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Generalized weak peripheral multiplicativity in algebras of Lipschitz functions

Research Article

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Abstract: Let (X, d_X) and (Y, d_Y) be pointed compact metric spaces with distinguished base points e_X and e_Y. The Banach algebra of all $\mathbb{K}$-valued Lipschitz functions on X – where $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$ – that map the base point e_X to 0 is denoted by $\mathrm{Lip}_0(X)$. The peripheral range of a function $f \in \mathrm{Lip}_0(X)$ is the set $\mathrm{Ran}_\pi(f) = \{f(x) : |f(x)| = \|f\|_\infty\}$ of range values of maximum modulus. We prove that if $T_1, T_2 \colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ and $S_1, S_2 \colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(X)$ are surjective mappings such that

$$\mathrm{Ran}_\pi(T_1(f)\,T_2(g)) \cap \mathrm{Ran}_\pi(S_1(f)\,S_2(g)) \neq \varnothing$$

for all $f, g \in \mathrm{Lip}_0(X)$, then there are mappings $\varphi_1, \varphi_2 \colon Y \to \mathbb{K}$ with $\varphi_1(y)\varphi_2(y) = 1$ for all $y \in Y$ and a base point-preserving Lipschitz homeomorphism $\psi \colon Y \to X$ such that $T_j(f)(y) = \varphi_j(y) S_j(f)(\psi(y))$ for all $f \in \mathrm{Lip}_0(X)$, $y \in Y$, and $j = 1, 2$. In particular, if S_1 and S_2 are identity functions, then T_1 and T_2 are weighted composition operators.

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1. Introduction and background

Spectral preserver problems involve analyzing mappings between Banach algebras that preserve certain spectral properties. Molnár [13] initiated the study in algebras of continuous functions by showing that if X were a first-countable, compact Hausdorff space and $T\colon C(X) \to C(X)$ were a surjection with $\sigma(T(f)T(g)) = \sigma(fg)$ for all $f, g \in C(X)$, then T is a weighted composition operator and that, if $T(1) = 1$, then T is a sup-norm isometric algebra isomorphism. This result was reminiscent of the classical Banach–Stone Theorem by demonstrating a connection between the spectral structure of $C(X)$ and its linear and multiplicative structures, as well as to the underlying topological structure of X. A wide range of spectral preserver problems have now been studied, and a variety of spectrum-type properties have also been shown to relate to the linear and multiplicative structures of uniform algebras [4, 11, 14], but also to more general unital, semi-simple commutative Banach algebras [3, 5, 6]. See [2] for a recent survey of spectral preservers.

It is an important and separate question whether the results proven for uniform algebras carry over to function algebras with norms other than the uniform norm. In this work, we explore a question in algebras of Lipschitz functions on compact metric spaces, and in this case there are several layers of structure to be analyzed that uniform algebras do not have.

In a uniform algebra, the range of a function need not be equal to its spectrum, so the spectral condition considered by Molnár was not equivalent to a range condition. Nonetheless, the *peripheral range*,

$$\mathrm{Ran}_\pi(f) = \big\{ f(x) : x \in X,\, |f(x)| = \|f\|_\infty \big\},$$

i.e. the set of range values of f of maximum modulus, is equal to the *peripheral spectrum*, the set of spectral values of maximum modulus [12]. Spectral preserver problems then progressed from spectral conditions like Molnár's to related peripheral spectrum conditions [12, 16], and it is natural to view these as peripheral range conditions, allowing the results for uniform algebras to be adapted to non-unital algebras, such as pointed Lipschitz algebras. Given a compact metric space (X, d) with distinguished base point e_X, the pointed Lipschitz algebra on (X, d) is the set

$$\mathrm{Lip}_0(X) = \left\{ f \in C(X) : \sup_{x,y \in X, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} < \infty,\ f(e_X) = 0 \right\}$$

of $\mathbb{K}$-valued Lipschitz functions mapping the base point to 0, where $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$. The Lipschitz constant is a norm on this space, making $\mathrm{Lip}_0(X)$ into a weak commutative Banach algebra in the sense that there exists $K > 0$ such that $L_{d_X}(fg) \le K L_{d_X}(f) L_{d_X}(g)$, for all $f, g \in \mathrm{Lip}_0(X)$, where $L_{d_X}(\cdot)$ denotes the Lipschitz constant.

In [9], it was shown that if $T\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is a surjection satisfying $\mathrm{Ran}_\pi(T(f)T(g)) = \mathrm{Ran}_\pi(fg)$ for all $f, g \in \mathrm{Lip}_0(X)$, then T is a weighted composition operator and, potentially, an isometric algebra isomorphism for the sup-norm, under some slight further assumptions. Similar mappings between the collections of all Lipschitz functions (i.e. the set $\mathrm{Lip}(X)$) on a compact metric space were also characterized. In this setting, the spectrum $\sigma(f)$ coincides with its range, so $\mathrm{Ran}_\pi(f)$ is precisely the spectral values of maximum modulus.

These results of [9] were extended in [8], by showing that, in fact, it is not necessary to multiplicatively preserve the entire peripheral range, but rather only to satisfy $\mathrm{Ran}_\pi(T(f)T(g)) \cap \mathrm{Ran}_\pi(fg) \neq \varnothing$ for all $f, g \in \mathrm{Lip}_0(X)$. Such mappings are called *weakly peripherally multiplicative*, and in this work we generalize the notion of weak peripheral multiplicativity and show that the previous results fit within a more general framework.

Main Theorem.
Let (X, d_X) and (Y, d_Y) be pointed compact metric spaces. If $T_1, T_2\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ and $S_1, S_2\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(X)$ are surjective mappings that satisfy

$$\mathrm{Ran}_\pi(T_1(f)T_2(g)) \cap \mathrm{Ran}_\pi(S_1(f)S_2(g)) \neq \varnothing \tag{1}$$

for all $f, g \in \mathrm{Lip}_0(X)$, then there exist mappings $\varphi_1, \varphi_2\colon Y \to \mathbb{K}$ with $\varphi_1(y)\varphi_2(y) = 1$ for all $y \in Y$ and a base-point preserving Lipschitz homeomorphism $\psi\colon Y \to X$ such that

$$T_j(f)(y) = \varphi_j(y) S_j(f)(\psi(y)), \qquad j = 1, 2, \tag{2}$$

for all $f \in \mathrm{Lip}_0(X)$ and all $y \in Y$.

Notice that the converse of Main Theorem holds true. This is to say that, given mappings $\varphi_1, \varphi_2 \colon Y \to \mathbb{K}$ with $\varphi_1(y) \cdot \varphi_2(y) = 1$ for all $y \in Y$ and given a base-point preserving Lipschitz homeomorphism $\psi \colon Y \to X$ such that

$$T_j(f)(y) = \varphi_j(y)\, S_j(f)(\psi(y)), \qquad j = 1, 2,$$

for all $f \in \mathrm{Lip}_0(X)$ and all $y \in Y$, then the mappings T_1, T_2, S_1, and S_2 satisfy (1). Maps that satisfy (1) are known as *jointly weakly peripherally multiplicative*. Studying multiple mappings that jointly satisfy spectral conditions has recently received attention [3, 10, 15]. In addition to being a natural extension, studying multiple mappings at once answers a wide range of possible questions. For example, surjective mappings $T \colon C(X) \to C(Y)$ that satisfy $\mathrm{Ran}_\pi(\overline{T(f)}\, T(g)) = \mathrm{Ran}_\pi(\bar{f} g)$ for all $f, g \in C(X)$ where characterized by Honma in [7], and this situation can be converted into the four mapping case, where T_1 is the conjugation of T, $T_2 = T$, S_1 is conjugation, and S_2 is the identity mapping.

