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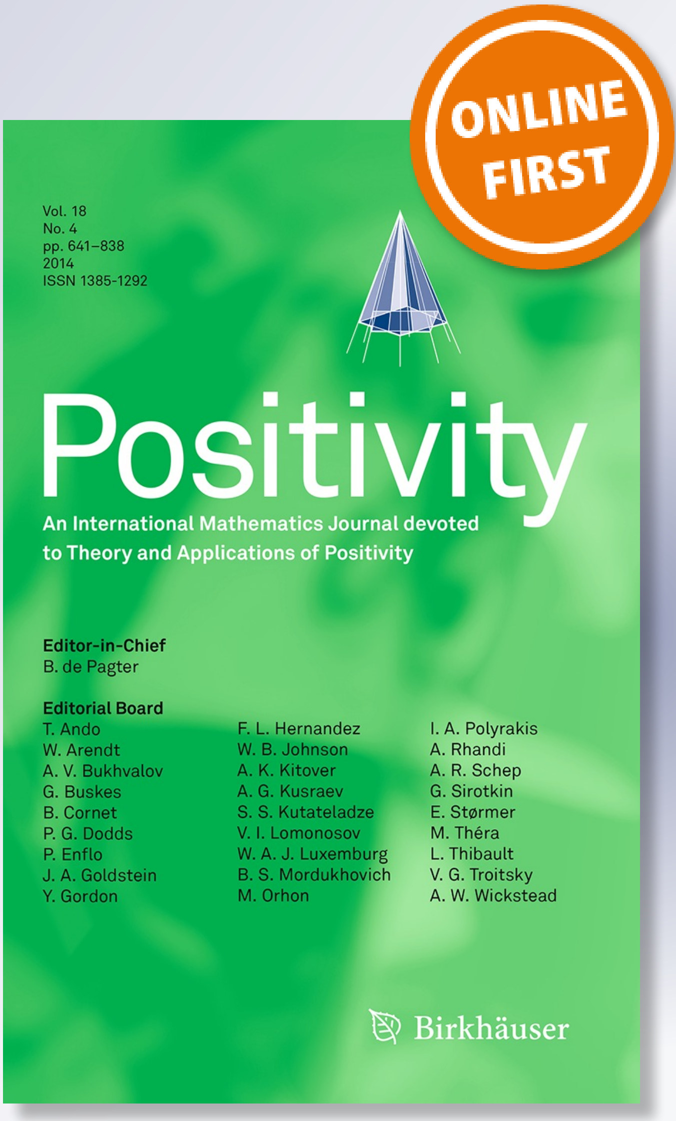

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Abstract Let X be a pointed compact metric space such that $\mathrm{lip}_0(X)$ has the uniform separation property. We prove that every weakly compact composition operator on spaces of Lipschitz functions $\mathrm{lip}_0(X)$ and $\mathrm{Lip}_0(X)$ is compact.

Keywords Composition operator · Weakly compact operator · Compact operator · Lipschitz function

Mathematics Subject Classification 47B33 · 47B07 · 26A16

1 Introduction

A *composition operator* C_ϕ on a function space $F(X)$ over a set X is a linear operator from $F(X)$ into itself defined by $C_\phi(f) = f \circ \phi$, where ϕ is a map from X into X called the *symbol* of C_ϕ. Boundedness and compactness of operators C_ϕ have been intensively studied in terms of the properties of ϕ for different function spaces. See the monograph by Singh and Manhas [7] and the references therein for a comprehensive treatment of this subject.

Our aim in this paper is to study weakly compact composition operators on spaces of Lipschitz functions. Let X and Y be metric spaces. We use the letter d to denote the distance in any metric space. A map $f : X \to Y$ is said to be *Lipschitz* if

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A. Jiménez-Vargas (✉)
Departamento de Matemáticas, Universidad de Almería, 04120 Almería, Spain
e-mail: ajimenez@ual.es

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$$\sup\left\{\frac{d(f(x),f(y))}{d(x,y)}: x,y\in X,\ x\neq y\right\}<\infty;$$

and *supercontractive* if

$$\lim_{d(x,y)\to 0}\frac{d(f(x),f(y))}{d(x,y)}=0,$$

meaning that the following property holds:

$$\forall\varepsilon>0,\ \exists\delta>0: x,y\in X,\ 0<d(x,y)<\delta\ \Rightarrow\ \frac{d(f(x),f(y))}{d(x,y)}<\varepsilon.$$

Constant maps are Lipschitz and supercontractive but, for example, the identity function on $\mathbb{R}$ is Lipschitz, but not supercontractive; whereas the function $n\mapsto n^2$ on $\mathbb{N}$ is supercontractive, but not Lipschitz. A supercontractive Lipschitz function is often called a *little Lipschitz function*.

