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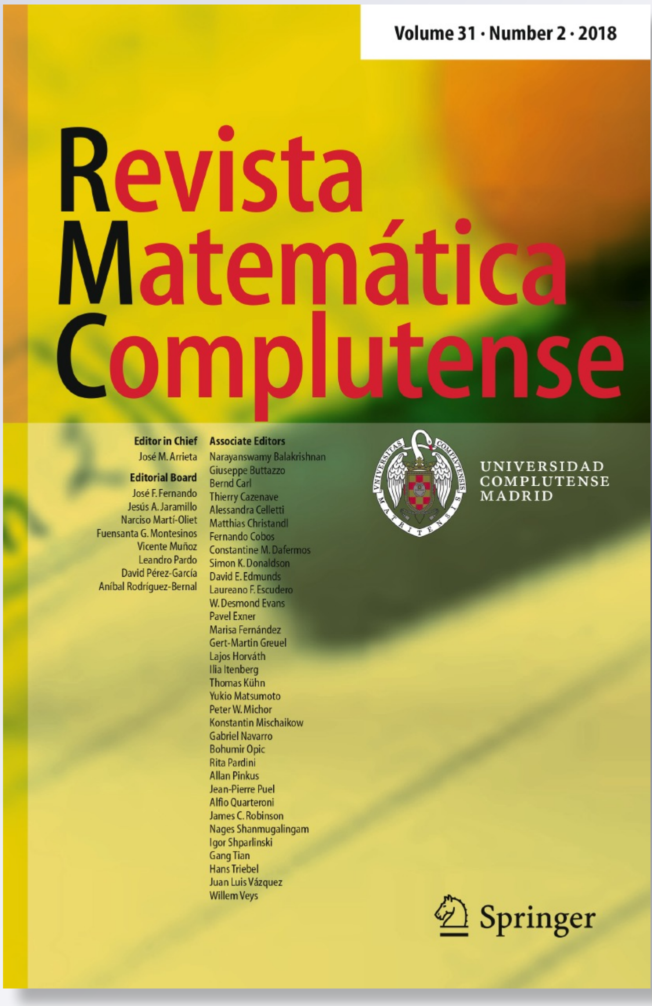


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# Isometric representations of weighted spaces of little Lipschitz functions

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**Abstract** Given a compact pointed metric space $X$ and a weight $v$ on the complement of the diagonal set in $X \times X$, we prove that the Banach space $\mathrm{lip}_v(X)$ of all weighted little Lipschitz scalar-valued functions on $X$ vanishing at the basepoint, equipped with the weighted Lipschitz norm, embeds almost isometrically into $c_0$. This result has many consequences on the structure of those Banach spaces and their duals. Moreover, we prove that this isomorphism can never be an isometric embedding whenever $X$ is a $\mathbb{T}$-balanced subset containing 0 and compact for some metrizable topology of a complex Banach space and $v$ is a radial 0-weight.

**Keywords** Lipschitz function · Little Lipschitz function · Weighted Banach space

**Mathematics Subject Classification** 46E15 · 46A20

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# 1 Introduction

Let $(X, d)$ be a pointed metric space with a basepoint $e$, let $\mathbb{K}$ be the field of real or complex numbers and let $\widetilde{X}$ be the set $\{(x, y) \in X \times X : x \neq y\}$.

The *Lipschitz space* $\mathrm{Lip}_0(X)$ is the Banach space of all functions $f$ from $X$ to $\mathbb{K}$ for which $f(e) = 0$ such that

$$\sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : (x, y) \in \widetilde{X} \right\} < \infty,$$

endowed with the *Lipschitz norm*:

$$\mathrm{Lip}(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : (x, y) \in \widetilde{X} \right\}.$$

The *little Lipschitz space* $\mathrm{lip}_0(X)$ is the closed linear subspace of $\mathrm{Lip}_0(X)$ consisting of all those functions $f$ which satisfy the property:

$$\forall \varepsilon > 0, \quad \exists \delta > 0 : (x, y) \in \widetilde{X},\ d(x, y) < \delta \quad \Rightarrow \quad \frac{|f(x) - f(y)|}{d(x, y)} < \varepsilon.$$

Sometimes, we will write $\mathrm{Lip}_0(X, d)$ and $\mathrm{Lip}(f, d)$ instead of $\mathrm{Lip}_0(X)$ and $\mathrm{Lip}(f)$, respectively. We denote by $\mathbb{T}$ and $\mathbb{D}$ the unit sphere and the unit closed ball of $\mathbb{C}$, respectively.

Let us recall that a function $\omega : [0, \infty) \to [0, \infty)$ is a *gauge* if it is continuous, increasing and subadditive with $\omega(0) = 0$ and $\omega(t) \geq t$ for all $0 \leq t \leq 1$. A gauge $\omega$ is *normalized* if $\omega(1) = 1$ and *nontrivial* if $\lim_{t \to 0} \omega(t)/t = \infty$. The most important examples of normalized nontrivial gauges are $\omega(t) = t^\alpha$ and $\omega(t) = \max\{t, t^\alpha\}$ with $\alpha \in (0, 1)$. A normalized nontrivial gauge permits to replace the metric $d$ on $X$ with the new metric $\omega \circ d$ and define so the *generalized spaces of Hölder functions* $\mathrm{Lip}_0(X, \omega \circ d)$. In the special case $\omega(t) = t^\alpha$, we write $\mathrm{Lip}_0^{(\alpha)}(X)$.

The isomorphic representation of $\mathrm{lip}_0^{(\alpha)}$ spaces with $0 < \alpha < 1$ has been widely studied. See, for example, the paper [23] by Kalton and the references therein. A known result of Bonic et al. [5] (corrected in [28]) asserts that $\mathrm{lip}_0^{(\alpha)}(X)$ is isomorphic to $c_0$ whenever $X$ is a compact subset of the Euclidean space $\mathbb{R}^n$. Weaver [28, page 98] asked whether this is true for every compact metric space. In [23, Section 8], Kalton answered this question negatively by showing that if $X$ is a compact convex subset of $\ell_2$ containing 0, then $\mathrm{lip}_0^{(\alpha)}(X)$ is isomorphic to $c_0$ if and only if $X$ is finite-dimensional. Moreover, he conjectured that if $X$ is an infinite-dimensional compact convex subset of any Banach space, then $\mathrm{lip}_0^{(\alpha)}(X)$ cannot be isomorphic to $c_0$, and obtained some general results of this type.

On the other hand, the isometric representation of those spaces was also dealt in Wulbert's article [29]. If $X$ is a compact metric space and $0 < \alpha \leq 1$, his surprising result (corrected in [3]) states that a point separating space $\mathrm{lip}_0^{(\alpha)}(X)$ can only be isometrically isomorphic to $c_0$ if $\alpha = 1$ and $X$ is isometric to a nowhere dense subset of $\mathbb{R}$.

In this paper, we will prove that $\mathrm{lip}_0^{(\alpha)}(X)$ $(0 < \alpha < 1)$ is not isometrically isomorphic to a subspace of $c_0$ whenever $X$ is a $\mathbb{T}$-balanced subset containing 0 and compact for some metrizable topology of a complex Banach space. We say that $X$ is $\mathbb{T}$-*balanced* if $\lambda x \in X$ for all $x \in X$ and $\lambda \in \mathbb{T}$. In particular, we show that $\mathrm{lip}_0^{(\alpha)}(\mathbb{T} \cup \{0\})$ and $\mathrm{lip}_0^{(\alpha)}(\mathbb{D})$ are not isometrically isomorphic to subspaces of $c_0$. In fact, we will prove much more general results in the context of weighted Banach spaces of Lipschitz functions that we present below. The isometric structure of weighted spaces of analytic functions and harmonic functions was studied by Boyd and the second author in a series of papers (see [8–10] and the references therein).