Section 2 contains basic material on Lipschitz algebras and some preliminary results that hold in general (pointed) Lipschitz algebras, including a new characterization of Lipschitz functions in terms of sequences of function values. Results characterizing jointly weakly peripherally multiplicative maps are outlined in Section 3, with the proof of Main Theorem being given in Section 4. Some immediate corollaries of Main Theorem – including sufficient conditions to ensure that T_1 and T_2 are sup-norm isometric algebra isomorphisms – are given in subsection 4.1.

2. Preliminaries and prior results

In this section we outline the properties of Lipschitz algebras that will be required for the proof of Main Theorem.

2.1. Background on Lipschitz algebras

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f \colon X \to Y$ is said to be *Lipschitz* if there exists a constant $k \geq 0$ such that

$$d_Y(f(x), f(y)) \leq k \cdot d_X(x, y) \qquad \text{for all} \quad x, y \in X.$$

A map $f \colon X \to Y$ is called a *Lipschitz homeomorphism* if f is bijective and both f and f^{-1} are Lipschitz. If X and Y are pointed metric spaces with distinguished base points e_X and e_Y, it is said that $f \colon X \to Y$ is *base point-preserving* if $f(e_X) = e_Y$. For each $x \in X$ and $\delta > 0$, we denote by $B_\delta(x)$ the open ball of radius δ centered at x, and the diameter of (X, d_X) is denoted by $\mathrm{diam}(X)$.

Let (X, d_X) be a compact metric space. For a continuous function $f \colon X \to \mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{C}$ or $\mathbb{R}$, let

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}, \qquad \text{and} \qquad L_{d_X}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d_X(x, y)} : x, y \in X,\, x \neq y\right\}.$$

We denote by $\mathrm{Lip}(X)$ the Banach algebra of all real- or complex-valued Lipschitz functions f on X, with the norm

$$\|f\|_{d_X} = \max\{\|f\|_\infty, L_{d_X}(f)\}.$$

If, in addition, X has a distinguished base point e_X, then $\mathrm{Lip}_0(X)$ is the (weak) Banach algebra of all scalar-valued Lipschitz functions f on X such that $f(e_X) = 0$, endowed with the norm $L_{d_X}(\cdot)$. Every Lip_0 space is clearly a subspace of a Lip space, but it is also well known that every Lip space can be identified with a Lip_0 space, see [17, Section 1.7].

2.2. A Lipschitz version of Bishop's Lemma for uniform algebras

Given $f \in \mathrm{Lip}_0(X)$, the *maximizing set of* f is the set $M(f) = \{x \in X : |f(x)| = \|f\|_\infty\}$, and the *peripheral range of* f is the image of the maximizing set, that is

$$\mathrm{Ran}_\pi(f) = \left\{f(x) : x \in X,\, |f(x)| = \|f\|_\infty\right\}.$$

A function $h \in \mathrm{Lip}_0(X)$ is a *peaking function* if $\mathrm{Ran}_\pi(h) = \{1\}$, and the set of all peaking functions is denoted by $\mathcal{P}(\mathrm{Lip}_0(X))$. Peaking functions can be used to isolate points in the underlying domain, so, given $x \in X \setminus \{e_X\}$, the peaking functions that peak at x are denoted by

$$\mathcal{P}_x(\mathrm{Lip}_0(X)) = \{h \in \mathcal{P}(\mathrm{Lip}_0(X)) : x \in M(h)\}.$$

We begin with a lemma that demonstrates the existence of peaking functions with special properties. This result, which is essentially the pointed Lipschitz algebra version of Bishop's Lemma for uniform algebras [1, Theorem 2.4.1], is proven in [8, Lemma 2.1].

Lemma 2.1.
Let (X, d_X) be a compact metric space with distinguished base point e_X and let $f \in \mathrm{Lip}_0(X)$. Then for each $x_0 \in X \setminus \{e_X\}$,

(a) *There exists a peaking function $h \in \mathcal{P}_{x_0}(\mathrm{Lip}_0(X))$ such that $M(h) = \{x_0\}$.*
(b) *If $f(x_0) \neq 0$, then there exists a peaking function $h \in \mathcal{P}_{x_0}(\mathrm{Lip}_0(X))$ such that $M(h) = M(fh) = \{x_0\}$. In particular, $\mathrm{Ran}_\pi(fh) = \{f(x_0)\}$.*
(c) *If $f(x_0) = 0$, then, given $\varepsilon > 0$, there exists a peaking function $h \in \mathcal{P}_{x_0}(\mathrm{Lip}_0(X))$ such that $\|fh\|_\infty < \varepsilon$.*

Following the arguments in [8], for each $x \in X$ we define the set

$$F_x(\mathrm{Lip}_0(X)) = \big\{f \in \mathrm{Lip}_0(X) : \|f\|_\infty = |f(x)| = 1\big\}.$$

Notice that $\mathcal{P}_x(\mathrm{Lip}_0(X)) \subset F_x(\mathrm{Lip}_0(X))$, and, if $f, g \in F_x(\mathrm{Lip}_0(X))$, then $fg \in F_x(\mathrm{Lip}_0(X))$. A useful property of these sets is that they single out elements of X, as shown by the following lemma.

Lemma 2.2.
Let (X, d_X) be a compact metric space with distinguished base point e_X and $x, x' \in X \setminus \{e_X\}$. Then $F_x(\mathrm{Lip}_0(X)) \subset F_{x'}(\mathrm{Lip}_0(X))$ if and only if $x = x'$.

Proof. Suppose that $F_x(\mathrm{Lip}_0(X)) \subset F_{x'}(\mathrm{Lip}_0(X))$ and $x \neq x'$. By Lemma 2.1 (a), there exists a peaking function $h \in \mathcal{P}_x(\mathrm{Lip}_0(X))$ such that $M(h) = \{x\}$, thus $|h(x')| < 1$, contradicting $F_x(\mathrm{Lip}_0(X)) \subset F_{x'}(\mathrm{Lip}_0(X))$. The reverse direction is clear. □

2.3. A characterization of Lipschitz functions

In the proof of Main Theorem we will use the following result, which is of more general interest as it gives a new characterization of Lipschitz functions.

Lemma 2.3.
Let (X, d_X) and (Y, d_Y) be compact metric spaces, and let $\psi\colon Y \to X$ be a continuous map. If ψ is not Lipschitz, then there exist sequences $\{y_n\}$ and $\{z_n\}$ in Y converging to a point $y \in Y$ such that $y_n \neq z_n$ and

$$n < \frac{d_X(\psi(y_n), \psi(z_n))}{d_Y(y_n, z_n)}$$

for all $n \in \mathbb{N}$ and a function $f \in \mathrm{Lip}(X)$ such that $f(\psi(y_n)) = d_X(\psi(y_n), \psi(z_n))$ and $f(\psi(z_n)) = 0$ for all $n \in \mathbb{N}$.

Proof. Since ψ is not Lipschitz, we can find sequences $\{p_n\}$ and $\{q_n\}$ in Y such that $p_n \neq q_n$ and $n < d_X(\psi(p_n), \psi(q_n))/d_Y(p_n, q_n)$ for all $n \in \mathbb{N}$. Note that $\psi(p_n) \neq \psi(q_n)$ for all $n \in \mathbb{N}$. By the compactness of Y, taking a subsequence if necessary, we may suppose that $\{p_n\}$ converges to a point $y \in Y$. Since

$$d_Y(q_n, y) \leq d_Y(p_n, y) + \frac{\mathrm{diam}(X)}{d_X(\psi(p_n), \psi(q_n))} d_Y(p_n, q_n) \leq d_Y(p_n, y) + \frac{\mathrm{diam}(X)}{n}$$

for all $n \in \mathbb{N}$, it follows that $\{q_n\}$ also converges to y.