Let X be a pointed compact metric space with a base point which we always will denote by 0, and let $\mathbb{K}$ be the field of real or complex numbers. The *Lipschitz space* $\mathrm{Lip}_0(X)$ is the Banach space of all Lipschitz functions $f\colon X\to\mathbb{K}$ for which $f(0)=0$, endowed with the Lipschitz norm

$$\mathrm{Lip}_d(f)=\sup\left\{\frac{|f(x)-f(y)|}{d(x,y)}: x,y\in X,\ x\neq y\right\},$$

and the *little Lipschitz space* $\mathrm{lip}_0(X)$ is the closed subspace of $\mathrm{Lip}_0(X)$ formed by all little Lipschitz functions. These spaces have been largely investigated along the time. We refer the reader to Weaver's book [8] for a complete study on them.

There are lip_0 spaces containing only the zero function as, for instance, $\mathrm{lip}_0[0,1]$ with the usual metric, but there exist also some large classes of lip_0 spaces which separate points, even uniformly, in the sense introduced by Weaver [8, Definition 3.2.1] as follows. Given a pointed compact metric space X, it is said that $\mathrm{lip}_0(X)$ *separates points uniformly* if there exists a constant $a>1$ such that, for every $x,y\in X$, there exists $f\in\mathrm{lip}_0(X)$ with $\mathrm{Lip}_d(f)\leq a$ such that $f(x)=d(x,y)$ and $f(y)=0$. This happens, for example, when X is *uniformly discrete* meaning that $d(x,y)\geq\delta$ for all $x,y\in X$ with $x\neq y$ for some $\delta>0$; or when X is the middle-thirds Cantor set with the metric inherited from $[0,1]$. Also, $\mathrm{lip}_0(X^\alpha)$ has the uniform separation property, where $X^\alpha=(X,d^\alpha)$ for some $0<\alpha<1$ (see [8, Proposition 3.2.2]). Lipschitz functions on X^α are called *Hölder functions of exponent* α. It is worth to point out that, by [8, Corollary 4.4.9], for any pointed compact metric space X, there exists a pointed compact metric space Y such that $\mathrm{lip}_0(Y)$ has the uniform separation property and $\mathrm{lip}_0(X)$ is isometrically isomorphic to $\mathrm{lip}_0(Y)$.

The composition operators on spaces of Lipschitz functions have been studied by some authors. N. Weaver characterized in [8, Proposition 1.8.2] the boundedness of composition operators C_ϕ on $\mathrm{Lip}_0(X)$ by means of the Lipschitz condition of their symbols, when X is a pointed complete metric space. The completeness on

X is not restrictive in view of [8, Proposition 1.2.3]. Assuming that X is a compact metric space and in terms of the supercontractive property of their symbols, Kamowitz and Scheinberg [6] gave a complete description of compact composition operators C_ϕ on both spaces $\mathrm{Lip}(X)$ of scalar-valued Lipschitz functions on X with the norm $\max\{\mathrm{Lip}_d\cdot\|\cdot\|_\infty\}$ and spaces $\mathrm{lip}(X^\alpha)$ $(0<\alpha<1)$ of scalar-valued little Lipschitz functions on X^α with the norm $\max\{\mathrm{Lip}_{d^\alpha}\cdot\|\cdot\|_\infty\}$. For pointed metric spaces X, not necessarily compact, this characterization was extended in [5] to both spaces $\mathrm{lip}_0(X)$ satisfying the uniform separation property on bounded subsets of X and spaces $\mathrm{Lip}_0(X)$. Botelho and Jamison [3] provided a characterization of compact weighted composition operators on spaces of vector valued Lipschitz functions.