Let us recall that a function $v\colon \widetilde{X} \to \mathbb{R}$ is called a *weight* on $\widetilde{X}$ if it is (strictly) positive and continuous. The *weighted Lipschitz space* $\mathrm{Lip}_v(X)$ is the Banach space of all functions $f$ in $\mathrm{Lip}_0(X)$ such that

$$\sup\left\{v(x,y)\frac{|f(x)-f(y)|}{d(x,y)} : (x,y) \in \widetilde{X}\right\} < \infty,$$

under the *weighted Lipschitz norm*:

$$\mathrm{Lip}_v(f) = \sup\left\{v(x,y)\frac{|f(x)-f(y)|}{d(x,y)} : (x,y) \in \widetilde{X}\right\}.$$

The *weighted little Lipschitz space* $\mathrm{lip}_v(X)$ is the closed linear subspace of $\mathrm{Lip}_v(X)$ formed by all those functions $f$ such that

$$\lim_{t\to 0}\sup\left\{v(x,y)\frac{|f(x)-f(y)|}{d(x,y)} : 0 < d(x,y) < t\right\} = 0.$$

Thus $\mathrm{Lip}_v(X)$ may be regarded as all Lipschitz scalar-valued functions $f$ on $X$ vanishing at $e$ such that $|f(x) - f(y)|/d(x,y)$ satisfies a growth condition of order $O(1/v(x,y))$ while $\mathrm{lip}_v(X)$ are those functions for which $|f(x)-f(y)|/d(x,y)$ has a growth rate of order $o(1/v(x,y))$.

Note that for $v$ being the function constantly 1 on $\widetilde{X}$, the spaces $\mathrm{Lip}_v(X)$ and $\mathrm{lip}_v(X)$ are just $\mathrm{Lip}_0(X)$ and $\mathrm{lip}_0(X)$, respectively. Therefore, weighted spaces of Lipschitz functions recover Lipschitz spaces. However, weighted Lipschitz spaces provide a much refined way to see any Lipschitz space in the following sense: we can use the weights in order to keep the original metric on the space and then to see a Lipschitz space as a weighted space of Lipschitz functions with respect to the original metric. More concretely, given a pointed metric space $(X, d)$, we can identify any space $\mathrm{Lip}_0(X, d')$, being $d'$ a metric on $X$, with the space $\mathrm{Lip}_v(X)$ taking the weight $v = d/d'$ on $\widetilde{X}$. From this approach, it is possible to give a distinguished representation as a $\mathrm{Lip}_v$ space of this type for the generalized spaces of Hölder functions: if $\omega$ is a gauge, notice that $\mathrm{Lip}_0(X, \omega \circ d) = \mathrm{Lip}_v(X)$ and $\mathrm{Lip}(f, \omega \circ d) = \mathrm{Lip}_v(f)$ for all $f \in \mathrm{Lip}_0(X, \omega \circ d)$, being $v = d/(\omega \circ d)$ on $\widetilde{X}$.

Little is known about the Banach spaces $\mathrm{Lip}_v(X)$ and $\mathrm{lip}_v(X)$, except their duality theory addressed in [20]. Moreover, those spaces appear closely connected to the classical strict topology $\beta$ on $\mathrm{Lip}_0(X)$ (see [19, Definition 3.2]) introduced by Buck

[11]. We will prove in this note that the space $\mathrm{lip}_v(X)$ is isomorphic to a closed subspace of $c_0$ whenever $X$ is compact. In fact, $\mathrm{lip}_v(X)$ is *almost isometric* to a subspace of $c_0$. This means that, for any $\epsilon > 0$, there exists a closed subspace $X_\epsilon$ of $c_0$ such that $d(\mathrm{lip}_v(X), X_\epsilon) \leq 1+\epsilon$, where

$$d(\mathrm{lip}_v(X), X_\epsilon) = \inf\left\{\|T\|\left\|T^{-1}\right\| \mid T\colon \mathrm{lip}_v(X) \to X_\epsilon \text{ is an onto isomorphism}\right\}$$

denotes the *Banach–Mazur distance* of $\mathrm{lip}_v(X)$ and $X_\epsilon$. This theorem extends the related result for $\mathrm{lip}_0$ spaces (see [23, Theorem 6.6]). Analogous results were stated for the little Bloch space by Kalton and Werner [22], weighted spaces of holomorphic functions by Bonet and Wolf [4], and weighted spaces of harmonic functions by Jordá and Zarco [21]. We will apply our result to obtain a series of properties on the Banach spaces $\mathrm{lip}_v(X)$ and their duals (see Corollaries 1–11).

## 2 Almost isometric representations

We begin with the main result of this section. We borrow the strategy to prove it from the proof of Theorem 6.6 in the work [23] of Kalton, who in turn acknowledges that Yoav Benyamini showed this proof to him.

**Theorem 1** *Let $X$ be a compact pointed metric space and let $v$ be a weight on $\widetilde{X}$. Then $\mathrm{lip}_v(X)$ is isomorphic to a closed subspace of $c_0$. In fact, $\mathrm{lip}_v(X)$ embeds almost isometrically into $c_0$. More precisely, for any $\epsilon > 0$, there exists a linear mapping $T\colon \mathrm{lip}_v(X) \to c_0$ such that*

$$(1-\epsilon)\mathrm{Lip}_v(f) \leq \|T(f)\|_\infty \leq \mathrm{Lip}_v(f)$$

*for all $f \in \mathrm{lip}_v(X)$.*

*Proof* We may suppose $\epsilon \in ]0,1[$. Consider the set $X \times X$ with the metric

$$d((x,y),(x',y')) = \max\left\{d(x,x'), d(y,y')\right\}.$$

For $(x,y) \in X \times X$ and $r > 0$, we denote

$$D((x,y),r) = \left\{(x',y') \in X \times X\colon d((x',y'),(x,y)) \leq r\right\}.$$

For each $k \in \mathbb{Z}$, define the set

$$C_k = \left\{(x,y) \in X \times X\colon d(x,y) \leq 2^k\right\}.$$

Clearly, $C_k$ is a compact subset of $X^2$ and $C_k \subset C_{k+1}$ for all $k \in \mathbb{Z}$. For each $k \in \mathbb{Z}$, we denote

$$m_k = \min\left\{v(x,y)\colon (x,y) \in C_k\right\},$$
$$M_k = \max\left\{v(x,y)\colon (x,y) \in C_k\right\}.$$

For each $f \in \mathrm{lip}_v(X)$ and $k \in \mathbb{Z}$, note that

$$m_k \mathrm{Lip}^{(k)}(f) \leq \mathrm{Lip}_v^{(k)}(f),$$

where

$$\mathrm{Lip}^{(k)}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} : (x,y) \in C_k\right\},$$
$$\mathrm{Lip}_v^{(k)}(f) = \sup\left\{v(x,y)\frac{|f(x) - f(y)|}{d(x,y)} : (x,y) \in C_k\right\}.$$

For each $k \in \mathbb{Z}$, consider now the compact set

$$D_k = \left\{(x,y) \in X \times X : 2^{k-1} \leq d(x,y) \leq 2^k\right\}.$$

It is clear that

$$D_k \subset \bigcup_{(x,y)\in D_k} \left\{(x',y') \in \widetilde{X} : d((x',y'),(x,y)) < \delta_k,\ \left|v(x',y') - v(x,y)\right| < \delta_k\right\},$$

where $\delta_k$ is chosen satisfying that

$$0 < \delta_k < 2^{k-3}\epsilon, \qquad \delta_k\left(\frac{1}{m_k} + \frac{M_k}{2^{k-2}m_{k+3}}\right) < \frac{\epsilon}{2}.$$

Since $D_k$ is compact, there is a finite set $F_k \subset D_k$ such that

$$D_k \subset \bigcup_{(x,y)\in F_k} \left\{(x',y') \in \widetilde{X} : d((x',y'),(x,y)) < \delta_k,\ \left|v(x',y') - v(x,y)\right| < \delta_k\right\}.$$

Then the set $F = \cup_{k\in\mathbb{Z}} F_k$ is countable.

Define now the mapping $T : \mathrm{lip}_v(X) \to \mathbb{K}^F$ by

$$T(f) = \left(v(x,y)\frac{f(x) - f(y)}{d(x,y)}\right)_{(x,y)\in F}.$$

We claim that $T(f) \in c_0(F)$ for each $f \in \mathrm{lip}_v(X)$. Indeed, let $f \in \mathrm{lip}_v(X)$ and $\epsilon' > 0$. Hence there exists $\delta > 0$ such that $v(x,y)|f(x) - f(y)|/d(x,y) < \epsilon'$ whenever $0 < d(x,y) < \delta$. Let $m \in \mathbb{N}$ be such that $2^{-m} < \delta$ and let $k \in \mathbb{Z}$ be with $k \leq -m$. If $(x,y) \in F_k$, then $d(x,y) \leq 2^k < \delta$, hence $|T(f)(x,y)| < \epsilon'$, and this proves our claim.