Next, we construct two sequences $\{y_n\}$ and $\{z_n\}$ in Y converging to y such that

$$y_n \neq z_n, \qquad n < \frac{d_X(\psi(y_n), \psi(z_n))}{d_Y(y_n, z_n)}, \qquad d_X(\psi(z_n), \psi(y)) \leq d_X(\psi(y_n), \psi(y))$$

holds for all $n \in \mathbb{N}$. In addition, we will show that there exists pairwise disjoint balls $B_{r_n}(\psi(y_n))$, where $r_n = (1/2)\min\{d_X(\psi(y_n), \psi(y)), d_X(\psi(y_n), \psi(z_n))\}$, such that

$$\psi(z_n) \notin \bigcup_{j=1}^{\infty} B_{r_j}(\psi(y_j)),$$

for all $n \in \mathbb{N}$. To do this, we distinguish two cases.

Case 1. Suppose that $\{n \in \mathbb{N} : \psi(p_n) = \psi(y)\}$ or $\{n \in \mathbb{N} : \psi(q_n) = \psi(y)\}$ are infinite. If the first set is infinite, then there exists a strictly increasing mapping $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $\psi(p_{\sigma(n)}) = \psi(y)$ for all $n \in \mathbb{N}$. Note that $\psi(q_{\sigma(n)}) \neq \psi(p_{\sigma(n)}) = \psi(y)$ for each $n \in \mathbb{N}$ and $\psi(q_{\sigma(n)}) \to \psi(y)$, as $q_{\sigma(n)} \to y$. Thus there exists a subsequence $\{q_{\sigma(\upsilon(n))}\}$ such that

$$d_X\big(\psi\big(q_{\sigma(\upsilon(n+1))}\big), \psi(y)\big) < \frac{1}{3} d_X\big(\psi\big(q_{\sigma(\upsilon(n))}\big), \psi(y)\big) \qquad \text{for all} \quad n \in \mathbb{N}.$$

Given $n \in \mathbb{N}$, let $y_n = q_{\sigma(\upsilon(n))}$ and $z_n = p_{\sigma(\upsilon(n))}$, then $y_n \neq z_n$,

$$n \leq \sigma(\upsilon(n)) < \frac{d_X(\psi(y_n), \psi(z_n))}{d_Y(y_n, z_n)}, \qquad d_X(\psi(z_n), \psi(y)) = 0 < d_X(\psi(y_n), \psi(y)).$$

Moreover, $d_X(\psi(y_{n+1}), \psi(y)) < (1/3)d_X(\psi(y_n), \psi(y))$ for all $n \in \mathbb{N}$. Set $r_n = (1/2)\min\{d_X(\psi(y_n), \psi(y)), d_X(\psi(y_n), \psi(z_n))\}$ for each $n \in \mathbb{N}$. As $\psi(z_n) = \psi(y)$ for all $n \in \mathbb{N}$, it follows that $r_n = d_X(\psi(y_n), \psi(y))/2$. Note that if $n < m$, then $r_m < r_n/3$ and $d_X(x, \psi(y)) < 3r_m < r_n$ for any $x \in B_{r_m}(\psi(y_m))$. This implies that for each $n \in \mathbb{N}$ and any $m > n$, we have $B_{r_m}(\psi(y_m)) \subset B_{r_n}(\psi(y))$. As $B_{r_n}(\psi(y_n)) \cap B_{r_n}(\psi(y)) = \varnothing$ for all $n \in \mathbb{N}$, we conclude that the balls $B_{r_n}(\psi(y_n))$ are pairwise disjoint and $\psi(z_n) = \psi(y) \notin \bigcup_{j=1}^{\infty} B_{r_j}(\psi(y_j))$ for all $n \in \mathbb{N}$. Therefore $\{y_n\}$ and $\{z_n\}$ satisfy the required conditions. The same argument applies if $\{n \in \mathbb{N} : \psi(q_n) = \psi(y)\}$ is infinite.

Case 2. Suppose that the sets $\{n \in \mathbb{N} : \psi(p_n) = \psi(y)\}$ and $\{n \in \mathbb{N} : \psi(q_n) = \psi(y)\}$ are both finite. Let M be the maximum of the union of these sets. Note that $\psi(p_{n+M}) \neq \psi(y)$ and $\psi(q_{n+M}) \neq \psi(y)$ for all $n \in \mathbb{N}$. Define the sequences $\{t_n\}$ and $\{s_n\}$ by

$$t_n = \begin{cases} p_{n+M} & \text{if} \quad d_X(\psi(q_{n+M}), \psi(y)) \leq d_X(\psi(p_{n+M}), \psi(y)), \\ q_{n+M} & \text{if} \quad d_X(\psi(p_{n+M}), \psi(y)) < d_X(\psi(q_{n+M}), \psi(y)), \end{cases}$$

$$s_n = \begin{cases} q_{n+M} & \text{if} \quad d_X(\psi(q_{n+M}), \psi(y)) \leq d_X(\psi(p_{n+M}), \psi(y)), \\ p_{n+M} & \text{if} \quad d_X(\psi(p_{n+M}), \psi(y)) < d_X(\psi(q_{n+M}), \psi(y)). \end{cases}$$

Note that $d_X(\psi(s_n),\psi(y)) \leq d_X(\psi(t_n),\psi(y))$ holds for all $n\in\mathbb{N}$. As $\{t_n\}$ converges to y, we can find a subsequence $\{t_{\sigma(n)}\}$ such that

$$d_X\big(\psi(t_{\sigma(n+1)}),\psi(y)\big) < \frac{1}{3}\min\big\{d_X\big(\psi(s_{\sigma(n)}),\psi(y)\big), d_X\big(\psi(t_{\sigma(n)}),\psi(s_{\sigma(n)})\big)\big\}$$

for all $n\in\mathbb{N}$. Let $y_n = t_{\sigma(n)}$ and $z_n = s_{\sigma(n)}$, then $y_n \neq z_n$, $n\leq \sigma(n) < d_X(\psi(y_n),\psi(z_n))/d_Y(y_n,z_n)$, and $d_X(\psi(z_n),\psi(y)) \leq d_X(\psi(y_n),\psi(y))$ holds for all $n\in\mathbb{N}$. Moreover, a straightforward induction yields that, for each $n\in\mathbb{N}$ and any $m>n$, we have

$$d_X(\psi(y_m),\psi(y)) < \frac{1}{3}\min\big\{d_X(\psi(z_n),\psi(y)), d_X(\psi(y_n),\psi(z_n))\big\}.$$