Our aim in this paper is to prove that every weakly compact composition operator C_ϕ on both $\mathrm{lip}_0(X)$ and $\mathrm{Lip}_0(X)$ is compact provided that $\mathrm{lip}_0(X)$ has the uniform separation property. The key tool to prove this result is the fact, stated in [8, Corollary 3.3.5], that $\mathrm{lip}_0(X)$ has the uniform separation property if and only if $\mathrm{lip}_0(X)^{**}$ is isometrically isomorphic to $\mathrm{Lip}_0(X)$.

2 The results

We prepare the proof of our main result by stating first a new characterization of supercontractive functions. Given a metric space X, for $x\in X$ and $r>0$, we denote by $B(x,r)$ the open ball in X of center x and radius r. See [4] for an analogous characterization of Lipschitz functions.

Lemma 2.1 *Let X be a pointed compact metric space and let $\phi\colon X\to X$ be a continuous map. If ϕ is not supercontractive, then there exist a real number $\varepsilon>0$, two sequences $\{x_n\}$ and $\{y_n\}$ in X converging to a point $x_0\in X$ such that $0<d(x_n,y_n)<1/n$ and $\varepsilon<d(\phi(x_n),\phi(y_n))/d(x_n,y_n)$ for all $n\in\mathbb{N}$, and a function $f\in\mathrm{Lip}_0(X)$ such that $f(\phi(x_n))=d(\phi(x_n),\phi(y_n))$ and $f(\phi(y_n))=0$ for all $n\in\mathbb{N}$.*

Proof Since ϕ is not supercontractive, we can take a real number $\varepsilon>0$ and two sequences $\{p_n\}$ and $\{q_n\}$ in X such that

$$0<d(p_n,q_n)<\frac{1}{n},\quad \varepsilon<\frac{d(\phi(p_n),\phi(q_n))}{d(p_n,q_n)}$$

for all $n\in\mathbb{N}$. Since X is compact, passing to a subsequence if necessary, we may suppose that $\{p_n\}$ converges to a point x_0 in X. It is clear that $\{q_n\}$ also converges to x_0.

We will construct two sequences $\{x_n\}$ and $\{y_n\}$ in X converging to x_0 such that

$$0<d(x_n,y_n)<\frac{1}{n},\quad \varepsilon<\frac{d(\phi(x_n),\phi(y_n))}{d(x_n,y_n)},\quad d(\phi(y_n),\phi(x_0))\le d(\phi(x_n),\phi(x_0))$$

for all $n\in\mathbb{N}$, and a sequence of pairwise disjoint open balls $\{B(\phi(x_n),r_n)\}$ such that

$$\phi(y_n)\notin\bigcup_{j=1}^{\infty}B(\phi(x_j),r_j)$$

for all $n \in \mathbb{N}$. To this end, we consider the sets

$$A = \{n \in \mathbb{N} \colon \phi(p_n) = \phi(x_0)\}, \quad B = \{n \in \mathbb{N} \colon \phi(q_n) = \phi(x_0)\}$$

and distinguish two cases.

Case 1 Suppose that A or B are infinite. If A is infinite, then there exists a strictly increasing map $\sigma \colon \mathbb{N} \to \mathbb{N}$ such that $\phi(p_{\sigma(n)}) = \phi(x_0)$ for all $n \in \mathbb{N}$. Notice that $\phi(q_{\sigma(n)}) \neq \phi(p_{\sigma(n)}) = \phi(x_0)$ for each $n \in \mathbb{N}$ and $\{\phi(q_{\sigma(n)})\} \to \phi(x_0)$. Hence there is a subsequence $\{q_{\sigma(\upsilon(n))}\}$ such that $d(\phi(q_{\sigma(\upsilon(n+1))}), \phi(x_0)) < (1/3)d(\phi(q_{\sigma(\upsilon(n))}), \phi(x_0))$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define $x_n = q_{\sigma(\upsilon(n))}$ and $y_n = p_{\sigma(\upsilon(n))}$. Then we have

$$0 < d(x_n, y_n) < \frac{1}{\sigma(\upsilon(n))} \leq \frac{1}{n}, \quad \varepsilon < \frac{d(\phi(x_n), \phi(y_n))}{d(x_n, y_n)},$$
$$d(\phi(y_n), \phi(x_0)) = 0 < d(\phi(x_n), \phi(x_0)).$$