Clearly, $\|T(f)\|_\infty \leq \mathrm{Lip}_v(f)$ for all $f \in \mathrm{lip}_v(X)$. Our next aim is to show that $(1-\epsilon)\mathrm{Lip}_v(f) \leq \|T(f)\|_\infty$ for all $f \in \mathrm{lip}_v(X)$. For it, let $f \in \mathrm{lip}_v(X)$ and $(x,y) \in \widetilde{X}$. If $(x,y) \in F$, we trivially have

$$v(x,y)\frac{|f(x) - f(y)|}{d(x,y)} \leq \|T(f)\|_\infty \leq \frac{\epsilon}{2}\mathrm{Lip}_v(f) + \left(1 - \frac{\epsilon}{2}\right)^{-1}\|T(f)\|_\infty.$$

Assume that $(x, y) \notin F$. We can find $k \in \mathbb{Z}$ such that $(x, y) \in D_k$ and therefore there exists a point $(z, w) \in F_k$ with $(z, w) \neq (x, y)$ such that $d((z, w), (x, y)) < \delta_k$ and $|v(z, w) - v(x, y)| < \delta_k$. Since

$$d(z, y) \leq d(z, x) + d(x, y) \leq \delta_k + 2^k < 2^{k-1}\epsilon + 2^k < 2^{k+1}$$

we have $(x, z) \in D((x, y), 2^{k+1})$. Similarly, $(w, y) \in D((x, y), 2^{k+1})$. Besides, any $(a, b) \in D((x, y), 2^{k+1})$ satisfies

$$d(a, b) \leq d(a, x) + d(x, y) + d(y, b) \leq 2^{k+1} + 2^k + 2^{k+1} < 2^{k+3},$$

and then $D((x, y), 2^{k+1}) \subset C_{k+3}$. Finally, note that

$$\begin{aligned} d(x, y) &\geq d(z, w) - d(z, x) - d(y, w) > d(z, w) - 2\delta_k \\ &> d(z, w) - \frac{\epsilon}{2}2^{k-1} \geq d(z, w)\left(1 - \frac{\epsilon}{2}\right). \end{aligned}$$

We next assume $z \neq x$ and $w \neq y$. In the others two cases, $z \neq x$ and $w = y$, or $z = x$ and $w \neq y$, the following inequality can be obtained similarly. We have

$$\begin{aligned} v(x, y)\frac{|f(x) - f(y)|}{d(x, y)} &\leq |v(x, y) - v(z, w)|\,\frac{|f(x) - f(y)|}{d(x, y)} + v(z, w)\frac{|f(x) - f(y)|}{d(x, y)} \\ &\leq \delta_k\frac{\mathrm{Lip}_v^{(k)}(f)}{m_k} + v(z, w)\left(\frac{|f(x) - f(z)|}{d(x, y)} + \frac{|f(w) - f(y)|}{d(x, y)}\right) \\ &\quad + v(z, w)\frac{|f(z) - f(w)|}{d(x, y)} \\ &\leq \delta_k\frac{\mathrm{Lip}_v^{(k)}(f)}{m_k} + \frac{M_k\delta_k}{d(x, y)}\left(\frac{|f(x) - f(z)|}{d(x, z)} + \frac{|f(w) - f(y)|}{d(w, y)}\right) \\ &\quad + \left(1 - \frac{\epsilon}{2}\right)^{-1} v(z, w)\frac{|f(z) - f(w)|}{d(z, w)} \\ &\leq \delta_k\frac{\mathrm{Lip}_v^{(k)}(f)}{m_k} + \frac{M_k\delta_k}{2^{k-1}}\frac{2\mathrm{Lip}_v^{(k+3)}(f)}{m_{k+3}} + \left(1 - \frac{\epsilon}{2}\right)^{-1} v(z, w)\frac{|f(z) - f(w)|}{d(z, w)} \\ &\leq \delta_k\left(\frac{1}{m_k} + \frac{M_k}{2^{k-2}m_{k+3}}\right)\mathrm{Lip}_v(f) + \left(1 - \frac{\epsilon}{2}\right)^{-1}\|T(f)\|_\infty \\ &\leq \frac{\epsilon}{2}\mathrm{Lip}_v(f) + \left(1 - \frac{\epsilon}{2}\right)^{-1}\|T(f)\|_\infty, \end{aligned}$$

and taking supremum over $(x, y)$, we infer that

$$\left(1 - \frac{\epsilon}{2}\right)\mathrm{Lip}_v(f) \leq \left(1 - \frac{\epsilon}{2}\right)^{-1}\|T(f)\|_\infty.$$

This implies that $(1 - \epsilon)\mathrm{Lip}_v(f) \leq \|T(f)\|_\infty$, as required. □

*Remark.* Observe that the proof of Theorem 1 can be adapted to show that $\mathrm{Lip}_v(X)$ is almost isometrically isomorphic to a subspace of $\ell_\infty$. Furthermore, note that if $X$ is a compact pointed metric space, then $\widetilde{X}$ has a countable dense subset $D$ and the map

$$f \mapsto \left( v(x, y) \frac{f(x) - f(y)}{d(x, y)} \right)_{(x,y)\in D}$$

is an isometric embedding of $\mathrm{Lip}_v(X)$ into $\ell_\infty(D)$.

In the rest of this section, we give some corollaries from Theorem 1. The first corollary gathers some simple, but interesting consequences.

**Corollary 1** *Let $X$ be a compact pointed metric space, let $v$ be a weight on $\widetilde{X}$ and let $A_v(X)$ be a closed infinite-dimensional subspace of* $\mathrm{lip}_v(X)$.

(i) *$A_v(X)$ has a complemented subspace almost isometric to $c_0$.*
(ii) *$A_v(X)$ is not reflexive.*
(iii) *$A_v(X)$ is not weakly sequentially complete.*
(iv) *$A_v(X)$ fails the Radon–Nikodým property.*
(v) *$A_v(X)$ is not complemented in $A_v(X)^{**}$.*
(vi) *$A_v(X)$ is not isomorphic with a dual space.*
(vii) *$A_v(X)$ is not injective.*
(viii) *$A_v(X)$ is not a Grothendieck space.*

*Proof* (i) By Theorem 1, $A_v(X)$ is a closed infinite-dimensional subspace of $c_0$. Hence $A_v(X)$ contains a complemented subspace isomorphic to $c_0$ by [24, Proposition 2.a.2]. Now we obtain (i) by [16, Proposition 1].

(ii), (iii) and (iv) follow from the fact that $c_0$ is not reflexive, is not weak sequentially complete and does not have the Radon–Nikodým property and that these properties are stable by taking closed subspaces and are invariant under isomorphisms.

(v) and (vi) follow from [12, Proposition 2.4.5], and (vii) from [12, Proposition 2.5.7].

Since a Grothendieck space cannot contain a complemented copy of $c_0$, (i) shows that (viii) holds. □

In order to obtain some properties of the fixed-point theory for $\mathrm{lip}_v(X)$, we recall the following concepts.

A Banach space $X$ is said to *contain an asymptotically isometric copy of* $c_0$ if there is a null sequence $(\epsilon_n)$ in $(0, 1)$ and a sequence $(x_n)$ in $X$ such that

$$\sup_{n\in\mathbb{N}}(1 - \epsilon_n)\,|t_n| \le \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \le \sup_{n\in\mathbb{N}} |t_n|$$

for all $(t_n) \in c_0$.

A Banach space $X$ is said to *contain an asymptotically isometric copy of* $\ell_1$ if there is a null sequence $(\epsilon_n)$ in $(0, 1)$ and a sequence $(x_n)$ in $X$ such that

$$\sum_{n=1}^{\infty}(1 - \epsilon_n)\,|t_n| \le \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \le \sum_{n=1}^{\infty} |t_n|$$

for all $(t_n) \in \ell_1$.