Let $r_n = (1/2)\min\{d_X(\psi(y_n),\psi(y)), d_X(\psi(y_n),\psi(z_n)\}$ for each $n\in\mathbb{N}$. Fix $n,m\in\mathbb{N}$ such that $m>n$. As $d_X(\psi(y_m),\psi(y)) < d_X(\psi(z_n),\psi(y))/3 \leq d_X(\psi(y_n),\psi(y))/3$ and $d_X(\psi(y_m),\psi(y)) < d_X(\psi(y_n),\psi(z_n))/3$, we have $d_X(\psi(y_m),\psi(y)) < 2r_n/3$. Also, we have that $r_m \leq d_X(\psi(y_m),\psi(y))/2 < r_n/3$, hence it is easy to check that $B_{r_m}(\psi(y_m)) \subset B_{r_n}(\psi(y))$. Since $B_{r_n}(\psi(y_n)) \cap B_{r_n}(\psi(y)) = \varnothing$, we have that $B_{r_n}(\psi(y_n)) \cap B_{r_m}(\psi(y_m)) = \varnothing$. Moreover, as $d_X(\psi(z_m),\psi(y)) \leq d_X(\psi(y_m),\psi(y)) < 2r_n/3$, it is clear that $\psi(z_m)\notin B_{r_n}(\psi(y_n))$. Finally, from the inequalities,

$$\begin{aligned} r_m \leq \frac{d_X(\psi(y_m),\psi(y))}{2} < \frac{d_X(\psi(z_n),\psi(y))}{6} &< d_X(\psi(z_n),\psi(y)) - \frac{d_X(\psi(z_n),\psi(y))}{3} \\ &< d_X(\psi(z_n),\psi(y)) - d_X(\psi(y_m),\psi(y)) \leq d_X(\psi(z_n),\psi(y_m)) \end{aligned}$$

we deduce that $\psi(z_n)\notin B_{r_m}(\psi(y_m))$. Therefore, we can conclude that the balls $B_{r_n}(\psi(y_n))$ are pairwise disjoint and $\psi(z_n)\notin\bigcup_{j=1}^{\infty} B_{r_j}(\psi(y_j))$ for all $n\in\mathbb{N}$.

Finally, we show that there exists a function $f\in \mathrm{Lip}(X)$ satisfying $f(\psi(y_n)) = d_X(\psi(y_n),\psi(z_n))$ and $f(\psi(z_n)) = 0$ for all $n\in\mathbb{N}$. Indeed, for each n, let $h_n(x) = \max\{0, 1 - d_X(x,\psi(y_n))/r_n\}$. Note that h_n is Lipschitz with $L_{d_X}(h_n) \leq 1/r_n$, $h_n(\psi(y_n)) = 1$ and $h_n(x) = 0$ for all $x\in X\setminus B_{r_n}(\psi(y_n))$ [8, Lemma 2.1]. Define $f\colon X\to\mathbb{K}$ by

$$f(x) = \sum_{n=1}^{\infty} d_X(\psi(y_n),\psi(z_n))\, h_n(x).$$

Note that $f(x) = 0$ for any $x\notin\bigcup_{j=1}^{\infty} B_{r_j}(\psi(y_j))$. As the balls $B_{r_n}(\psi(y_n))$ are disjoint, if $x\in\bigcup_{j=1}^{\infty} B_{r_j}(\psi(y_j))$, then $f(x) = d_X(\psi(y_m),\psi(z_m))\, h_m(x)$ for some fixed $m\in\mathbb{N}$ (depending only on x). In particular, $f(\psi(y_n)) = d_X(\psi(y_n),\psi(z_n))$. Finally, as $d_X(\psi(y_n),\psi(z_n)) \leq 2d_X(\psi(y_n),\psi(y))$, we have that $d_X(\psi(y_n),\psi(z_n)) \leq 4r_n$, hence it must be that $L_{d_X}(d_X(\psi(y_n),\psi(z_n))\, h_n) \leq 4$. Therefore f is Lipschitz and satisfies the required conditions. □

Next we adapt the previous lemma to pointed Lipschitz algebras.

Lemma 2.4.
Let (X,d_X) and (Y,d_Y) be pointed compact metric spaces, and let $\psi\colon Y\to X$ be a continuous map. If ψ is not Lipschitz, then there exist sequences $\{y_n\}$ and $\{z_n\}$ in Y converging to a point $y\in Y$ such that $y_n\neq z_n$ and $n < d_X(\psi(y_n),\psi(z_n))/d_Y(y_n,z_n)$ for all $n\in\mathbb{N}$ and a function $f\in\mathrm{Lip}_0(X)$ such that $f(\psi(y_n)) = d_X(\psi(y_n),\psi(z_n))$ and $f(\psi(z_n)) = 0$ for all $n\in\mathbb{N}$.

Proof. If ψ is not Lipschitz, by Lemma 2.3 we have two sequences $\{y_n\}$ and $\{z_n\}$ in Y converging to a point $y\in Y$ and a function $g\in\mathrm{Lip}(X)$ satisfying $g(\psi(y_n)) = d_X(\psi(y_n),\psi(z_n))$, $g(\psi(z_n)) = 0$, $y_n\neq z_n$, and $n < d_X(\psi(y_n),\psi(z_n))/d_Y(y_n,z_n)$ for all $n\in\mathbb{N}$.

We distinguish two cases. Firstly, if $\psi(y) = e_X$, then $g(\psi(z_n)) \to g(e_X)$ by the continuity of ψ and g, but since $g(\psi(z_n)) = 0$ for all n, it follows that $g(e_X) = 0$. Hence we can take $f = g$ and the lemma follows. Secondly, if $\psi(y)\neq e_X$, take $\varepsilon = d_X(\psi(y),e_X)/2 > 0$. Since $\{\psi(y_n)\}$ converges to $\psi(y)$, there exists an $m\in\mathbb{N}$ such that $\varepsilon\leq d_X(\psi(y_{n+m}),e_X)$ for all $n\in\mathbb{N}$. Then the sequences $\{y_{n+m}\}$ and $\{z_{n+m}\}$ and the function $f(x) = (1-\max\{0,1-d(x,e_X)/\varepsilon\})\cdot g(x)$ satisfy the required conditions of the lemma. □

3. Jointly sup-norm multiplicative maps

Given compact metric spaces (X, d_X) and (Y, d_Y) with distinguished base points e_X and e_Y, four surjections $T_1, T_2\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ and $S_1, S_2\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(X)$ are called *jointly sup-norm multiplicative* if

$$\|T_1(f)\,T_2(g)\|_\infty = \|S_1(f)\,S_2(g)\|_\infty \tag{3}$$

for all $f, g \in \mathrm{Lip}_0(X)$. In this section we prove a collection of results that are generalized from [10] and hold for any jointly sup-norm multiplicative surjections. We assume throughout this section that (3) holds.

Lemma 3.1.
Let $f, g \in \mathrm{Lip}_0(X)$ and $j \in \{1, 2\}$. Then $|S_j(f)(x)| \le |S_j(g)(x)|$ for all $x \in X$ if and only if $|T_j(f)(y)| \le |T_j(g)(y)|$ for all $y \in Y$.

Proof. Fix the pair $(j, i) = (1, 2)$ or $(j, i) = (2, 1)$. Suppose that $|S_j(f)(x)| \le |S_j(g)(x)|$ for all $x \in X$, then $\|S_j(f)\,h\|_\infty \le \|S_j(g)\,h\|_\infty$ holds for all $h \in \mathrm{Lip}_0(X)$. If $k \in \mathcal{P}(\mathrm{Lip}_0(Y))$ and $h \in \mathrm{Lip}_0(X)$ is such that $T_i(h) = k$, then, by (3),

$$\|T_j(f)\,k\|_\infty = \|T_j(f)\,T_i(h)\|_\infty = \|S_j(f)\,S_i(h)\|_\infty \le \|S_j(g)\,S_i(h)\|_\infty = \|T_j(g)\,T_i(h)\|_\infty = \|T_j(g)\,k\|_\infty.$$

Since $k \in \mathcal{P}(\mathrm{Lip}_0(Y))$ was chosen arbitrarily, we have that $|T_j(f)(y)| \le |T_j(g)(y)|$ for all $y \in Y$, see e.g. [8, Lemma 2.2].