Moreover, $d(\phi(x_{n+1}), \phi(x_0)) < (1/3)d(\phi(x_n), \phi(x_0))$ for all $n \in \mathbb{N}$. Set

$$r_n = \frac{1}{2} \min\{d(\phi(x_n), \phi(x_0)), d(\phi(x_n), \phi(y_n))\}$$

for each $n \in \mathbb{N}$. As $\phi(y_n) = \phi(x_0)$ for all $n \in \mathbb{N}$, it follows that $r_n = d(\phi(x_n), \phi(x_0))/2$. Note that if $n < m$, then $r_m < r_n/3$ and $d(x, \phi(x_0)) < 3r_m < r_n$ for any $x \in B(\phi(x_m), r_m)$. This implies that, for each $n \in \mathbb{N}$ and any $m > n$, we have $B(\phi(x_m), r_m) \subset B(\phi(x_0), r_n)$. As $B(\phi(x_n), r_n) \cap B(\phi(x_0), r_n) = \varnothing$ for all $n \in \mathbb{N}$, we conclude that the balls $B(\phi(x_n), r_n)$ are pairwise disjoint and $\phi(y_n) = \phi(x_0) \notin \cup_{j=1}^{\infty} B(\phi(x_j), r_j)$ for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ and $\{y_n\}$ satisfy the required conditions. The same argument works if B is infinite.

Case 2 Suppose that A and B are both finite. Let $r = \max(A \cup B)$. Note that $\phi(p_{n+r}) \neq \phi(x_0)$ and $\phi(q_{n+r}) \neq \phi(x_0)$ for all $n \in \mathbb{N}$. Define the sequences $\{t_n\}$ and $\{s_n\}$ by

$$t_n = \begin{cases} p_{n+r} & \text{if } d(\phi(q_{n+r}), \phi(x_0)) \leq d(\phi(p_{n+r}), \phi(x_0)), \\ q_{n+r} & \text{if } d(\phi(p_{n+r}), \phi(x_0)) < d(\phi(q_{n+r}), \phi(x_0)), \end{cases}$$
$$s_n = \begin{cases} q_{n+r} & \text{if } d(\phi(q_{n+r}), \phi(x_0)) \leq d(\phi(p_{n+r}), \phi(x_0)), \\ p_{n+r} & \text{if } d(\phi(p_{n+r}), \phi(x_0)) < d(\phi(q_{n+r}), \phi(x_0)). \end{cases}$$

Note that $d(\phi(s_n), \phi(x_0)) \leq d(\phi(t_n), \phi(x_0))$ for all $n \in \mathbb{N}$. As $\{t_n\}$ converges to x_0, take a subsequence $\{t_{\sigma(n)}\}$ for which

$$d(\phi(t_{\sigma(n+1)}), \phi(x_0)) < \frac{1}{3} \min\{d(\phi(s_{\sigma(n)}), \phi(x_0)), d(\phi(t_{\sigma(n)}), \phi(s_{\sigma(n)}))\}$$

for all $n \in \mathbb{N}$. Let $x_n = t_{\sigma(n)}$ and $y_n = s_{\sigma(n)}$. Then

$$0 < d(x_n, y_n) < \frac{1}{\sigma(n)+r} < \frac{1}{\sigma(n)} \leq \frac{1}{n}, \quad \varepsilon < \frac{d(\phi(x_n), \phi(y_n))}{d(x_n, y_n)},$$
$$d(\phi(y_n), \phi(x_0)) \leq d(\phi(x_n), \phi(x_0))$$

for all $n \in \mathbb{N}$. Moreover, a straightforward induction yields that, for each $n \in \mathbb{N}$ and any $m > n$,

$$d(\phi(x_m), \phi(x_0)) < \frac{1}{3} \min \{d(\phi(y_n), \phi(x_0)), d(\phi(x_n), \phi(y_n))\}.$$