Let $X$ be a Banach space. A mapping $T : C \subset X \to X$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The *fixed point set* of $T$ is $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$. We say that the space $X$ has the *fixed point property* if for every nonempty closed bounded convex subset $C$ of X and every nonexpansive mapping $T : C \to C$, we have $\mathrm{Fix}(T) \neq \emptyset$. A mapping $T : X \to X$ is said to be *asymptotically nonexpansive* if $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in X$ and for all $n \in \mathbb{N}$, where $(k_n)$ is a sequence of real numbers converging to 1.

A Banach space $X$ has the *generalized Gossez–Lami Dozo property* if, whenever $(x_n)$ is a weakly null sequence in $X$ that is not norm null, then $\liminf_n \|x_n\| < \limsup_n \limsup_m \|x_n - x_m\|$. A Banach space $X$ has *property asymptotic (P)* if, whenever $(x_n)$ is a weakly null sequence in $X$ that is not norm null, then $\liminf_n \|x_n\| < \mathrm{diama}\{x_n\}$, where $\mathrm{diama}\{x_n\} = \lim_n \mathrm{diam}\{x_k : k \geq n\}$ is the *asymptotic diameter* of the sequence $(x_n)$. Sims and Smyth [27] have shown that the generalized Gossez–Lami Dozo property and property asymptotic (P) are equivalent.

A nonempty bounded and convex subset $K$ of a Banach space $X$ is said to *have normal structure* if for every convex subset $H$ of $K$ that contains more than one point, there is a point $x_0 \in H$ such that $\sup\{\|x_0 - y\| : y \in H\} < \mathrm{diam}(H)$, where $\mathrm{diam}(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the *diameter* of H. A Banach space $X$ is said to *have normal structure* if every bounded convex subset of $X$ has normal structure. A Banach space $X$ is said to *have weak normal structure* if for each weakly compact convex set $K$ of $X$ that contains more than one point has normal structure.

**Corollary 2** *Let $X$ be a compact pointed metric space, let $v$ be a weight on $\widetilde{X}$ and let $A_v(X)$ be a closed infinite-dimensional subspace of* $\mathrm{lip}_v(X)$.

(i) *$A_v(X)$ contains an asymptotically isometric copy of $c_0$.*
(ii) *Neither $A_v(X)$ nor $A_v(X)^*$ has the fixed point property, and $A_v(X)^{**}$ cannot even be renormed to have the fixed point property.*
(iii) *$A_v(X)$ fails to have the fixed point property for asymptotically nonexpansive mappings.*
(iv) *$A_v(X)$ fails to have the generalized Gossez–Lami Dozo property.*
(v) *$A_v(X)$ fails to have weak normal structure.*
(vi) *$A_v(X)^*$ contains an asymptotically isometric copy of $\ell_1$.*
(vii) *$A_v(X)^{**}$ contains an isometric copy of $L_1([0, 1])$.*
(viii) *$A_v(X)^{**}$ contains an isometric copy of $C([0, 1])^*$.*

*Proof* (i) follow by [15, Theorem 2.5], (ii) by [15, Corollary 2.33], (iii) by [15, Theorem 2.27], (iv) by [15, Theorem 2.29], (v) by [15, Theorem 2.30] and (vi) by [15, Theorem 2.32]. Finally, [14, Theorem 2] shows that (vi), (vii) and (viii) are equivalent. □

Let us recall that a Banach space $X$ is said to be *almost reflexive* if every bounded sequence in $X$ has a weak Cauchy subsequence.

**Corollary 3** *Let $X$ be a compact pointed metric space and let $v$ be a weight on $\widetilde{X}$. Then* $\mathrm{lip}_v(X)$ *is almost reflexive.*

*Proof* By Rosenthal's $\ell_1$-theorem [26], a Banach space $X$ is almost reflexive if and only it does not contain a subspace isomorphic to $\ell_1$. Since $c_0$ does not contain a subspace isomorphic to $\ell_1$, so also does $\mathrm{lip}_v(X)$ by Theorem 1. □

Let us recall that a closed subspace $J$ of a Banach space $X$ is called an *M-ideal* if there is a closed subspace $J_0$ of $X^*$ such that $X^*$ is the $\ell_1$-sum $J^\perp \oplus_1 J_0$, where $J^\perp$ is the annihilator of $J$ in $X^*$. This notion was introduced by Alfsen and Effros in [1]. Given a Banach space $X$, we will denote by $B(X)$ the closed unit ball of $X$. By [17, Theorem I.2.2], a closed subspace $J$ of a Banach space $X$ is an M-ideal in $X$ if and only if $J$ satisfies the *(restricted) 3-ball property*, that is, for all $y_1, y_2, y_3 \in B(J)$, all $x \in B(X)$ and all $\epsilon > 0$, there is $y \in J$ such that $\|x + y_i - y\| \leq 1+\epsilon$ for $i = 1, 2, 3$.

**Corollary 4** *Let $X$ be a compact pointed metric space and let $v$ be a weight on $\widetilde{X}$. Then $\mathrm{lip}_v(X)$ is an M-ideal in its bidual.*

*Proof* Let $g_1, g_2, g_3 \in B(\mathrm{lip}_v(X))$, $f \in B(\mathrm{lip}_v(X)^{**})$ and $\epsilon > 0$. By Theorem 1, there exists a closed subspace $X_\epsilon$ of $c_0$ and an isomorphism $T\colon \mathrm{lip}_v(X) \to X_\epsilon$ such that $\|T\| \leq 1$ and $\left\|T^{-1}\right\| \leq 1+\epsilon$. Then $T(g_1), T(g_2), T(g_3)$ are in $B(X_\epsilon)$ and $T^{**}(f)$ in $B(X_\epsilon^{**})$. Since $c_0$ is an M-ideal in its bidual by [17, Examples III.1.4 (a)] and the property of being an M-ideal in its bidual passes to subspaces by [17, Theorem III.1.6 (a)], it follows that $X_\epsilon$ is an M-ideal in its bidual. Hence there exists $h \in X_\epsilon^{**}$ such that $\|T^{**}(f) + T(g_i) - h\| \leq 1 + \epsilon$ for $i = 1, 2, 3$. Therefore, $\left\|f + g_i - T^{-1}(h)\right\| \leq (1 + \epsilon)^2$ for $i = 1, 2, 3$. This proves that $\mathrm{lip}_v(X)$ is an M-ideal in its bidual. □

In [20], it is studied the biduality problem as to when $\mathrm{Lip}_v(X)$ is naturally isometrically isomorphic to the bidual of $\mathrm{lip}_v(X)$ for pointed compact metric spaces $X$, and was showed that this is the case whenever $\mathrm{lip}_v(X)$ is an M-ideal in $\mathrm{Lip}_v(X)$ and

$$\sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : f \in B(\mathrm{lip}_v(X))\right\} = \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : f \in B(\mathrm{Lip}_v(X))\right\}$$

for all $(x, y) \in \widetilde{X}$.

The next application is relative to the property (V) introduced by Pełczyński in [25]. A series $\sum_{n\in\mathbb{N}} x_n$ in a Banach space $X$ is called *weakly unconditionally Cauchy* (wuC for short) if it satisfies $\sum_{n=1}^{\infty} |f(x_n)| < \infty$ for all $f \in X^*$. A Banach space $X$ has *property (V)* if for any (bounded) non relatively weakly compact set $H \subset X^*$, there is a wuC-series $\sum_{n\in\mathbb{N}} x_n$ in $X$ such that $\sup_{f\in H} |f(x_n)|$ does not converge to zero.

**Corollary 5** *Let $X$ be a compact pointed metric space and let $v$ be a weight on $\widetilde{X}$. Then $\mathrm{lip}_v(X)$ has property (V).*

*Proof* It follows from Theorem 1 because $c_0$ has property (V) and this property is stable under isomorphisms and passes to closed subspaces [25]. □

We now give some applications for bounded linear operators on $\mathrm{lip}_v(X)$. A linear operator between Banach spaces $T\colon X \to Y$ is said to be:

(i) *compact* if $T$ maps bounded sets into relatively norm compact sets.
(ii) *completely continuous* if $T$ maps weakly compact sets into relatively norm compact sets.
(iii) *weakly compact* if $T$ maps bounded sets into relatively weakly compact sets.
(iv) *almost weakly compact* if, whenever $T$ has a bounded inverse on a closed subspace $M$ of $X$, then $M$ is reflexive.
(v) *strictly singular* if for each infinite dimensional subspace $X_1 \subset X$, $T$ is not an isomorphism on $X_1$.
(vi) *unconditionally converging* if $T$ does not fix a copy of $c_0$. It is said that $T$ fixes a copy of $c_0$ if $X$ contains a subspace isomorphic to $c_0$ such that the restriction of $T$ to this subspace is an isomorphism.
(vii) *conditionally weakly compact* if $T$ maps bounded sets into weakly precompact sets.
(viii) *strictly cosingular* if a closed subspace $Y_1 \subset Y$ has finite codimension whenever $Q \circ T$ is a surjection, where $Q$ is the quotient map from $Y$ onto $Y/Y_1$.