Conversely, suppose that $|T_j(f)(y)| \le |T_j(g)(y)|$ for all $y \in Y$, then $\|T_j(f)\,T_i(h)\|_\infty \le \|T_j(g)\,T_i(h)\|_\infty$ for any $h \in \mathrm{Lip}_0(X)$. If $k \in \mathcal{P}(\mathrm{Lip}_0(X))$ and $h \in \mathrm{Lip}_0(X)$ is such that $S_i(h) = k$, then

$$\|S_j(f)\,k\|_\infty = \|S_j(f)\,S_i(h)\|_\infty = \|T_j(f)\,T_i(h)\|_\infty \le \|T_j(g)\,T_i(h)\|_\infty = \|S_j(g)\,S_i(h)\|_\infty = \|S_j(g)\,k\|_\infty.$$

As $k \in \mathcal{P}(\mathrm{Lip}_0(X))$ was arbitrarily chosen, we have that $|S_j(f)(x)| \le |S_j(g)(x)|$ for all $x \in X$, proving the result. □

Given $h, k \in \mathrm{Lip}_0(X)$ such that $S_1(h), S_2(k) \in F_x(\mathrm{Lip}_0(X))$, then (3) implies that $\|T_1(h)\,T_2(k)\|_\infty = \|S_1(h)\,S_2(k)\|_\infty = 1$ and therefore $M(T_1(h)\,T_2(k)) = \{y \in Y : |T_1(h)(y)\,T_2(k)(y)| = 1\}$. For each $x \in X \setminus \{e_X\}$, we define $\mathcal{A}_1 = S_1^{-1}[F_x(\mathrm{Lip}_0(X))]$, $\mathcal{A}_2 = S_2^{-1}[F_x(\mathrm{Lip}_0(X))]$, and

$$A_x = \bigcap_{h \in \mathcal{A}_1, k \in \mathcal{A}_2} M(T_1(h)\,T_2(k)).$$

Lemma 3.2.
For each $x \in X \setminus \{e_X\}$, the set A_x is nonempty.

Proof. Let $h_1, \dots, h_n \in \mathcal{A}_1$ and let $k_1, \dots, k_n \in \mathcal{A}_2$. As $S_1(h_1) \cdot \ldots \cdot S_1(h_n) \in F_x(\mathrm{Lip}_0(X))$ and $S_2(k_1) \cdot \ldots \cdot S_2(k_n) \in F_x(\mathrm{Lip}_0(X))$, there exist $h \in \mathcal{A}_1$ and $k \in \mathcal{A}_2$ such that $S_1(h) = S_1(h_1) \cdot \ldots \cdot S_1(h_n)$ and $S_2(k) = S_2(k_1) \cdot \ldots \cdot S_2(k_n)$. Since $|S_1(h_i)| \le 1$ and $|S_2(k_i)| \le 1$ for all $1 \le i \le n$, $|S_1(h)| \le |S_1(h_i)|$ and $|S_2(k)| \le |S_2(k_i)|$ for any $1 \le i \le n$. Lemma 3.1 implies that $|T_1(h)(y)| \le |T_1(h_i)(y)|$ and $|T_2(k)(y)| \le |T_2(k_i)(y)|$ for any $1 \le i \le n$ and all $y \in Y$. Since Y is compact, there exists $y_0 \in M(T_1(h)\,T_2(k))$. Hence $1 = |T_1(h)(y_0)\,T_2(k)(y_0)| \le |T_1(h_i)(y_0)\,T_2(k_i)(y_0)| \le 1$ for each $1 \le i \le n$, thus $|T_1(h_i)(y_0)\,T_2(k_i)(y_0)| = 1$ for each $1 \le i \le n$. So, it must be that $y_0 \in \bigcap_{i=1}^n M(T_1(h_i)\,T_2(k_i))$. Therefore, $\{M(T_1(h)\,T_2(k)) : h \in \mathcal{A}_1, k \in \mathcal{A}_2\}$ has the finite intersection property, and, since maximizing sets are closed subsets of the compact set Y, A_x is nonempty. □

Notice that $e_Y \notin A_x$ for any $x \in X \setminus \{e_X\}$.

Lemma 3.3.
Let $f, g \in \mathrm{Lip}_0(X)$. Then for each $x \in X \setminus \{e_X\}$ and each $y \in A_x$, $S_1(f)\,S_2(g) \in F_x(\mathrm{Lip}_0(X))$ if and only if $T_1(f)\,T_2(g) \in F_y(\mathrm{Lip}_0(Y))$.

Proof. Let $x \in X \setminus \{e_X\}$; let $y \in A_x$; and suppose that $T_1(f)T_2(g) \in F_y(\mathrm{Lip}_0(Y))$. Then $1 = \|T_1(f)T_2(g)\|_\infty = \|S_1(f)S_2(g)\|_\infty$, and we need only to show $|S_1(f)(x)S_2(g)(x)| = 1$. If $S_1(f)(x)S_2(g)(x) = 0$, then, without loss of generality, we can assume that $S_1(f)(x) = 0$. Hence Lemma 2.1 (c) implies that there exists a peaking function $h \in \mathcal{P}_x(\mathrm{Lip}_0(X))$ such that $\|S_1(f)h\|_\infty < 1/\|S_2(g)\|_\infty$. Let $k_1, k_2 \in \mathrm{Lip}_0(X)$ be such that $S_1(k_1) = S_2(k_2) = h$. As $y \in A_x$, then $y \in M(T_1(k_1)T_2(k_2))$, and since $\|T_1(k_1)T_2(k_2)\|_\infty = \|S_1(k_1)S_2(k_2)\|_\infty = \|h\|_\infty^2 = 1$ this implies that $T_1(k_1)T_2(k_2) \in F_y(\mathrm{Lip}_0(Y))$. Thus

$$\begin{aligned} 1 &= \|T_1(f)T_2(g)T_1(k_1)T_2(k_2)\|_\infty \le \|T_1(f)T_2(k_2)\|_\infty \cdot \|T_1(k_1)T_2(g)\|_\infty \\ &= \|S_1(f)h\|_\infty \cdot \|S_2(g)h\|_\infty < \frac{1}{\|S_2(g)\|_\infty} \cdot \|S_2(g)\|_\infty = 1, \end{aligned}$$

which is a contradiction. Hence $S_1(f)(x) \neq 0 \neq S_2(g)(x)$, and by Lemma 2.1 (b) there exist functions $h_1, h_2 \in \mathcal{P}_x(\mathrm{Lip}_0(X))$ such that $M(h_2) = M(S_1(f)h_2) = \{x\}$ and $M(h_1) = M(S_2(g)h_1) = \{x\}$. If $k_1, k_2 \in \mathrm{Lip}_0(X)$ are such that $S_1(k_1) = h_1$ and $S_2(k_2) = h_2$, then since $y \in A_x$, the definition of A_x implies that $T_1(k_1)T_2(k_2) \in F_y(\mathrm{Lip}_0(Y))$, so

$$|S_1(f)(x)S_2(g)(x)| = \|S_1(f)h_2\|_\infty \cdot \|S_2(g)h_1\|_\infty = \|T_1(f)T_2(k_2)\|_\infty \cdot \|T_1(k_1)T_2(g)\|_\infty \ge \|T_1(f)T_2(g)T_1(k_1)T_2(k_2)\|_\infty = 1.$$

Therefore $|S_1(f)(x)S_2(g)(x)| = 1$, showing $S_1(f)S_2(g) \in F_x(\mathrm{Lip}_0(X))$. The reverse implication follows analogously. □

Not only is A_x nonempty, but the following lemma shows that it contains only a single point.

Lemma 3.4.

For each $x \in X \setminus \{e_X\}$, the set A_x is a singleton.

