{-1}(f(\phi(y)) - g(\phi(y)))\right| \\ &= \left|\delta_y(\Delta(f) - \Delta(g)) + \lambda a^{-1}\delta_{\phi(y)}(f - g)\right| \\ &\leq \|\delta_y\| \,\text{Lip}(\Delta(f) - \Delta(g)) + a^{-1} \left\|\delta_{\phi(y)}\right\| \text{Lip}(f - g) \\ &= \left(d_Y(y, e_Y) + a^{-1}d_X(\phi(y), e_X)\right) \text{Lip}(f - g). \quad \square \end{aligned}$$

In relation to Theorem 3.1, notice that the basepoint of Y is not necessarily in the set Y_0, but for a suitable choice of basepoint in Y_0, we can see that every 2-local isometry from $\text{Lip}_0(X)$ to $\text{Lip}_0(Y)$ induces a linear isometry from $\text{Lip}_0(X)$ onto $\text{Lip}_0(Y_0)$, as follows.

Corollary 3.3. *Let X and Y be uniformly concave complete pointed metric spaces and let Δ be a 2-local isometry from $\text{Lip}_0(X)$ to $\text{Lip}_0(Y)$. Then there exists an uniformly concave complete pointed metric space Y_0 such that if $R \colon \text{Lip}_0(Y) \to \text{Lip}_0(Y_0)$ is the map given by $R(f) = f|_{Y_0} - f(e_{Y_0})$ for all $f \in \text{Lip}_0(Y)$, then $R \circ \Delta \colon \text{Lip}_0(X) \to \text{Lip}_0(Y_0)$ is a surjective linear isometry.*

Proof. By Theorem 3.1, we have

$$\Delta(f)(y_1) - \Delta(f)(y_2) = \lambda a^{-1} \left(f(\phi(y_1)) - f(\phi(y_2))\right)$$

for all $y_1, y_2 \in Y_0$ and $f \in \text{Lip}_0(X)$, where Y_0 is a subspace of Y, isometric to Y, $\lambda \in S_{\mathbb{K}}$ and $\phi \colon Y_0 \to X$ is a surjective a-dilation. Consider Y_0 as a pointed metric space with the metric induced by d_Y and basepoint $e_{Y_0} := \phi^{-1}(e_X)$, and let $R \colon \text{Lip}_0(Y) \to \text{Lip}_0(Y_0)$ be the map defined in the statement. From above we deduce that

$$(R \circ \Delta)(f)(y) = \lambda a^{-1} f(\phi(y)) \qquad \left(y \in Y_0,\ f \in \text{Lip}_0(X)\right),$$

and therefore $R \circ \Delta$ is a linear isometry from $\text{Lip}_0(X)$ onto $\text{Lip}_0(Y_0)$. $\square$

4. 2-Iso-reflexivity

In this section, we shall prove that every 2-local isometry from $\text{Lip}_0(X)$ to $\text{Lip}_0(Y)$ is a surjective linear isometry whenever X and Y are separable complete uniformly concave pointed metric spaces, and therefore $\text{Lip}_0(X)$ will be 2-iso-reflexive.

For its proof we shall need some peaking functions with additional properties. The construction of such functions begins in the next lemma.

Lemma 4.1. *Let X be a concave pointed metric space, $(x_1, x_2) \in \widetilde{X}$ and $0 < \delta < d(x_1, x_2)$. Consider the functions $g_1, g_2, g_3 \colon X \to \mathbb{R}$ defined by*

$$\begin{aligned} g_1(z) &= \frac{2d(x_1,x_2)-\delta}{2d(x_1,x_2)} \max\{0, d(x_1,x_2) - d(z,x_1)\} - \frac{\delta}{2d(x_1,x_2)} \max\{0, d(x_1,x_2) - d(z,x_2)\}, \\ g_2(z) &= \max\left\{g_1(z), -\frac{1}{2}\max\{0, \delta - d(z,x_2)\}\right\}, \\ g_3(z) &= \min\left\{g_2(z), \frac{4d(x_1,x_2)-2\delta}{4d(x_1,x_2)-\delta} \max\left\{0, d(x_1,x_2) - \frac{\delta}{4} - d(z,x_1)\right\}\right\}. \end{aligned}$$