It is well-known that the sets of compact, weakly compact, conditionally weakly compact, strictly singular and strictly cosingular operators form ideals of operators.

Pełczyński [25] showed that $X$ having the property (V) is equivalent to the fact that any bounded linear operator $T$ from $X$ into a Banach space $Y$ is either weakly compact or fixes a copy of $c_0$. Hence, from Corollary 5, we deduce the two following results.

**Corollary 6** *Let $X$ be a compact pointed metric space, let $v$ be a weight on $\widetilde{X}$ and let $Y$ be a Banach space. Then every bounded linear operator $T : \mathrm{lip}_v(X) \to Y$ is either compact or fixes a copy of $c_0$.*

**Corollary 7** *Let $X$ be a compact pointed metric space, let $v$ be a weight on $\widetilde{X}$ and let $Y$ be a Banach space not containing $c_0$. Then every bounded linear operator $T : \mathrm{lip}_v(X) \to Y$ is compact.*

Let us recall that a Banach space $X$ is said to have the *Dunford–Pettis property* if for each Banach space $Y$, every weakly compact operator from $X$ to $Y$ is completely continuous. We refer to the survey [13] about the Dunford–Pettis property by Diestel.

**Corollary 8** *Let $X$ be a compact pointed metric space and let $v$ be a weight on $\widetilde{X}$. Then $\mathrm{lip}_v(X)$ has the Dunford–Pettis property.*

*Proof* For any set $\Gamma$, every closed linear subspace of $c_0(\Gamma)$ has the Dunford–Pettis property by [13, Theorem 4]. Hence the corollary follows from Theorem 1. □

Let us recall that a Banach space $X$ is said to have the *Schur property* if every weakly convergent sequence in $X$ is norm convergent in $X$.

**Corollary 9** *Let $X$ be a compact pointed metric space and let $v$ be a weight on $\widetilde{X}$. Then $\mathrm{lip}_v(X)^*$ has the Schur property and, therefore, the Dunford–Pettis property as well.*

*Proof* Since $\mathrm{lip}_v(X)$ is almost reflexive and has the Dunford–Pettis property by Corollaries 3 and 8, we apply [13, Theorem 3]. □

Since $c_0$ has not the Schur property and this property is stable under isomorphisms and passes to subspaces, we deduce from Corollary 9 that $\mathrm{lip}_v(X)^*$ does not contain a subspace isomorphic to $c_0$. Moreover, we have:

**Corollary 10** *Let $X$ be a compact pointed metric space and let $v$ be a weight on $\widetilde{X}$. Then every infinite-dimensional closed subspace of* $\mathrm{lip}_v(X)^*$ *contains a subspace isomorphic to $\ell_1$.*

*Proof* Since $\mathrm{lip}_v(X)^*$ has the Schur property by Corollary 9, the first assertion follows from Rosenthal's $\ell_1$-theorem [26]. □

We next characterize compact operators on $\mathrm{lip}_v(X)$.

**Corollary 11** *Let $X$ be a compact pointed metric space, let $v$ be a weight on $\widetilde{X}$ and let $Y$ be a Banach space. For any bounded linear operator $T\colon \mathrm{lip}_v(X) \to Y$, the following assertions are equivalent:*

(i) *$T$ is compact.*
(ii) *$T$ is completely continuous.*
(iii) *$T$ is weakly compact.*
(iv) *$T$ is almost weakly compact.*
(v) *$T$ is strictly singular.*
(vi) *$T$ is unconditionally converging.*
(vii) *$T$ is conditionally weakly compact.*
(viii) *$T$ is strictly cosingular.*

*Proof* It is known that every compact operator between Banach spaces satisfies all the properties (ii)–(viii). For the converse, note that the conditions (ii)–(vi) are equivalent by [18, Theorem 2.3] since $\mathrm{lip}_v(X)$ has properties (V) and Dunford–Pettis by Corollaries 5 and 8. Since $\mathrm{lip}_v(X)$ is also almost reflexive by Corollary 3, we have that (vi) implies (i) by [18, Corollary 2.5]. Finally, (vii) [respectively, (viii)] implies (i) by Corollary 6. □

## 3 Isometric representations

Concerning Theorem 1, our objective in this section is to prove that such an almost isometric isomorphism of $\mathrm{lip}_v(X)$ onto a closed closed of $c_0$ cannot become an isometry whenever $X$ is a $\mathbb{T}$-balanced subset containing 0 and compact for some metrizable topology of a complex Banach space and $v$ is a radial 0-weight on $\widetilde{X}$. In fact, we establish a more general result: $\mathrm{lip}_v(X)$ cannot be isometrically isomorphic to a linear subspace of $C(K)$ for some scattered compact Hausdorff space $K$. Let us recall that a closed subset of a topological space $K$ is called *perfect* if it has no isolated points, and if $K$ contains no perfect subsets it is said to be *scattered (dispersed)*.

For this we need to fix some notation and recall and introduce some concepts. Given a Banach space $E$ over $\mathbb{K}$, we denote by $S(E)$, $B(E)$ and $\mathrm{Ext}(B(E))$ the unit sphere of $E$, the unit closed ball of $E$ and the set of extreme points of $B(E)$, respectively. In the particular case $E = \mathbb{C}$, we will write $\mathbb{T}$ and $\mathbb{D}$ instead of $S(\mathbb{C})$ and $B(\mathbb{C})$.

**Definition 1** Let $X$ be a pointed metric space. A weight $v$ on $\widetilde{X}$ is said to be a 0-weight if for every $\epsilon > 0$, there exists $\delta > 0$ such that $v(x, y) < \epsilon$ whenever $0 < d(x, y) < \delta$.

Let $X$ be a nonempty subset of a complex Banach space. Let us recall that $X$ is said to be *balanced* if $\lambda x \in X$ for all $x \in X$ and $\lambda \in \mathbb{D}$. We will say that $X$ is $\mathbb{T}$-*balanced* if $\lambda x \in X$ for all $x \in X$ and $\lambda \in \mathbb{T}$. If $X$ is a $\mathbb{T}$-balanced subset of a complex Banach space containing 0 (in particular, if $X$ is a nonempty balanced subset), we can consider $X$ as a pointed metric space with the metric induced by its norm and basepoint 0.

**Definition 2** Let $X$ be a $\mathbb{T}$-balanced subset of a complex Banach space containing 0. We say that a weight $v$ on $\widetilde{X}$ is radial if $v(\lambda x, \lambda y) = v(x, y)$ for all $\lambda \in \mathbb{T}$ and $(x, y) \in \widetilde{X}$.

For instance, if $X$ is a $\mathbb{T}$-balanced subset of a complex Banach space containing 0 and $\omega$ is a normalized nontrivial gauge, then $v = \|\cdot\| / (\omega \circ \|\cdot\|)$ is a radial 0-weight on $\widetilde{X}$.

Our approach requires a previous study of the extreme points of $B(\text{lip}_v(X)^*)$. This viewpoint is inspired by the papers [6,7] where analogous results are stated in the context of holomorphic functions. As usual, $C(X)$ denotes the Banach space of all scalar-valued bounded continuous functions on a compact Hausdorff space $X$, and $C_0(X)$ the Banach space of all scalar-valued continuous functions which vanish at infinity on a locally compact Hausdorff space $X$, both spaces endowed with the supremum norm.