Proof. Fix $x \in X \setminus \{e_X\}$, and let $y, y' \in A_x$. If $y \neq y'$, then, by Lemma 2.1 (a), there exists a peaking function $k \in \mathcal{P}_y(\mathrm{Lip}_0(Y))$ such that $M(k) = \{y\}$, implying $|k(y')| < 1$. If $h_1, h_2 \in \mathrm{Lip}_0(X)$ are such that $T_1(h_1) = T_2(h_2) = k$, then Lemma 3.3 implies that $S_1(h_1)S_2(h_2) \in F_x(\mathrm{Lip}_0(X))$. Again, by Lemma 3.3, $k^2 = T_1(h_1)T_2(h_2) \in F_{y'}(\mathrm{Lip}_0(Y))$, which is a contradiction. Therefore $y = y'$, i.e. A_x is a singleton. □

Given the correspondence between x and the singleton A_x, define the map $\tau\colon X \to Y$ by $\tau(e_X) = e_Y$ and

$$\{\tau(x)\} = A_x \tag{4}$$

for $x \in X \setminus \{e_X\}$. Note that $\tau(x) \neq e_Y$ for any $x \neq e_X$. If the mappings T_1, T_2, S_1, and S_2 were all injective, then we could follow a similar construction with their formal inverses to construct $\psi\colon Y \to X$ that acts as the analogue of τ. We could then show directly that τ and ψ are inverses to gain that τ is a bijection. In fact, it is not necessary for us to assume that any of the four maps is injective; we can construct ψ nonetheless and show that τ and ψ are mutual inverses.

Lemma 3.5.

The map $\tau\colon X \to Y$ defined by (4) is bijective.

Proof. Let $x, x' \in X$. If either $x = e_X$ or $x' = e_X$, then $\tau(x) = \tau(x')$ implies that $x' = x$. Suppose that $x, x' \in X \setminus \{e_X\}$ and choose $h \in F_x(\mathrm{Lip}_0(X))$. Let $h_1, h_2 \in \mathrm{Lip}_0(X)$ be such that $S_1(h_1) = S_2(h_2) = h$, then, by Lemma 3.3, $T_1(h_1)T_2(h_2) \in F_{\tau(x)}(\mathrm{Lip}_0(Y))$. If $\tau(x) = \tau(x')$, then $T_1(h_1)T_2(h_2) \in F_{\tau(x')}(\mathrm{Lip}_0(Y))$, which, again by Lemma 3.3, gives that $h^2 = S_1(h_1)S_2(h_2) \in F_{x'}(\mathrm{Lip}_0(X))$ and thus $h \in F_{x'}(\mathrm{Lip}_0(X))$. Lemma 2.2 then gives $x = x'$, i.e. τ is injective.

Now, we prove that τ is surjective. Let $y \in Y \setminus \{e_Y\}$. Given $h, k \in F_y(\mathrm{Lip}_0(Y))$, let $f, g \in \mathrm{Lip}_0(X)$ be such that $T_1(f) = h$ and $T_2(g) = k$, then $\|S_1(f)S_2(g)\|_\infty = \|T_1(f)T_2(g)\|_\infty = 1$, implying $M(S_1(f)S_2(g)) = \{x \in X : |S_1(f)(x)S_2(g)(x)| = 1\}$. Let $\mathcal{B}_1 = T_1^{-1}[F_y(\mathrm{Lip}_0(Y))]$, $\mathcal{B}_2 = T_2^{-1}[F_y(\mathrm{Lip}_0(Y))]$, and define the set

$$B_y = \bigcap_{f \in \mathcal{B}_1, g \in \mathcal{B}_2} M(S_1(f)S_2(g)). \tag{5}$$

We will show that the family $\{M(S_1(f)S_2(g)) : f \in \mathcal{B}_1, g \in \mathcal{B}_2\}$ has the finite intersection property. Let $f_1, \ldots, f_n \in \mathcal{B}_1$ and let $g_1, \ldots, g_n \in \mathcal{B}_2$. As $T_1(f_1) \cdot \ldots \cdot T_1(f_n) \in F_y(\mathrm{Lip}_0(Y))$ and $T_2(g_1) \cdot \ldots \cdot T_2(g_n) \in F_y(\mathrm{Lip}_0(Y))$, there exists $f \in \mathcal{B}_1$ and $g \in \mathcal{B}_2$ such that $T_1(f) = T_1(f_1) \cdot \ldots \cdot T_1(f_n)$ and $T_2(g) = T_2(g_1) \cdot \ldots \cdot T_2(g_n)$. Since $|T_1(f_i)| \leq 1$ and $|T_2(g_i)| \leq 1$ for all $1 \leq i \leq n$, $|T_1(f)| \leq |T_1(f_i)|$ and $|T_2(g)| \leq |T_2(g_i)|$ for any $1 \leq i \leq n$. Lemma 3.1 implies that $|S_1(f)(x)| \leq |S_1(f_i)(x)|$ and $|S_2(g)(x)| \leq |S_2(g_i)(x)|$ for any $1 \leq i \leq n$ and all $x \in X$. Since X is compact, there exists $x_0 \in M(S_1(f)S_2(g))$. Hence $1 = |S_1(f)(x_0)S_2(g)(x_0)| \leq |S_1(f_i)(x_0)S_2(g_i)(x_0)| \leq 1$ for each $1 \leq i \leq n$, thus $|S_1(f_i)(x_0)S_2(g_i)(x_0)| = 1$ for each $1 \leq i \leq n$. So, it must be that $x_0 \in \bigcap_{i=1}^n M(S_1(f_i)S_2(g_i))$. Therefore, $\{M(S_1(f)S_2(g)) : f \in \mathcal{B}_1, g \in \mathcal{B}_2\}$ has the finite intersection property as claimed, and, since maximizing sets are closed subsets of the compact set X, B_y is nonempty.

Let $x \in B_y$ and let $k \in F_y(\mathrm{Lip}_0(Y))$. If $h_1, h_2 \in \mathrm{Lip}_0(X)$ are such that $T_1(h_1) = T_2(h_2) = k$, then, by (5), $S_1(h_1)S_2(h_2) \in F_x(\mathrm{Lip}_0(Y))$. Lemma 3.3 implies that $k^2 = T_1(h_1)T_2(h_2) \in F_{\tau(x)}(\mathrm{Lip}_0(Y))$, thus $k \in F_{\tau(x)}(\mathrm{Lip}_0(Y))$. Consequently, by Lemma 2.2, $\tau(x) = y$, i.e. τ is surjective. □

Lemma 3.6.

Let $f, g \in \mathrm{Lip}_0(X)$ and $x \in X$, then $|T_1(f)(\tau(x))\, T_2(g)(\tau(x))| = |S_1(f)(x)\, S_2(g)(x)|$.

Proof. If any of $S_1(f)$, $S_2(g)$, $T_1(f)$, $T_2(g)$ is identically 0, then the result follows by (3), so we may assume that none of $S_1(f)$, $S_2(g)$, $T_1(f)$, $T_2(g)$ is identically 0. Since $\tau(e_X) = e_Y$, it is true that

$$S_1(f)(e_X)\, S_2(g)(e_X) = 0 = T_1(f)(\tau(e_X))\, T_2(g)(\tau(e_X)),$$

and we may assume that $x \neq e_X$.