For each $n \in \mathbb{N}$, take

$$r_n = \frac{1}{2} \min \{d(\phi(x_n), \phi(x_0)), d(\phi(x_n), \phi(y_n))\}.$$

Fix $n, m \in \mathbb{N}$ such that $m > n$. As $d(\phi(x_m), \phi(x_0)) < d(\phi(y_n), \phi(x_0))/3 \leq d(\phi(x_n), \phi(x_0))/3$ and $d(\phi(x_m), \phi(x_0)) < d(\phi(x_n), \phi(y_n))/3$, we have $d(\phi(x_m), \phi(x_0)) < 2r_n/3$. Also, $r_m \leq d(\phi(x_m), \phi(x_0))/2 < r_n/3$ and it is easy to check that $B(\phi(x_m), r_m) \subset B(\phi(x_0), r_n)$. Since $B(\phi(x_n), r_n) \cap B(\phi(x_0), r_n) = \varnothing$, it follows that $B(\phi(x_n), r_n) \cap B(\phi(x_m), r_m) = \varnothing$. Moreover, as $d(\phi(y_m), \phi(x_0)) \leq d(\phi(x_m), \phi(x_0)) < 2r_n/3$, it is clear that $\phi(y_m) \notin B(\phi(x_n), r_n)$. Finally, from the inequalities

$$r_m \leq \frac{d(\phi(x_m), \phi(x_0))}{2} < \frac{d(\phi(y_n), \phi(x_0))}{6} < d(\phi(y_n), \phi(x_0)) - \frac{d(\phi(y_n), \phi(x_0))}{3}$$
$$< d(\phi(y_n), \phi(x_0)) - d(\phi(x_m), \phi(x_0)) \leq d(\phi(y_n), \phi(x_m)),$$

we infer that $\phi(y_n) \notin B(\phi(x_m), r_m)$. Then we can conclude that the balls $B(\phi(x_n), r_n)$ are pairwise disjoint and $\phi(y_n) \notin \cup_{j=1}^{\infty} B(\phi(x_j), r_j)$ for all $n \in \mathbb{N}$.

We now prove that there exists a function $f \in \mathrm{Lip}_0(X)$ such that $f(\phi(x_n)) = d(\phi(x_n), \phi(y_n))$ and $f(\phi(y_n)) = 0$ for all $n \in \mathbb{N}$. Indeed, for each n, let

$$h_n(x) = \max \left\{0, 1 - \frac{d(x, \phi(x_n))}{r_n}\right\}.$$

Notice that h_n is Lipschitz with $\mathrm{Lip}_d(h_n) \leq 1/r_n$, $h_n(\phi(x_n)) = 1$ and $h_n(x) = 0$ for all $x \in X \backslash B(\phi(x_n), r_n)$. Define $g\colon X \to \mathbb{R}$ by

$$g(x) = \sum_{n=1}^{\infty} d(\phi(x_n), \phi(y_n)) h_n(x).$$

Observe that $g(x) = 0$ for any $x \notin \cup_{j=1}^{\infty} B(\phi(x_j), r_j)$ and so $g(\phi(y_n)) = 0$ for all $n \in \mathbb{N}$. As the balls $B(\phi(x_n), r_n)$ are pairwise disjoint, if $x \in \cup_{j=1}^{\infty} B(\phi(x_j), r_j)$, then $g(x) = d(\phi(x_m), \phi(y_m)) h_m(x)$ for some fixed $m \in \mathbb{N}$ (depending only on x) and, in particular, $g(\phi(x_n)) = d(\phi(x_n), \phi(y_n))$ for all $n \in \mathbb{N}$. Moreover, as $d(\phi(x_n), \phi(y_n)) \leq 2d(\phi(x_n), \phi(x_0))$, it follows that $d(\phi(x_n), \phi(y_n)) \leq 4r_n$, hence $\mathrm{Lip}_d(d(\phi(x_n), \phi(y_n))h_n) \leq 4$ and so g is Lipschitz.