Let $X$ be a compact pointed metric space and consider the weak* topology $w^*$ on $\text{lip}_v(X)^*$. By the Krein–Milman theorem, the set $\text{Ext}(B(\text{lip}_v(X)^*))$ is nonempty. For every $x \in X$, $\delta_x$ denotes the evaluation functional at $x$ defined on $\text{lip}_v(X)$, and for any $(x, y) \in \widetilde{X}$, we denote by $\delta^v_{(x,y)}$ the functional on $\text{lip}_v(X)$ given by

$$\delta^v_{(x,y)} = v(x, y)\frac{\delta_x - \delta_y}{d(x, y)}.$$

Note that $\delta^v_{(x,y)} \in B(\text{lip}_v(X)^*)$. We first have a simple result which gives a somewhat description of extreme points of $B(\text{lip}_v(X)^*)$.

**Proposition 1** *Let $X$ be a compact pointed metric space and let $v$ be a weight on $\widetilde{X}$. Then* $\text{Ext}(B(\text{lip}_v(X)^*))$ *is contained in the set* $\{\lambda\delta^v_{(x,y)} : \lambda \in S(\mathbb{K}),\ (x, y) \in \widetilde{X}\}$.

*Proof* Let $\gamma \in \text{Ext}(B(\text{lip}_v(X)^*))$. Since the map $\Phi_v : \text{lip}_v(X) \to C_0(\widetilde{X})$, given by

$$\Phi_v(f)(x, y) = v(x, y)\frac{f(x) - f(y)}{d(x, y)}$$

for all $f \in \text{lip}_v(X)$ and $(x, y) \in \widetilde{X}$, is an isometric linear embedding, there exists $F \in \text{Ext}(B(C_0(\widetilde{X})^*)$ such that $\Phi^*_v(F) = \gamma$. By the Arens–Kelley theorem, $F$ is of the form $\lambda\psi_{(x,y)}$ where $\lambda \in S(\mathbb{K})$ and $\psi_{(x,y)}$ is the evaluation functional at a point $(x, y) \in \widetilde{X}$ defined on $C_0(\widetilde{X})$. An easy verification gives

$$\gamma(f) = \Phi^*_v(\lambda\psi_{(x,y)})(f) = \lambda\psi_{(x,y)}(\Phi_v(f)) = \lambda\Phi_v(f)(x, y) = \lambda v(x, y)\frac{\delta_x - \delta_y}{d(x, y)}(f)$$

for all $f \in \mathrm{lip}_v(X)$, and thus $\gamma$ has the desired form. □

For any $(x, y) \in \widetilde{X}$, it is easy to see that $\delta^v_{(x,y)} \in \mathrm{Ext}(B(\mathrm{lip}_v(X)^*))$ if and only if $\lambda\delta^v_{(x,y)} \in \mathrm{Ext}(B(\mathrm{lip}_v(X)^*))$ for all $\lambda \in S(\mathbb{K})$. Taking this observation into account, we introduce the following concept.

**Definition 3** Let $X$ be a compact pointed metric space and let $v$ be a weight on $\widetilde{X}$. The little Lipschitz $v$-boundary of $\widetilde{X}$, denoted by $\mathcal{B}^l_v(\widetilde{X})$, is the nonempty set $\{(x, y) \in \widetilde{X} : \delta^v_{(x,y)} \in \mathrm{Ext}(B(\mathrm{lip}_v(X)^*))\}$.

**Lemma 1** *Let $X$ be a compact pointed metric space and let $v$ be a 0-weight on $\widetilde{X}$. Then $\mathrm{Lip}_0(X) \subset \mathrm{lip}_v(X)$ and $\mathrm{Lip}_v(f) \leq \|v\|_\infty \mathrm{Lip}(f)$ for all $f \in \mathrm{Lip}_0(X)$. As a consequence, $\mathrm{lip}_v(X)$ separates the points of $X$.*

*Proof* Since $v$ is a 0-weight and $X$ is compact, we have that $v$ is bounded on $\widetilde{X}$. Let $f \in \mathrm{Lip}_0(X)$. Then $f \in \mathrm{Lip}_v(X)$ also because

$$v(x, y)\frac{|f(x) - f(y)|}{d(x, y)} \leq \|v\|_\infty \mathrm{Lip}(f)$$

for all $(x, y) \in \widetilde{X}$, and $\mathrm{Lip}_v(f) \leq \|v\|_\infty \mathrm{Lip}(f)$. But even more is true: as $v$ is a 0-weight, $f$ belongs to $\mathrm{lip}_v(X)$ because

$$v(x, y)\frac{|f(x) - f(y)|}{d(x, y)} \leq v(x, y)\mathrm{Lip}(f)$$

for all $(x, y) \in \widetilde{X}$. For the consequence, if $(x, y) \in \widetilde{X}$, then the function $f_y : z \mapsto d(z, y) - d(e, y)$ defined on $X$, is in $\mathrm{Lip}_0(X)$ and $f_y(x) - f_y(y) = d(x, y)$. This completes the proof. □

**Lemma 2** *Let $X$ be a compact pointed metric space and let $v$ be a 0-weight on $\widetilde{X}$.*

(i) *If $(x, y), (x', y') \in \widetilde{X}$ and $\lambda, \lambda' \in S(\mathbb{K})$, then $\lambda\delta^v_{(x,y)} = \lambda'\delta^v_{(x',y')}$ implies that either $\lambda = \lambda'$ and $(x, y) = (x', y')$ or, $\lambda = -\lambda'$ and $(x, y) = (y', x')$.*
(ii) *If $(x, y), (x', y') \in \widetilde{X}$, then $\delta^v_{(x,y)} = \delta^v_{(x',y')}$ implies that $(x, y) = (x', y')$.*

*Proof* (i) Suppose that $\lambda\delta^v_{(x,y)} = \lambda'\delta^v_{(x',y')}$. We first show that $\{x, y\} = \{x', y'\}$. Otherwise, there would exist at least a point in $\{x, y, x', y'\}$ which is distinct of the others three. There is no loss of generality in assuming that such a point is $x$. Let $\epsilon = d(x, \{y, x', y'\}) > 0$ and define $g : X \to \mathbb{R}$ by $g(z) = h(z) - h(e)$ where $h(z) = \max\{0, \epsilon - d(z, x)\}$ for all $z \in X$. It is easily seen that $g$ is in $\mathrm{Lip}_0(X)$, hence in $\mathrm{lip}_v(X)$ by Lemma 1, and an easy computation shows that $\delta^v_{(x,y)}(g) = \epsilon v(x, y)/d(x, y)$ and $\delta^v_{(x',y')}(g) = 0$. This contradicts that $v(x, y)\epsilon \neq 0$ and so $\{x, y\} = \{x', y'\}$, as required. This implies that either $(x, y) = (x', y')$ or $(x, y) = (y', x')$. In the former case we would have $\lambda\delta^v_{(x,y)} = \lambda'\delta^v_{(x,y)}$ which implies $\lambda = \lambda'$ because $\mathrm{lip}_v(X)$ separates the points of $X$ by Lemma 1. In the latter case we would have $\lambda\delta^v_{(x,y)} = \lambda'\delta^v_{(y,x)}$ which implies $v(x, y) = v(y, x)$, hence $\lambda = -\lambda'$ and this completes the proof of (i). The assertion (ii) follows from (i). □

Finally, we give an easy observation which permits to simplify the next proof.

**Lemma 3** *Let $X$ be a compact pointed metric space and let $v$ be a weight on $\widetilde{X}$. Define $v_m(x, y) = \max\{v(x, y), v(y, x)\}$ for any $(x, y) \in \widetilde{X}$. Then $v_m$ is a symmetric weight on $\widetilde{X}$ (that is, $v_m(x, y) = v_m(y, x)$ for all $(x, y) \in \widetilde{X}$),* $\mathrm{Lip}_v(X) = \mathrm{Lip}_{v_m}(X)$, $\mathrm{lip}_v(X) = \mathrm{lip}_{v_m}(X)$ *and* $\mathrm{Lip}_v = \mathrm{Lip}_{v_m}$.

We are now in good condition to prove the following result.