If $S_1(f)(x)S_2(g)(x) = 0$, then, without loss of generality, we can assume that $S_1(f)(x) = 0$. Given an $\varepsilon > 0$, Lemma 2.1 (c) implies that there exists a peaking function $h \in \mathcal{P}_x(\mathrm{Lip}_0(X))$ such that $\|S_1(f)h\|_\infty < \varepsilon / \|S_2(g)\|_\infty$. Let $h_1, h_2 \in \mathrm{Lip}_0(X)$ be such that $S_1(h_1) = S_2(h_2) = h$, then Lemma 3.3 implies $T_1(h_1)T_2(h_2) \in F_{\tau(x)}(\mathrm{Lip}_0(Y))$, thus

$$\begin{aligned}|T_1(f)(\tau(x))\, T_2(g)(\tau(x))| &\leq \|T_1(f)\, T_2(g)\, T_1(h_1)\, T_2(h_2)\|_\infty \leq \|T_1(f)\, T_2(h_2)\|_\infty \cdot \|T_1(h_1)\, T_2(g)\|_\infty \\ &= \|S_1(f)\, h\|_\infty \cdot \|S_2(g)\, h\|_\infty < \frac{\varepsilon}{\|S_2(g)\|_\infty} \cdot \|S_2(g)\|_\infty = \varepsilon.\end{aligned}$$

Therefore $T_1(f)(\tau(x))\, T_2(g)(\tau(x)) = 0$, by the liberty of the choice of ε. A symmetric argument shows that $T_1(f)(\tau(x))\, T_2(g)(\tau(x)) = 0$ implies $S_1(f)(x)\, S_2(g)(x) = 0$.

If $S_1(f)(x)S_2(g)(x) \neq 0$, then $S_1(f)(x), S_2(g)(x) \neq 0$. Hence, by Lemma 2.1 (b), there exist peaking functions $h_1, h_2 \in \mathcal{P}_x(\mathrm{Lip}_0(X))$ such that $M(h_2) = M(S_1(f)h_2) = \{x\}$ and $M(h_1) = M(S_2(g)h_1) = \{x\}$. Let $k_1, k_2 \in \mathrm{Lip}_0(X)$ be such that $S_1(k_1) = h_1$ and $S_2(k_2) = h_2$. Since $S_1(k_1)S_2(k_2) \in F_x(\mathrm{Lip}_0(X))$, Lemma 3.3 implies that $T_1(k_1)T_2(k_2) \in F_{\tau(x)}(\mathrm{Lip}_0(Y))$, hence

$$\begin{aligned}|T_1(f)(\tau(x))\, T_2(g)(\tau(x))| &\leq \|T_1(f)\, T_2(g)\, T_1(k_1)\, T_2(k_2)\|_\infty \leq \|T_1(f)\, T_2(k_2)\|_\infty \cdot \|T_1(k_1)\, T_2(g)\|_\infty \\ &= \|S_1(f)\, h_2\|_\infty \cdot \|S_2(g)\, h_1\|_\infty = |S_1(f)(x)\, S_2(g)(x)|.\end{aligned}$$

Since $T_1(f)(\tau(x))\, T_2(g)(\tau(x)) = 0$ if and only if $S_1(f)(x)S_2(g)(x) = 0$, we have $S_1(f)(x)\, S_2(g)(x) \neq 0$ implies that $T_1(f)(\tau(x))\, T_2(g)(\tau(x)) \neq 0$. Therefore, we have that $T_1(f)(\tau(x)) \neq 0$ and $T_2(g)(\tau(x)) \neq 0$, and Lemma 2.1 (b) implies that there exists $k_1, k_2 \in \mathcal{P}_{\tau(x)}(\mathrm{Lip}_0(Y))$ such that $M(k_2) = M(T_1(f)k_2) = \{\tau(x)\}$ and $M(k_1) = M(T_2(g)k_1) = \{\tau(x)\}$. Let $h_1, h_2 \in \mathrm{Lip}_0(X)$ be such that $T_1(h_1) = k_1$ and $T_2(h_2) = k_2$. As $k_1 k_2 \in F_{\tau(x)}(\mathrm{Lip}_0(Y))$, Lemma 3.3 gives $S_1(h_1)\, S_2(h_2) \in F_x(\mathrm{Lip}_0(X))$, so

$$\begin{aligned}|S_1(f)(x)\, S_2(g)(x)| &\leq \|S_1(f)\, S_2(g)\, S_1(h_1)\, S_2(h_2)\|_\infty \leq \|S_1(f)\, S_2(h_2)\|_\infty \cdot \|S_2(g)\, S_1(h_1)\|_\infty \\ &= \|T_1(f)\, k_2\|_\infty \cdot \|T_2(g)\, k_1\|_\infty = |T_1(f)(\tau(x))\, T_2(g)(\tau(x))|.\end{aligned}$$

Therefore $|T_1(f)(\tau(x))\, T_2(g)(\tau(x))| = |S_1(f)(x)\, S_2(g)(x)|$ for all $x \in X$. □

Denoting the formal inverse of τ by ψ, Lemma 3.6 implies that

$$|T_1(f)(y)\,T_2(g)(y)| = |S_1(f)(\psi(y))\,S_2(g)(\psi(y))|$$

for all $f, g \in \mathrm{Lip}_0(X)$ and $y \in Y$.

Lemma 3.7.
The map $\tau\colon X \to Y$ defined by (4) is a homeomorphism.

Proof. As τ is bijective, X is compact and Y is Hausdorff, it is only yet to show that τ is continuous. Let $x_0 \in X \setminus \{e_X\}$ and let U be an open neighborhood of $\tau(x_0)$. As $\tau(x_0) \neq e_Y$, Lemma 2.1 (a) implies that there exists a peaking function $h \in \mathcal{P}_{\tau(x_0)}(\mathrm{Lip}_0(Y))$ such that $M(h) = \{\tau(x_0)\}$, thus there exists an even $n \in \mathbb{N}$ such that $|h^n| < 1/2$ on $Y \setminus U$. Set $k = h^{n/2}$. Let $f, g \in \mathrm{Lip}_0(X)$ be such that $T_1(f) = k$ and $T_2(g) = k$, and let $V = \{x \in X : |S_1(f)(x)\,S_2(g)(x)| > 1/2\}$. For each $\zeta \in V$, we have

$$|k^2(\tau(\zeta))| = |T_1(f)(\tau(\zeta))\,T_2(g)(\tau(\zeta))| = |S_1(f)(\zeta)\,S_2(g)(\zeta)| > \frac{1}{2}.$$

As $|k^2| = |h^n| < 1/2$ on $Y \setminus U$, it must be that $\tau(\zeta) \in U$, hence $\zeta \in \tau^{-1}[U]$. Therefore V is an open set such that $x_0 \in V \subset \tau^{-1}[U]$, and it follows that τ is continuous at x_0.

We now demonstrate the continuity of τ at e_X. Let $\{x_n\} \subset X$ be such that $x_n \to e_X$ and let $g \in \mathrm{Lip}_0(Y)$ be the function defined by $g(y) = d_Y(y, e_Y)$. If $f_1, f_2 \in \mathrm{Lip}_0(X)$ are such that $T_1(f_1) = T_2(f_2) = g$, then Lemma 3.6 implies that

$$|g(\tau(x_n))|^2 = |T_1(f_1)(\tau(x_n))\,T_2(f_2)(\tau(x_n))| = |S_1(f_1)(x_n)\,S_2(f_2)(x_n)|$$

for all $n \in \mathbb{N}$. As $S_1(f_1)(x_n)\,S_2(f_2)(x_n) \to 0$, it follows that

$$d_Y(\tau(x_n), \tau(e_X)) = d_Y(\tau(x_n), e_Y) = g(\tau(x_n)) \to 0.$$

Therefore τ is continuous at e_X. □

4. Jointly weakly peripherally multiplicative maps

Suppose that $T_1, T_2\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ and $S_1, S_2\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(X)$ are surjective mappings that satisfy

$$\mathrm{Ran}_\pi(T_1(f)\,T_2(g)) \cap \mathrm{Ran}_\pi(S_1(f)\,S_2(g)) \neq \varnothing \tag{6}$$

for all $f, g \in \mathrm{Lip}_0(X)$. Since any such foursome of maps automatically satisfies (3), we can apply the results of Section 3. Given $h \in S_1^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$ and $k \in S_2^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$, (6) implies that $1 \in \mathrm{Ran}_\pi(T_1(h)\,T_2(k))$, so there exists $y \in Y$ such that $T_1(h)(y)\,T_2(k)(y) = 1$. As the next lemma shows, y can be chosen such that $y = \tau(x)$.