Then, for each $k \in \{1, 2, 3\}$, the function g_k is Lipschitz with

$$\frac{g_k(x_1) - g_k(x_2)}{d(x_1,x_2)} = 1$$

and enjoys the condition i):

$$\frac{|g_k(z) - g_k(w)|}{d(z,w)} < 1, \quad \forall (z,w) \in \widetilde{X} \setminus \{(x_1,x_2),(x_2,x_1)\}.$$

Furthermore, g_3 satisfies the conditions:

ii) *$g_3(z) = 0$ if $d(z,x_1) \geq d(x_1,x_2) - \delta/4$ and $d(z,x_2) \geq \delta$,*
iii) *$g_3(z) \geq 0$ if $d(z,x_2) \geq \delta$,*
iv) *$g_3(z) \geq -\delta/2$ for all $z \in X$.*

Proof. Evaluating g_k for $k = 1, 2, 3$ at x_1 and x_2, it is immediate that

$$\frac{g_k(x_1) - g_k(x_2)}{d(x_1,x_2)} = 1.$$

We now prove that g_1 has the property i). Given $(z,w) \in \widetilde{X} \setminus \{(x_1,x_2),(x_2,x_1)\}$, we can distinguish 16 cases according to the relations between (z,w) and (x_1,x_2). We only check 5 cases and the rest can be verified similarly.

1) If $\max\{d(z,x_1), d(z,x_2), d(w,x_1), d(w,x_2)\} \leq d(x_1,x_2)$, we have

$$g_1(z) = \frac{2d(x_1,x_2)-\delta}{2d(x_1,x_2)} (d(x_1,x_2) - d(z,x_1)) + \frac{\delta}{2d(x_1,x_2)} (-d(x_1,x_2) + d(z,x_2))$$

and therefore

$$|g_1(z) - g_1(w)| \leq \frac{2d(x_1,x_2)-\delta}{2d(x_1,x_2)} |d(w,x_1) - d(z,x_1)| + \frac{\delta}{2d(x_1,x_2)} |d(z,x_2) - d(w,x_2)|.$$

Since $(x_1,x_2) \neq (z,w) \neq (x_2,x_1)$, it follows that

$$|d(w,x_1) - d(z,x_1)| < d(z,w)$$

or

$$|d(z,x_2)-d(w,x_2)|<d(z,w)$$

by the concavity of X. Hence

$$\frac{|g_1(z)-g_1(w)|}{d(z,w)}<\frac{2d(x_1,x_2)-\delta+\delta}{d(z,w)2d(x_1,x_2)}d(z,w)=1.$$

2) If $\max\{d(z,x_1),d(z,x_2),d(w,x_1)\}\leq d(x_1,x_2)$ and $d(w,x_2)>d(x_1,x_2)$, we have

$$g_1(w)=\frac{2d(x_1,x_2)-\delta}{2d(x_1,x_2)}\left(d(x_1,x_2)-d(w,x_1)\right),$$

hence

$$g_1(z)-g_1(w)=\frac{2d(x_1,x_2)-\delta}{2d(x_1,x_2)}\left(d(w,x_1)-d(z,x_1)\right)-\frac{\delta}{2d(x_1,x_2)}\left(d(x_1,x_2)-d(z,x_2)\right),$$

and therefore

$$\begin{aligned}\frac{|g_1(z)-g_1(w)|}{d(z,w)}&\leq\frac{2d(x_1,x_2)-\delta}{2d(x_1,x_2)}+\frac{\delta}{2d(x_1,x_2)}\frac{|d(x_1,x_2)-d(z,x_2)|}{d(z,w)}\\&<\frac{2d(x_1,x_2)-\delta}{2d(x_1,x_2)}+\frac{\delta}{2d(x_1,x_2)}\frac{d(w,x_2)-d(z,x_2)}{d(z,w)}\leq 1.\end{aligned}$$