Finally, if $\phi(x_0) = 0$, then $\{g(\phi(y_n))\} \to g(0)$, but since $g(\phi(y_n)) = 0$ for all n, it follows that $g(0) = 0$ and so $g \in \mathrm{Lip}_0(X)$. Hence take $f = g$ and the lemma follows. Otherwise, if $\phi(x_0) \neq 0$, take $\varepsilon = d(\phi(x_0), 0)/2 > 0$. Since $\{\phi(x_n)\}$ converges to $\phi(x_0)$, there exists $m \in \mathbb{N}$ such that $\varepsilon \leq d(\phi(x_{n+m}), 0)$ for all $n \in \mathbb{N}$. Then the sequences $\{x_{n+m}\}$ and $\{y_{n+m}\}$ and the function $f : X \to \mathbb{R}$, defined by

$$f(x) = \left(1 - \max\left\{0, 1 - \frac{d(x, 0)}{\varepsilon}\right\}\right) g(x),$$

satisfy the required conditions in the lemma. □

A second tool that we will use to prove the main result is the fact that the Arens product on $\mathrm{lip}_0(X)^{**}$ coincides with the pointwise product on $\mathrm{lip}_0(X)^{**}$ whenever $\mathrm{lip}_0(X)$ separates points uniformly (see [2, Theorem 3.8] for the case $\mathrm{lip}(X^\alpha)$ with $0 < \alpha < 1$).

Let A be a commutative Banach algebra. The *Arens product* on A^{**} is defined in stages as follows (see [1]). For any $a, b \in A$, $f \in A^*$ and $F, G \in A^{**}$, define

$$(f \diamond a)(b) = f(ab), \quad (F \diamond f)(a) = F(f \diamond a), \quad (F \diamond G)(f) = F(G \diamond f).$$

Then A^{**} is a Banach algebra under this product and it is denoted by $(A^{**}, \diamond)$. The algebra A is said to be *Arens regular* if the algebra $(A^{**}, \diamond)$ is commutative.

By [8, Theorems 3.3.3 and 2.2.2], $\mathrm{lip}_0(X)^{**}$ is isometrically isomorphic to $\mathrm{Lip}_0(X)$ provided that $\mathrm{lip}_0(X)$ separates points uniformly, via the map $\Phi : \mathrm{lip}_0(X)^{**} \to \mathrm{Lip}_0(X)$ defined by

$$\Phi(F)(x) = F(\delta_x) \quad (F \in \mathrm{lip}_0(X)^{**},\ x \in X). \tag{1}$$

In fact, this identification is the most natural since

$$\Phi(Q_X(f))(x) = (Q_X(f))(\delta_x) = \delta_x(f) = f(x) \quad (f \in \mathrm{lip}_0(X),\ x \in X), \tag{2}$$

where Q_X denotes the canonical injection from $\mathrm{lip}_0(X)$ into $\mathrm{lip}_0(X)^{**}$. We now can prove the following.

Lemma 2.2 *Let X be a pointed compact metric space and assume that* $\mathrm{lip}_0(X)$ *separates points uniformly. Then the Arens product on* $\mathrm{lip}_0(X)^{**}$ *coincides with the pointwise product on* $\mathrm{lip}_0(X)^{**}$*, and so* $\mathrm{lip}_0(X)$ *is Arens regular.*

Proof Let $x \in X$, $f, g \in \mathrm{lip}_0(X)$ and $F, G \in \mathrm{lip}_0(X)^{**}$. First, we have

$$(\delta_x \diamond f)(g) = \delta_x(fg) = f(x)g(x) = f(x)\delta_x(g),$$

and so $\delta_x \diamond f = f(x)\delta_x$. Now, using (1), we compute

$$(G \diamond \delta_x)(f) = G(\delta_x \diamond f) = G(f(x)\delta_x) = f(x)G(\delta_x) = G(\delta_x)\delta_x(f),$$

and so $G \diamond \delta_x = G(\delta_x)\delta_x$. Finally, we have

$$(F \diamond G)(\delta_x) = F(G \diamond \delta_x) = F(G(\delta_x)\delta_x) = F(\delta_x)G(\delta_x) = (FG)(\delta_x).$$

Since the closed linear hull of the set $\{\delta_x : x \in X\}$ in $\mathrm{Lip}_0(X)^*$ coincides with $\mathrm{lip}_0(X)^*$ by [8, Theorem 3.3.3], we infer that $(F \diamond G)(\gamma) = (FG)(\gamma)$ for all $\gamma \in \mathrm{lip}_0(X)^*$, and so $F \diamond G = FG$, as desired. □

We now are ready to prove the main result of this paper.