**Theorem 2** *Let $X$ be a compact pointed metric space and let $v$ be a 0-weight on $\widetilde{X}$. If $\mathrm{lip}_v(X)$ is isometrically isomorphic to a linear subspace of $C(K)$ for some scattered compact Hausdorff space $K$, then $\mathcal{B}_v^l(\widetilde{X})$ is a countable subset of $\widetilde{X}$.*

*Proof* By Lemma 3, we can assume that $v$ is symmetric. Let $K$ be a scattered compact Hausdorff space and let $M$ be a linear subspace of $C(K)$. Assume that there is an isometric isomorphism $T$ from $\mathrm{lip}_v(X)$ onto $M$. Then $T^*$ is an isometric isomorphism from $M^*$ onto $\mathrm{lip}_v(X)^*$ that maps $\mathrm{Ext}(B(M^*))$ onto $\mathrm{Ext}(B(\mathrm{lip}_v(X)^*))$. Consider $C(K)^*$, $M^*$ and $\mathrm{lip}_v(X)^*$ endowed with the weak* topology.

The idea of the proof is to transfer the property of being scattered from $K$ to $\mathcal{B}_v^l(\widetilde{X})$. To do so, we are going to construct the following chain of homeomorphisms:

$$K \overset{\delta}{\longleftrightarrow} \frac{\mathrm{Ext}(B(C(K)^*))}{\sim} \supset \overline{\left(\frac{\mathrm{Ext}(B(M^*))}{\sim}\right)} \overset{\widetilde{T}^*}{\longleftrightarrow} \overline{\left(\frac{\mathrm{Ext}(B(\mathrm{lip}_v(X)^*))}{\approx}\right)} \overset{\widetilde{\delta}^v}{\longleftrightarrow} \overline{\left(\frac{\mathcal{B}_v^l(\widetilde{X})}{R}\right)}.$$

Our main task is to give sense to all involved symbols and create such a chain.

It is clear that the map $\delta^v \colon \widetilde{X} \to \mathrm{lip}_v(X)^*$, defined by $\delta^v(x, y) = \delta^v_{(x,y)}$, is a continuous map from $\widetilde{X}$ to $(\mathrm{lip}_v(X)^*, w^*)$ and, by Lemma 2 (ii), it is injective. Fortunately, the map $\delta^v$ can be extended to $X \times X$ just defining $\delta^v(x, x) = 0$ for all $x \in X$. Let us see that $\delta^v \colon X \times X \to (\mathrm{lip}_v(X)^*, w^*)$ is continuous. If $(x_0, y_0) \in X \times X$, consider the subbasic neighbourhood of $\delta^v_{(x_0,y_0)}$ in $(\mathrm{lip}_v(X)^*, w^*)$ given by

$$N(\delta^v_{(x_0,y_0)}; f, \varepsilon) = \left\{\varphi \in \mathrm{lip}_v(X)^* \colon \left|\varphi(f) - \delta^v_{(x_0,y_0)}(f)\right| < \varepsilon\right\},$$

where $f \in \mathrm{lip}_v(X)$ and $\varepsilon > 0$. Since the map $\Phi_v(f)$ defined in the proof of the Proposition 1 belongs to $C_0(\widetilde{X})$, it extends to a continuous functions on $X \times X$ which is zero on the diagonal. Then the set

$$N((x_0, y_0); \varepsilon) = \{(x, y) \in X \times X \colon |\Phi_v(f)(x, y) - \Phi_v(f)(x_0, y_0)| < \varepsilon\}$$

is an open neighbourhood of $(x_0, y_0)$ in $X \times X$ and $\delta^v(N((x_0, y_0); \varepsilon)) \subset N(\delta^v_{(x_0,y_0)}; f, \varepsilon)$.

Next, we introduce a pair of equivalence relations: for $\phi_1, \phi_2 \in \mathrm{lip}_v(X)^*$, define $\phi_1 \approx \phi_2$ if and only if there exists $\lambda \in S(\mathbb{K})$ such that $\phi_2 = \lambda\phi_1$; and for $(x, y), (x', y') \in X \times X$, define $(x, y)R(x', y')$ if and only if $(x, y) = (x', y')$ or

$(x, y) = (y', x')$. Then the map $\delta^v$ induces a continuous map $\widetilde{\delta}^v : (X \times X)/R \to \mathrm{lip}_v(X)^*/\approx$ when both spaces are endowed with their respective quotient topologies, defined by $\widetilde{\delta}^v([(x, y)]) = [\delta^v_{(x,y)}]$. In the proof of that $\widetilde{\delta}^v$ is well defined, we have used that $v$ is symmetric. By Lemma 2 (i), $\widetilde{\delta}^v : (X \times X)/R \to \mathrm{lip}_v(X)^*/\approx$ is injective. As $(X \times X)/R$ is compact, it follows that $\widetilde{\delta}^v$ is a uniform homeomorphism from $(X \times X)/R$ onto its image.

Note that if $(x, y) \in \mathcal{B}^l_v(\widetilde{X})$, we have $[(x, y)] \subset \mathcal{B}^l_v(\widetilde{X})$; and if $\phi \in \mathrm{Ext}(B(\mathrm{lip}_v(X)^*))$, also $[\phi] \subset \mathrm{Ext}(B(\mathrm{lip}_v(X)^*))$. Hence $\widetilde{\delta}^v$ maps $\mathcal{B}^l_v(\widetilde{X})/R$ onto $\mathrm{Ext}(B(\mathrm{lip}_v(X)^*))/\approx$. Therefore the uniform homeomorphism $\widetilde{\delta}^v$ from $\mathcal{B}^l_v(\widetilde{X})/R$ onto $\mathrm{Ext}(B(\mathrm{lip}_v(X)^*))/\approx$ has an extension $\widetilde{\delta}^v : \overline{\mathcal{B}^l_v(\widetilde{X})/R} \to \overline{\mathrm{Ext}(B(\mathrm{lip}_v(X)^*))/\approx}$ that is also an onto uniform homeomorphism.

Define a similar equivalence relation $\sim$ on $\phi_1, \phi_2 \in C(K)^*$: $\phi_1 \sim \phi_2$ if and only if there exists $\lambda \in S(\mathbb{K})$ such that $\phi_2 = \lambda\phi_1$. Note that if $\lambda, \lambda' \in S(\mathbb{K})$ and $x, x' \in K$, then $\lambda\delta_x \sim \lambda'\delta_{x'}$ if and only if $x = x'$. Any $\lambda\delta_x$ is related to $\delta_x$ and $[\delta_x] = \cup_{\lambda\in S(\mathbb{K})}\lambda\delta_x$. By the Arens–Kelley theorem, we have

$$\mathrm{Ext}(B(C(K)^*)) = \{\lambda\delta_x : \lambda \in S(\mathbb{K}),\ x \in K\},$$

and so the set $(\mathrm{Ext}(B(C(K)^*))/\sim) = \{[\delta_x] : x \in K\}$ is contained in $C(K)^*/\sim$. Let endow the quotient space $C(K)^*/\sim$ with the quotient topology $q_{w^*}$. The map $\delta : K \to \mathrm{Ext}(B(C(K)^*))/\sim$, defined by $\delta(x) = [\delta_x]$, is bijective and continuous, hence it is a homeomorphism since $K$ is compact. As $K$ is a scattered compact Hausdorff space, then $\mathrm{Ext}(B(C(K)^*))/\sim$ is a scattered compact Hausdorff space.

Note that $\mathrm{Ext}(B(M^*))$ can be seen as a subset of $\mathrm{Ext}(B(C(K)^*))$, and that if $\lambda \in S(\mathbb{K})$ and $x \in K$ is such that $\delta_x \in \mathrm{Ext}(B(M^*))$, then $\lambda\delta_x \in \mathrm{Ext}(B(M^*))$ and hence $[\delta_x] \subset \mathrm{Ext}(B(M^*))$. Therefore $\mathrm{Ext}(B(M^*))/\sim$ can be seen as a subset of $\mathrm{Ext}(B(C(K)^*))/\sim$ and thus $\overline{\mathrm{Ext}(B(M^*))/\sim}$ is also a scattered compact Hausdorff space.

We now connect $\mathrm{Ext}(B(M^*))/\sim$ and $\mathrm{Ext}(B(\mathrm{lip}_v(X)^*))/\approx$ via the uniform homeomorphism $\widetilde{T}^*$ from $\mathrm{Ext}(B(M^*))/\sim$ onto $\mathrm{Ext}(B(\mathrm{lip}_v(X)^*))/\approx$, given by $\widetilde{T}^*([\phi]) = [T^*(\phi)]$ for all $\phi \in \mathrm{Ext}(B(M^*))$. Then $\overline{\mathrm{Ext}(B(\mathrm{lip}_v(X)^*))/\approx}$ is a scattered compact Hausdorff space too. Finally, we have that

$$\overline{\mathcal{B}^l_v(\widetilde{X})/R} = (\widetilde{\delta}^v)^{-1}(\overline{\mathrm{Ext}(B(\mathrm{lip}_v(X)^*))/\approx})$$

is a scattered compact Hausdorff space.