3) If $\max\{d(z,x_1),d(z,x_2)\}\leq d(x_1,x_2)$ and $\min\{d(w,x_1),d(w,x_2)\}>d(x_1,x_2)$, then $g_1(w)=0$ and $g_1(z)-g_1(w)=g_1(z)$. Hence

$$\begin{aligned}\frac{|g_1(z)-g_1(w)|}{d(z,w)}&\leq\frac{2d(x_1,x_2)-\delta}{2d(x_1,x_2)}\frac{d(x_1,x_2)-d(z,x_1)}{d(z,w)}+\frac{\delta}{2d(x_1,x_2)}\frac{d(x_1,x_2)-d(z,x_2)}{d(z,w)}\\&<\frac{2d(x_1,x_2)-\delta}{2d(x_1,x_2)}\frac{d(w,x_1)-d(z,x_1)}{d(z,w)}+\frac{\delta}{2d(x_1,x_2)}\frac{d(w,x_2)-d(z,x_2)}{d(z,w)}\leq 1.\end{aligned}$$

4) If $d(z,x_2)\leq d(x_1,x_2)$ and $\min\{d(z,x_1),d(w,x_1),d(w,x_2)\}>d(x_1,x_2)$, then

$$\frac{|g_1(z)-g_1(w)|}{d(z,w)}=\frac{\delta}{2d(x_1,x_2)}\frac{d(x_1,x_2)-d(z,x_2)}{d(z,w)}<\frac{d(w,x_2)-d(z,x_2)}{d(z,w)}\leq 1.$$

5) If $\min\{d(z,x_1),d(z,x_2),d(w,x_1),d(w,x_2)\}>d(x_1,x_2)$, then

$$\frac{|g_1(z)-g_1(w)|}{d(z,w)}=0<1.$$

We can check similarly that the functions g_2 and g_3 satisfy the property i). Finally, we prove the conditions ii), iii) and iv) for g_3. If $d(z,x_1)\geq d(x_1,x_2)-\delta/4$ and $d(z,x_2)\geq\delta$, then $g_2(z)=\max\{g_1(z),0\}\geq 0$ and thus $g_3(z)=\min\{g_2(z),0\}=0$; if $d(z,x_2)\geq\delta$, then $g_2(z)=\max\{g_1(z),0\}\geq 0$ and therefore $g_3(z)\geq 0$; and if $z\in X$, we have

$$-\frac{\delta}{2}\leq-\frac{\delta}{2d(x_1,x_2)}\max\{0,d(x_1,x_2)-d(z,x_2)\}\leq g_1(z)\leq g_2(z),$$

and therefore $g_3(z)\geq-\delta/2$. □

We are now ready to define the announced functions.

Lemma 4.2. *Let X be a concave metric space and let $x_1, x_2, x_3 \in X$ be three distinct points such that $d(x_1, x_2) = d(x_1, x_3)$. Given $\delta \in]0, d(x_1, x_2)[$, assume that the set*

$$C = \{z \in X \colon d(z, x_1) \geq d(x_1, x_2),\ d(z, x_2) \geq 3\delta,\ d(z, x_3) \geq 3\delta\}$$

contains a countable subset $R = \{r_n \colon n \in \mathbb{N}\}$ of pairwise distinct points. Then there exist two Lipschitz functions $f, g \colon X \to \mathbb{R}$ satisfying:

i) $(f(x_1) - f(x_2))/d(x_1, x_2) = 1 = (g(x_1) - g(x_3))/d(x_1, x_3)$,
ii) $|f(z) - f(w)|/d(z, w) < 1$ *for all* $(z, w) \in \widetilde{X} \setminus \{(x_1, x_2), (x_2, x_1)\}$,
iii) $|g(z) - g(w)|/d(z, w) < 1$ *for all* $(z, w) \in \widetilde{X} \setminus \{(x_1, x_3), (x_3, x_1)\}$,
iv) $\{x \in C \colon (f(x), g(x)) = (f(r_n), g(r_n))\} = \{r_n\}$ *for each* $n \in \mathbb{N}$.