Theorem 2.3 *Let X be a pointed compact metric space and let $\phi\colon X \to X$ be a base point-preserving Lipschitz map. Suppose that* $\mathrm{lip}_0(X)$ *separates points uniformly. Then every weakly compact operator* $C_\phi\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X)$ *is compact.*

Proof By Gantmacher's Theorem, the operator $C_\phi\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X)$ is weakly compact if and only if $C_\phi^{**}(\mathrm{lip}_0(X)^{**})$ is contained in $Q_X(\mathrm{lip}_0(X))$.

We claim that this inclusion means that $C_\phi(\mathrm{Lip}_0(X))$ is contained in $\mathrm{lip}_0(X)$. To prove this, consider $\mathrm{lip}_0(X)^{**}$ as a Banach algebra with the Arens product $\diamond$. Since C_ϕ is an algebra endomorphism of $\mathrm{lip}_0(X)$, then C_ϕ^{**} is an algebra endomorphism of $(\mathrm{lip}_0(X)^{**}, \diamond)$. Indeed, for any $F, G \in \mathrm{lip}_0(X)^{**}$ and $\gamma \in \mathrm{lip}_0(X)^*$, it is clear that $C_\phi^{**}(F)(\gamma) = (F \circ C_\phi^*)(\gamma) = F(\gamma \circ C_\phi)$, and, by applying Lemma 2.2, we have

$$\begin{aligned} C_\phi^{**}(F \diamond G)(\gamma) &= (F \diamond G)(\gamma \circ C_\phi) = (FG)(\gamma \circ C_\phi) = F(\gamma \circ C_\phi)G(\gamma \circ C_\phi) \\ &= C_\phi^{**}(F)(\gamma)C_\phi^{**}(G)(\gamma) = (C_\phi^{**}(F)C_\phi^{**}(G))(\gamma) \\ &= (C_\phi^{**}(F) \diamond C_\phi^{**}(G))(\gamma). \end{aligned}$$

Note also that Φ is an algebra homomorphism from $(\mathrm{lip}_0(X)^{**}, \diamond)$ to $\mathrm{Lip}_0(X)$ since

$$\begin{aligned} \Phi(F \diamond G)(x) &= (F \diamond G)(\delta_x) = (FG)(\delta_x) = F(\delta_x)G(\delta_x) \\ &= \Phi(F)(x)\Phi(G)(x) = (\Phi(F)\Phi(G))(x) \end{aligned}$$

for all $x \in X$. It follows that $\Phi C_\phi^{**}\Phi^{-1}$ is an algebra endomorphism of $\mathrm{lip}_0(X)$. By [8, Corollary 4.5.6], $\Phi C_\phi^{**}\Phi^{-1}$ is of the form C_φ for some base point-preserving Lipschitz map $\varphi\colon X \to X$. Using the equalities (2) and $Q_X C_\phi = C_\phi^{**} Q_X$, we infer that

$$C_\phi(f) = \Phi(Q_X(C_\phi(f))) = \Phi(C_\phi^{**}(Q_X(f))) = C_\varphi(\Phi(Q_X(f))) = C_\varphi(f)$$

for all $f \in \mathrm{lip}_0(X)$. Hence $C_\phi = C_\varphi$ on $\mathrm{lip}_0(X)$, which implies $\phi = \varphi$ since $\mathrm{lip}_0(X)$ separates points. It follows that

$$\begin{aligned} C_\phi^{**}(\mathrm{lip}_0(X)^{**}) \subset Q_X(\mathrm{lip}_0(X)) &\Leftrightarrow \Phi C_\phi^{**}\Phi^{-1}(\mathrm{Lip}_0(X)) \\ &\subset \mathrm{lip}_0(X) \Leftrightarrow C_\phi(\mathrm{Lip}_0(X)) \subset \mathrm{lip}_0(X), \end{aligned}$$

and this proves our claim. Hence $C_\phi\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X)$ is weakly compact if and only if $f \circ \phi \in \mathrm{lip}_0(X)$ for all $f \in \mathrm{Lip}_0(X)$.