Since $(X \times X)/R$ is a compact metric space and $\mathcal{B}^l_v(\widetilde{X})/R \subset (X \times X)/R$, we conclude that $\overline{\mathcal{B}^l_v(\widetilde{X})/R}$, and hence $\mathcal{B}^l_v(\widetilde{X})/R$, is countable. Therefore $\mathcal{B}^l_v(\widetilde{X})$ is countable. □

The authors are grateful to one of the referees for showing us the following simple proof of Theorem 2 when the linear subspace of $C(K)$ is strongly separating. Let us recall that a linear subspace $M$ of $C_0(K)$ is *strongly separating* if given any pair of distinct points $x, y$ of the locally compact Hausdorff space $K$, then there exists $f \in M$ such that $|f(x)| \neq |f(y)|$ (see [2]).

Let $X$ and $v$ be as in the statement of Theorem 2 and suppose that $\mathrm{lip}_v(X)$ is isometrically isomorphic to a strongly separating linear subspace of $C(K)$ for some scattered compact Hausdorff space $K$. We next prove that $\mathcal{B}_v^l(\widetilde{X})$ is a countable subset of $\widetilde{X}$.

Indeed, let $M$ be a strongly separating linear subspace of $C(K)$, $T$ a linear isometry from $\mathrm{lip}_v(X)$ onto $M$ and $\iota$ the inclusion map from $M$ into $C(K)$. Suppose that $(x, y) \in \mathcal{B}_v^l(\widetilde{X})$. Since $\iota \circ T$ is an isometric linear embedding from $\mathrm{lip}_v(X)$ into $C(K)$, there exist some $t \in K$ and $\lambda \in S(\mathbb{K})$ such that $(\iota \circ T)^*(\lambda\delta_t) = \delta^v_{(x,y)}$. In this way, we have a map $\phi\colon \mathcal{B}_v^l(\widetilde{X}) \to K$ given by $\phi(x, y) = t$. Next we check that $\phi$ is well-defined. Assume there are $t, t' \in K$ and $\lambda, \lambda' \in S(\mathbb{K})$ such that $(\iota \circ T)^*(\lambda\delta_t) = (\iota \circ T)^*(\lambda'\delta_{t'})$. Hence $\lambda T(f)(t) = \lambda' T(f)(t')$ for all $f \in \mathrm{lip}_v(X)$. Taking modules and taking into account that $M$ is strongly separating, we infer that $t = t'$ and thus $\phi$ is well-defined.

Finally, if $(x, y), (x', y') \in \mathcal{B}_v^l(\widetilde{X})$ and $\phi(x, y) = t = \phi(x', y')$, there are $\lambda, \lambda' \in S(\mathbb{K})$ so that $\delta^v_{(x,y)} = (\iota \circ T)^*(\lambda\delta_t)$ and $\delta^v_{(x',y')} = (\iota \circ T)^*(\lambda'\delta_t)$. Thus $\delta^v_{(x,y)} = (\lambda/\lambda')\delta^v_{(x',y')}$. By Lemma 2, this implies that either $(x, y) = (x', y')$ or $(x, y) = (y', x')$. Therefore $\#(\phi^{-1}(\{t\})) \le 2$ for all $t \in K$. Since $K$ is countable so is $\mathcal{B}_v^l(\widetilde{X})$, as required.

The following easy lemma will be crucial for our objective.

**Lemma 4** *If $X$ is a $\mathbb{T}$-balanced subset of a complex Banach space containing 0 and compact for some metrizable topology and $v$ is a radial weight on $\widetilde{X}$, then*

$$\mathcal{B}_v^l(\widetilde{X}) = \bigcup_{\lambda\in\mathbb{T}} \lambda\mathcal{B}_v^l(\widetilde{X}).$$

*Proof* Only one of the inclusions deserves attention. Take $(x, y) \in \widetilde{X}\backslash\mathcal{B}_v^l(\widetilde{X})$ and $\lambda \in \mathbb{T}$. Assume that $\delta^v_{(x,y)} = (1/2)(\varphi_1 + \varphi_2)$ for some $\varphi_1, \varphi_2 \in B(\mathrm{lip}_v(X)^*)$ with $\varphi_1 \ne \varphi_2$. For $i = 1, 2$, consider $(\varphi_i)_\lambda(f) = \varphi_i(f_\lambda)$ for all $f \in \mathrm{lip}_v(X)$, where $f_\lambda(x) = f(\lambda x)$ for any $x \in X$. It is easy to check that $\delta^v_{(\lambda x,\lambda y)} = (1/2)((\varphi_1)_\lambda + (\varphi_2)_\lambda)$ with $(\varphi_1)_\lambda, (\varphi_2)_\lambda \in B(\mathrm{lip}_v(X)^*)$ and $(\varphi_1)_\lambda \ne (\varphi_2)_\lambda$. Therefore $\lambda(x, y) \in \widetilde{X}\backslash\mathcal{B}_v^l(\widetilde{X})$ and this completes the proof. □

We now are ready to prove our announced result.

**Theorem 3** *Let $E$ be a complex Banach space, let $X$ be a $\mathbb{T}$-balanced subset of $E$ containing 0 and compact for some metrizable topology, and let be $v$ a radial 0-weight on $\widetilde{X}$. Then $\mathrm{lip}_v(X)$ cannot be isometrically isomorphic to a subspace of $C(K)$ with $K$ a scattered compact Hausdorff space. In particular, $\mathrm{lip}_v(X)$ cannot be isometrically isomorphic to a subspace of $c_0$.*

*Proof* By Lemma 4, $\mathcal{B}_v^l(\widetilde{X}) = \cup_{\lambda\in\mathbb{T}}\lambda\mathcal{B}_v^l(\widetilde{X})$. Theorem 2 gives now the result. □

Theorem 3 can be applied now to get a wide family of instances where the space $\mathrm{lip}_0(X)$ is not isometrically isomorphic to a subspace of $c_0$.

**Corollary 12** *If $X$ is $\mathbb{T}$-balanced subset of a complex Banach space containing 0 and compact for some metrizable topology, and $\omega$ is a normalized nontrivial gauge, then $\mathrm{lip}_0(X, \omega \circ \|\cdot\|)$ is not isometrically isomorphic to a subspace of $c_0$.*

*Proof* Apply Theorem 3 to the radial 0-weight $v = \|\cdot\| / (\omega \circ \|\cdot\|)$ on $\widetilde{X}$. □

Taking in Corollary 12 the normalized nontrivial gauge $\omega(t) = t^\alpha$ $(0 < \alpha < 1)$, we obtain the following.

**Corollary 13** *If $X$ is a $\mathbb{T}$-balanced subset of a complex Banach space containing* 0 *and compact for some metrizable topology, and* $0 < \alpha < 1$*, then* $\mathrm{lip}_0^{(\alpha)}(X)$ *is not isometrically isomorphic to a subspace of* $c_0$*. In particular, both* $\mathrm{lip}_0^{(\alpha)}(\mathbb{T} \cup \{0\})$ *and* $\mathrm{lip}_0^{(\alpha)}(\mathbb{D})$ *are not isometrically isomorphic to subspaces of* $c_0$*.*

Another application of Theorem 3 yields the following.

**Corollary 14** *If $E$ is a separable complex Banach space and $v$ is a radial* 0*-weight on* $\widetilde{B(E^*)}$ *endowed with the weak* topology, then* $\mathrm{lip}_v(B(E^*))$ *is not isometrically isomorphic to a subspace of* $c_0$*.*

**Corollary 15** *If $E$ is a reflexive complex Banach space with separable dual and $v$ is a radial* 0*-weight on* $\widetilde{B(E)}$ *endowed with the weak topology, then* $\mathrm{lip}_v(B(E))$ *is not isometrically isomorphic to a subspace of* $c_0$*.*

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