Proof. By [9, Proposition 3.2], there are two Lipschitz functions $f_0, g_0 \colon C \to [0, 1]$ with $\mathrm{Lip}(f_0) \leq 1$ and $\mathrm{Lip}(g_0) \leq 1$ such that

$$\{x \in C \colon (f_0(x), g_0(x)) = (f_0(r_n), g_0(r_n))\} = \{r_n\}$$

for each $n \in \mathbb{N}$. Consider the set

$$W = \left\{z \in X \colon d(z, x_1) < d(x_1, x_2) - \frac{\delta}{8}\right\} \cup \{z \in X \colon d(z, x_2) < 2\delta\} \cup \{z \in X \colon d(z, x_3) < 2\delta\}.$$

Note that $d(C, W) \geq \delta/8$, and define the functions $h_1, h_2 \colon C \cup W \to \mathbb{R}$ by

$$h_1(x) = \begin{cases} \frac{\delta^2}{16(\delta+1)^2} f_0(x), & \text{if} \quad x \in C, \\ \frac{-\delta^2}{16(\delta+1)^2}, & \text{if} \quad x \in W, \end{cases}$$

and

$$h_2(x) = \begin{cases} \frac{\delta^2}{16(\delta+1)^2} g_0(x), & \text{if} \quad x \in C, \\ \frac{-\delta^2}{16(\delta+1)^2}, & \text{if} \quad x \in W. \end{cases}$$

Clearly, h_1 and h_2 are Lipschitz and bounded, with

$$\mathrm{Lip}(h_k) \leq \max\left\{\frac{\delta^2}{16(\delta+1)^2}, \frac{\delta}{(\delta+1)^2}\right\} < 1, \qquad \|h_k\|_\infty \leq \frac{\delta^2}{16(\delta+1)^2}$$

for $k = 1, 2$. By [20, Theorem 1.33], for $k = 1, 2$ there exist a function $f_k \colon X \to \mathbb{R}$ such that $f_k|_{C \cup W} = h_k$, $\|f_k\|_\infty = \|h_k\|_\infty$ and $\mathrm{Lip}(f_k) = \mathrm{Lip}(h_k)$. By Lemma 4.1, we can take two Lipschitz functions $g_{(x_1,x_2,\delta)}, g_{(x_1,x_3,\delta)} \colon X \to \mathbb{R}$ such that

$$\frac{\left|g_{(x_1,x_2,\delta)}(z) - g_{(x_1,x_2,\delta)}(w)\right|}{d(z, w)} < 1 = \frac{g_{(x_1,x_2,\delta)}(x_1) - g_{(x_1,x_2,\delta)}(x_2)}{d(x_1, x_2)}, \quad \forall (z, w) \in \widetilde{X} \setminus \{(x_1, x_2), (x_2, x_1)\},$$

$$g_{(x_1,x_2,\delta)}(z) = 0, \quad \forall z \in X \colon d(z, x_1) \geq d(x_1, x_2) - \frac{\delta}{4},\ d(z, x_2) \geq \delta,$$

$$g_{(x_1,x_2,\delta)}(z) \geq 0, \quad \forall z \in X \colon d(z, x_2) \geq \delta,$$

$$g_{(x_1,x_2,\delta)}(z) \geq -\frac{\delta}{2}, \quad \forall z \in X,$$

and

$$\frac{\left|g_{(x_1,x_3,\delta)}(z) - g_{(x_1,x_3,\delta)}(w)\right|}{d(z,w)} < 1 = \frac{g_{(x_1,x_3,\delta)}(x_1) - g_{(x_1,x_3,\delta)}(x_3)}{d(x_1,x_3)}, \quad \forall (z,w) \in \widetilde{X} \setminus \{(x_1,x_3),(x_3,x_1)\},$$