Assume now that $C_\phi\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X)$ is not compact. Then ϕ is not supercontractive by the version of [5, Theorem 1.3] for lip_0 spaces (see [6] for the case $\mathrm{lip}(X^\alpha)$ with $0 < \alpha < 1$). By applying Lemma 2.1, there exist a real number $\varepsilon > 0$, two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $0 < d(x_n, y_n) < 1/n$ and $\varepsilon < d(\phi(x_n), \phi(y_n))/d(x_n, y_n)$ for all $n \in \mathbb{N}$ and a function $f \in \mathrm{Lip}_0(X)$ such that $f(\phi(x_n)) = d(\phi(x_n), \phi(y_n))$ and $f(\phi(y_n)) = 0$ for all $n \in \mathbb{N}$. Then we have

$$\frac{|(f \circ \phi)(x_n) - (f \circ \phi)(y_n))|}{d(x_n, y_n)} = \frac{d(\phi(x_n), \phi(y_n))}{d(x_n, y_n)} > \varepsilon$$

for all $n \in \mathbb{N}$. Since $\{d(x_n, y_n)\} \to 0$ as $n \to \infty$, we infer that $f \circ \phi$ is not in $\mathrm{lip}_0(X)$ and thus $C_\phi\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X)$ is not weakly compact. This completes the proof of the theorem. □

From Theorem 2.3 we derive the following.

Corollary 2.4 *Let X be a pointed compact metric space and let $\phi\colon X \to X$ be a base point-preserving Lipschitz map. Suppose that $\mathrm{lip}_0(X)$ separates points uniformly. Then every weakly compact operator $C_\phi\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(X)$ is compact.*

Proof Assume that $C_\phi\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(X)$ is weakly compact. Hence $\Phi^{-1}C_\phi\Phi\colon \mathrm{lip}_0(X)^{**} \to \mathrm{lip}_0(X)^{**}$ is weakly compact. Given $F \in \mathrm{lip}_0(X)^{**}$ and $x \in X$, we have

$$\begin{aligned}(\Phi^{-1}C_\phi\Phi)(F)(\delta_x) &= ((C_\phi\Phi)(F))(x) = C_\phi(\Phi(F))(x) = \Phi(F)(\phi(x))\\ &= F(\delta_{\phi(x)}) = F(\delta_x \circ C_\phi) = (F \circ C_\phi^*)(\delta_x) = C_\phi^{**}(F)(\delta_x).\end{aligned}$$

Since linear combinations of elements of the form δ_x are dense in $\mathrm{lip}_0(X)^*$, it follows that $\Phi^{-1}C_\phi\Phi = C_\phi^{**}$. Hence $C_\phi^{**}\colon \mathrm{lip}_0(X)^{**} \to \mathrm{lip}_0(X)^{**}$ is weakly compact and so also is $C_\phi\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X)$. Then, by Theorem 2.3, $C_\phi\colon \mathrm{lip}_0(X) \to \mathrm{lip}_0(X)$ is compact. Hence ϕ is supercontractive by the version of [5, Theorem 1.3] for lip_0 spaces, and finally $C_\phi\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(X)$ is compact by [5, Theorem 1.2]. □

Remark 2.1 Analogous versions of the preceding results can be established for composition operators C_ϕ from $\mathrm{lip}_0(X)$ to $\mathrm{lip}_0(Y)$ and from $\mathrm{Lip}_0(X)$ to $\mathrm{Lip}_0(Y)$. Moreover, every space $\mathrm{Lip}(X)$ is isometrically isomorphic to a certain space $\mathrm{Lip}_0(X_0)$ by [8, Theorem 1.7.2], and an analogous isometric isomorphism identifies the spaces $\mathrm{lip}(X)$ and $\mathrm{lip}_0(X_0)$ (see [8, p. 74]). Using these identifications, it is easy to show that Theorem 2.3 and Corollary 2.4 are valid also for composition operators from $\mathrm{lip}(X)$ to $\mathrm{lip}(Y)$ and from $\mathrm{Lip}(X)$ to $\mathrm{Lip}(Y)$.

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