$$g_{(x_1,x_3,\delta)}(z) = 0, \quad \forall z \in X\colon d(z,x_1) \geq d(x_1,x_3) - \frac{\delta}{4},\ d(z,x_3) \geq \delta,$$

$$g_{(x_1,x_3,\delta)}(z) \geq 0, \quad \forall z \in X\colon d(z,x_3) \geq \delta,$$

$$g_{(x_1,x_3,\delta)}(z) \geq -\frac{\delta}{2}, \quad \forall z \in X.$$

Consider the functions

$$f = f_1 + \frac{\delta^2}{16(\delta+1)^2} + g_{(x_1,x_2,\delta)}, \qquad g = f_2 + \frac{\delta^2}{16(\delta+1)^2} + g_{(x_1,x_3,\delta)}.$$

Note that

$$f|_{X\setminus W} = f_1|_{X\setminus W} + \frac{\delta^2}{16(\delta+1)^2}, \qquad g|_{X\setminus W} = f_2|_{X\setminus W} + \frac{\delta^2}{16(\delta+1)^2}$$

and

$$f|_W = g_{(x_1,x_2,\delta)}\big|_W, \qquad g|_W = g_{(x_1,x_3,\delta)}\big|_W.$$

We now prove that f and g satisfy the conditions i)–iv). Since $x_1, x_2, x_3 \in W$, i) holds. To prove ii), let $(z,w) \in \widetilde{X} \setminus \{(x_1,x_2),(x_2,x_1)\}$. If $z, w \in W$, we have

$$\frac{|f(z) - f(w)|}{d(z,w)} = \frac{\left|g_{(x_1,x_2,\delta)}(z) - g_{(x_1,x_2,\delta)}(w)\right|}{d(z,w)} < 1;$$

if $z, w \in X \setminus W$,

$$\frac{|f(z) - f(w)|}{d(z,w)} = \frac{|f_1(z) - f_1(w)|}{d(z,w)} \leq \mathrm{Lip}(f_1) = \mathrm{Lip}(h_1) < 1;$$

if $z \in X \setminus W$, $w \in W$ and $g_{(x_1,x_2,\delta)}(w) \geq f_1(z) + \delta^2/16(\delta+1)^2$, we have

$$\begin{aligned}\frac{|f(z) - f(w)|}{d(z,w)} &= \frac{\left|f_1(z) + \frac{\delta^2}{16(\delta+1)^2} - g_{(x_1,x_2,\delta)}(w)\right|}{d(z,w)} = \frac{g_{(x_1,x_2,\delta)}(w) - \left(f_1(z) + \frac{\delta^2}{16(\delta+1)^2}\right)}{d(z,w)} \\ &\leq \frac{g_{(x_1,x_2,\delta)}(w)}{d(z,w)} = \frac{\left|g_{(x_1,x_2,\delta)}(w) - g_{(x_1,x_2,\delta)}(z)\right|}{d(z,w)} < 1;\end{aligned}$$

if $z \in X \setminus W$, $w \in W$ and $0 \leq g_{(x_1,x_2,\delta)}(w) < f_1(z) + \delta^2/16(\delta+1)^2$, we have

$$\frac{|f(z) - f(w)|}{d(z,w)} = \frac{f_1(z) + \frac{\delta^2}{16(\delta+1)^2} - g_{(x_1,x_2,\delta)}(w)}{d(z,w)} \leq \frac{f_1(z) - f_1(w)}{d(z,w)} \leq \mathrm{Lip}(f_1) < 1;$$

and if $z \in X \setminus W$, $w \in W$ and $g_{(x_1,x_2,\delta)}(w) < 0$, then $d(w,x_2) < \delta$ and

$$\frac{|f(z)-f(w)|}{d(z,w)} = \frac{f_1(z)+\frac{\delta^2}{16(\delta+1)^2}-g_{(x_1,x_2,\delta)}(w)}{d(z,w)}$$
$$\leq \frac{\frac{\delta^2}{8(\delta+1)^2}+\frac{\delta}{2}}{d(z,w)} < \frac{\delta}{d(z,w)} = \frac{2\delta-\delta}{d(z,w)} < \frac{d(z,x_2)-d(w,x_2)}{d(z,w)} \leq 1.$$

Similarly, it is proved that g satisfies iii). Finally, given $n\in\mathbb{N}$ and $x\in C$ with $(f(x),g(x))=(f(r_n),g(r_n))$, it follows that $(f_0(x),g_0(x))=(f_0(r_n),g_0(r_n))$, hence $x=r_n$ and this proves iv). □

We shall also need the following result.

Lemma 4.3. *Let X and Y be uniformly concave complete pointed metric spaces and let Δ be a 2-local isometry from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$. Let $Y_0\subseteq Y$ be as in Theorem 3.1 and assume $|Y_0|\geq 3$. If $Y_0\neq Y$, $y\in Y\setminus Y_0$ and $y_1\in Y_0$, then there exists a sequence $\{z_n\}$ of points in Y_0 such that $d_Y(z_n,y_1)=d_Y(y,y_1)$ for all $n\in\mathbb{N}$, and $d_Y(z_n,z_m)\geq d_Y(y,Y_0)>0$ for all $n,m\in\mathbb{N}$ with $n\neq m$.*

Proof. Let $y_1,y_2,y_3\in Y_0$ be three distinct points and denote $x_k=\phi(y_k)$ for $k=1,2,3$. We shall first construct an isometry φ of Y onto Y_0 for which $\varphi(y_k)=y_k$ for $k=1,2,3$. The argument is similar to the proof of Step 11. We have

$$\Delta(f)(z_1)-\Delta(f)(z_2)=\lambda a^{-1}\left(f(\phi(z_1))-f(\phi(z_2))\right)$$

for all $z_1,z_2\in Y_0$ and $f\in\mathrm{Lip}_0(X)$, with Y_0,λ,a,ϕ being as in the statement of Theorem 3.1. Consider $g=h_{(x_1,x_2)},h=h_{(x_1,x_3)}\in\mathrm{Lip}_0(X)$ as in Lemma 2.2. By Step 1, there exist a number $\lambda_{g,h}\in S_{\mathbb{K}}$ and a surjective $a_{g,h}$-dilation $\phi_{g,h}\colon Y\to X$ such that

$$\frac{\Delta(f)(z_1)-\Delta(f)(z_2)}{d_Y(z_1,z_2)}=\lambda_{g,h}\frac{f(\phi_{g,h}(z_1))-f(\phi_{g,h}(z_2))}{d_X(\phi_{g,h}(z_1),\phi_{g,h}(z_2))}$$

for all $(z_1,z_2)\in\widetilde{Y}$ and $f\in\{g,h\}$. Hence

$$\lambda\frac{h_{(x_1,x_k)}(x_1)-h_{(x_1,x_k)}(x_k)}{d_X(x_1,x_k)}=\frac{\Delta(h_{(x_1,x_k)})(y_1)-\Delta(h_{(x_1,x_k)})(y_k)}{d_Y(y_1,y_k)}$$
$$=\lambda_{g,h}\frac{h_{(x_1,x_k)}(\phi_{g,h}(y_1))-h_{(x_1,x_k)}(\phi_{g,h}(y_k))}{d_X(\phi_{g,h}(y_1),\phi_{g,h}(y_k))}$$

for $k=2,3$. By Lemma 2.2, it follows that

$$(\phi_{g,h}(y_1),\phi_{g,h}(y_2))\in\{(x_1,x_2),(x_2,x_1)\}$$

and

$$(\phi_{g,h}(y_1),\phi_{g,h}(y_3))\in\{(x_1,x_3),(x_3,x_1)\}.$$

Therefore $\phi_{g,h}(y_k)=x_k$ for $k=1,2,3$. Define now the mapping $\varphi=\phi^{-1}\circ\phi_{g,h}\colon Y\to Y$. Clearly, $Y_0=\varphi(Y)$ and

$$d_Y(\varphi(z_1),\varphi(z_2))=a^{-1}d_X(\phi_{g,h}(z_1),\phi_{g,h}(z_2))=a^{-1}a_{g,h}d_Y(z_1,z_2)$$

for all $z_1,z_2\in Y$. In particular, we have

$$d_Y(\varphi(y_1),\varphi(y_2)) = a^{-1}a_{g,h}d_Y(y_1,y_2),$$

but

$$d_Y(\varphi(y_1),\varphi(y_2)) = d_Y(\phi^{-1}(\phi_{g,h}(y_1),\phi^{-1}(\phi_{g,h}(y_2))) = d_Y(\phi^{-1}(x_1),\phi^{-1}(x_2)) = d_Y(y_1,y_2).$$

Therefore $a^{-1}a_{g,h} = 1$, and thus φ is an isometry. Note that $\varphi(y_k) = y_k$ for $k = 1,2,3$.

Finally, assume $Y_0 \neq Y$ and let $y \in Y \setminus Y_0$ and $y_1 \in Y_0$. Define $z_n = \varphi^n(y) \in Y_0$ for all $n \in \mathbb{N}$. Clearly, $d_Y(z_n,y_1) = d_Y(\varphi^n(y),\varphi^n(y_1)) = d_Y(y,y_1)$ for all $n \in \mathbb{N}$, and $d_Y(z_n,z_m) = d_Y(\varphi^{n-m}(y),y) \geq d_Y(y,Y_0)$ for all $n,m \in \mathbb{N}$ with $n > m$. □

We are now in position to prove the announced result.

Theorem 4.4. *Let X and Y be uniformly concave complete pointed metric spaces and let Δ be a 2-local isometry from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$. Assume that X is also separable. Then $Y_0 = Y$ and Δ is a linear isometry from $\mathrm{Lip}_0(X)$ onto $\mathrm{Lip}_0(Y)$.*

Proof. By Theorem 3.1, there are a nonempty subspace Y_0 of Y which is isometric to Y, a number $\lambda \in S_{\mathbb{K}}$ and a surjective a-dilation $\phi\colon Y_0 \to X$ such that

$$\Delta(f)(y_1) - \Delta(f)(y_2) = \lambda a^{-1}\left(f(\phi(y_1)) - f(\phi(y_2))\right)$$

for all $y_1,y_2 \in Y_0$ and $f \in \mathrm{Lip}_0(X)$.

Since Y, X and Y_0 have the same cardinality, if X is finite, then $Y_0 = Y$, and we have finished by Theorem 2.1.

Suppose now that X is not finite. Assume, on the contrary, that there exists a point $y \in Y \setminus Y_0$. Given $y_1 \in Y_0$, by Lemma 4.3 there are two distinct points $y_2, y_3 \in Y_0$ for which $d_Y(y_2,y_1) = d_Y(y,y_1) = d_Y(y_3,y_1)$. Take $\delta = (a/6)d_Y(y,Y_0)$ and denote $x_1 = \phi(y_1)$, $x_2 = \phi(y_2)$ and $x_3 = \phi(y_3)$. Consider the set

$$C = \{z \in X\colon d_X(z,x_1) \geq d_X(x_1,x_2),\ d_X(z,x_2) \geq 3\delta,\ d_X(z,x_3) \geq 3\delta\}.$$

If $\{z_n\}$ is the sequence given in Lemma 4.3, it is easy to see that $X\setminus C$ contains at most two points of $\{\phi(z_n)\}$. Therefore C is infinite. Let $R = \{r_n\colon n \in \mathbb{N}\}$ be an infinite countable dense subset of pairwise distinct points of C. Apply Lemma 4.2 to the points x_1,x_2,x_3 and get the functions $f,g \in \mathrm{Lip}_0(X)$. By Theorem 2.1 and the definition of 2-local isometry, there are a number $\lambda_{f,g} \in S_{\mathbb{K}}$ and a surjective $a_{f,g}$-dilation $\phi_{f,g}\colon Y \to X$ such that

$$\Delta(h)(y) = \lambda_{f,g}a_{f,g}^{-1}\left(h\left(\phi_{f,g}(y)\right) - h\left(\phi_{f,g}(e_Y)\right)\right)$$

for all $y \in Y$ and $h \in \{f,g\}$. Define the mapping $\varphi = \phi^{-1}\circ\phi_{f,g}\colon Y \to Y_0$. Similarly as in the proof of Lemma 4.3, it is proved that $a_{f,g} = a$ and φ is an isometry with $\varphi(y_k) = y_k$ for $k = 1,2,3$. Then $\phi_{f,g}(y_k) = \phi(\varphi(y_k)) = x_k$ for $k = 1,2,3$, and we have

$$\begin{aligned}\lambda_{f,g} &= \lambda_{f,g}\frac{f(x_1)-f(x_2)}{d_X(x_1,x_2)} = \lambda_{f,g}a^{-1}\frac{f(\phi_{f,g}(y_1)) - f(\phi_{f,g}(y_2))}{d_Y(y_1,y_2)}\\ &= \frac{\Delta(f)(y_1)-\Delta(f)(y_2)}{d_Y(y_1,y_2)} = \lambda\frac{f(x_1)-f(x_2)}{d_X(x_1,x_2)} = \lambda.\end{aligned}$$

We now check that $\varphi(\phi^{-1}(r_n)) = \phi^{-1}(r_n)$ for all $n \in \mathbb{N}$. Indeed, given $n \in \mathbb{N}$, we have

$$\begin{aligned}\lambda a^{-1}(f(r_n)-f(x_1)) &= \Delta(f)(\phi^{-1}(r_n))) - \Delta(f)(y_1) \\ &= \lambda_{f,g} a_{f,g}^{-1}\left(f(\phi_{f,g}(\phi^{-1}(r_n))) - f(\phi_{f,g}(y_1))\right) \\ &= \lambda a^{-1}\left(f(\phi_{f,g}(\phi^{-1}(r_n))) - f(x_1)\right),\end{aligned}$$

which implies $f(\phi_{f,g}(\phi^{-1}(r_n))) = f(r_n)$. Similarly, we obtain $g(\phi_{f,g}(\phi^{-1}(r_n))) = g(r_n)$ for all $n \in \mathbb{N}$. It follows that $\phi_{f,g}(\phi^{-1}(r_n)) = r_n$, and thus $\varphi(\phi^{-1}(r_n)) = \phi^{-1}(r_n)$ for all $n \in \mathbb{N}$.

Observe that $\phi_{f,g}(y) \in C$ because

$$d_X(\phi_{f,g}(y), x_k) = d_X(\phi_{f,g}(y), \phi_{f,g}(y_k)) = a\, d_Y(y, y_k) \geq \begin{cases} a\, d_Y(y_1, y_2) = d_X(x_1, x_2) & \text{if} \quad k = 1, \\ a\, d_Y(y, Y_0) = 6\delta & \text{if} \quad k = 2, 3 \end{cases}$$

Therefore, by the density of $\{r_n : n \in \mathbb{N}\}$ in C, there is $n \in \mathbb{N}$ such that

$$d_Y(\varphi(y), \phi^{-1}(r_n)) = a^{-1} d_X(\phi_{f,g}(y), r_n) < d_Y(y, \varphi(y))/2.$$

Finally, since

$$\begin{aligned} d_Y(y, \varphi(y)) &\leq d_Y(y, \phi^{-1}(r_n)) + d_Y(\varphi(y), \phi^{-1}(r_n)) \\ &= d_Y(\varphi(y), \varphi(\phi^{-1}(r_n))) + d_Y(\varphi(y), \phi^{-1}(r_n)) \\ &= 2 d_Y(\varphi(y), \phi^{-1}(r_n)) < d_Y(y, \varphi(y)), \end{aligned}$$

we arrive at a contradiction. This proves that $Y_0 = Y$. □

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