Lemma 4.1.
Let $x \in X \setminus \{e_X\}$. Then $T_1(h)(\tau(x))\,T_2(k)(\tau(x)) = 1$ for all pairs (h, k) satisfying $h \in S_1^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$ and $k \in S_2^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$.

Proof. Let $h \in S_1^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$ and $k \in S_2^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$. By Lemma 3.6, we have

$$1 = |S_1(h)(x)\, S_2(k)(x)| = |T_1(h)(\tau(x))\, T_2(k)(\tau(x))|,$$

which gives that $T_1(h)(\tau(x))\, T_2(k)(\tau(x)) \neq 0$. Therefore Lemma 2.1 (b) implies that there exists a peaking function $g \in \mathcal{P}_{\tau(x)}(\mathrm{Lip}_0(Y))$ with $M(g) = M(T_1(h)\, T_2(k)\, g) = \{\tau(x)\}$. Notice that

$$|T_1(h)(y)\, T_2(k)(y)\, g(y)^2| \leq |T_1(h)(y)\, T_2(k)(y)\, g(y)| < |T_1(h)(\tau(x))\, T_2(k)(\tau(x))\, g(\tau(x))|$$

for all $y \in Y \setminus \{\tau(x)\}$, which implies $M(T_1(h)\, T_2(k)\, g) = \{\tau(x)\} = M(T_1(h)\, T_2(k)\, g^2)$.

Let $f_1, f_2 \in \mathrm{Lip}_0(X)$ be such that $T_1(f_1) = T_1(h)\, g$ and $T_2(f_2) = T_2(k)\, g$. If $x_0 \in M(S_1(f_1)\, S_2(k))$, then, by Lemma 3.6,

$$\begin{aligned} |T_1(h)(\tau(x_0))\, T_2(k)(\tau(x_0))\, g(\tau(x_0))| &= |T_1(f_1)(\tau(x_0))\, T_2(k)(\tau(x_0))| = |S_1(f_1)(x_0)\, S_2(k)(x_0)| \\ &= \|S_1(f_1)\, S_2(k)\|_\infty = \|T_1(f_1)\, T_2(k)\|_\infty = \|T_1(h)\, T_2(k)\, g\|_\infty. \end{aligned}$$

Since $M(T_1(h)\, T_2(k)\, g) = \{\tau(x)\}$, then $\tau(x_0) = \tau(x)$, and the injectivity of τ gives $x = x_0$. Therefore $M(S_1(f_1)\, S_2(k)) = \{x\}$ and $\mathrm{Ran}_\pi(S_1(f_1)\, S_2(k)) = \{S_1(f_1)(x)\}$. A similar argument implies that $M(S_1(h)\, S_2(f_2)) = \{x\}$ and $\mathrm{Ran}_\pi(S_1(h)\, S_2(f_2)) = \{S_2(f_2)(x)\}$.

If $x_0 \in M(S_1(f_1) S_2(f_2))$, then

$$|T_1(h)(\tau(x_0))\, T_2(k)(\tau(x_0))\, g^2(\tau(x_0))| = |S_1(f_1)(x_0)\, S_2(f_2)(x_0)| = \|S_1(f_1)\, S_2(f_2)\|_\infty = \|T_1(f_1)\, T_2(f_2)\|_\infty = \|T_1(h)\, T_2(k)\, g^2\|_\infty.$$

Since $M(T_1(h)\, T_2(k)\, g^2) = \{\tau(x)\}$, we have that $\tau(x_0) = \tau(x)$, which again implies that $x_0 = x$. Thus $M(S_1(f_1)\, S_2(f_2)) = \{x\}$ and $\mathrm{Ran}_\pi(S_1(f_1)\, S_2(f_2)) = \{S_1(f_1)(x)\, S_2(f_2)(x)\}$.

The following tabulates what has been proven thus far:

	f	$\mathrm{Ran}_\pi(f)$
(a)	$T_1(h)\, T_2(k)\, g^2 = T_1(f_1)\, T_2(f_2)$	$\{T_1(h)(\tau(x))\, T_2(k)(\tau(x))\}$
(b)	$T_1(h)\, T_2(k)\, g = T_1(f_1)\, T_2(k) = T_1(h)\, T_2(f_2)$	$\{T_1(h)(\tau(x))\, T_2(k)(\tau(x))\}$
(c)	$S_1(f_1)\, S_2(k)$	$\{S_1(f_1)(x)\}$
(d)	$S_1(h)\, S_2(f_2)$	$\{S_2(f_2)(x)\}$
(e)	$S_1(f_1)\, S_2(f_2)$	$\{S_1(f_1)(x)\, S_2(f_2)(x)\}$

By (6), the peripheral ranges of (a) and (e) coincide, so

$$T_1(h)(\tau(x))\, T_2(k)(\tau(x)) = S_1(f_1)(x)\, S_2(f_2)(x).$$

Similarly, the peripheral ranges of (b), (c), and (d) coincide, yielding

$$T_1(h)(\tau(x))\, T_2(k)(\tau(x)) = S_1(f_1)(x) = S_2(f_2)(x).$$

Therefore $T_1(h)(\tau(x))\, T_2(k)(\tau(x)) = (T_1(h)(\tau(x))\, T_2(k)(\tau(x)))^2$, which implies that $T_1(h)(\tau(x))\, T_2(k)(\tau(x)) = 1$. □

Given $h, k \in S_1^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$ and $f \in S_2^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$, where $x \in X \setminus \{e_X\}$, Lemma 4.1 implies that $T_1(h)(\tau(x))\, T_2(f)(\tau(x)) = 1 = T_1(k)(\tau(x))\, T_2(f)(\tau(x))$. Thus $T_1(h)(\tau(x)) = T_1(k)(\tau(x))$ holds for any pair $h, k \in S_1^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$, and we define the map $\rho_1 \colon X \to \mathbb{K}$ by $\rho_1(e_X) = 1$ and

$$\rho_1(x) = T_1(h)(\tau(x)) \tag{7}$$

for $x \in X \setminus \{e_X\}$ and $h \in S_1^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$. Note that this assignment is independent of the choice of h. Similarly, we define the map $\rho_2 \colon X \to \mathbb{K}$ by $\rho_2(e_X) = 1$ and

$$\rho_2(x) = T_2(h)(\tau(x)) \tag{8}$$

for $x \in X \setminus \{e_X\}$ and $h \in S_2^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$, which is again independent of the choice of h. Now, given an $x \in X \setminus \{e_X\}$, an $h \in S_1^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$, and a $k \in S_2^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$, Lemma 4.1 implies that $\rho_1(x)\rho_2(x) = T_1(h)(\tau(x))\,T_2(k)(\tau(x)) = 1$. Thus $\rho_1(x)\rho_2(x) = 1$ for all $x \in X \setminus \{e_X\}$, and since $\rho_1(e_x) = \rho_2(e_x) = 1$, we have that $\rho_1(x)\rho_2(x) = 1$ for all $x \in X$.

Lemma 4.2.

Let $f \in \mathrm{Lip}_0(X)$ and $x \in X$. Then $T_1(f)(\tau(x)) = \rho_1(x)S_1(f)(x)$ and $T_2(f)(\tau(x)) = \rho_2(x)S_2(f)(x)$.

Proof. Since $\rho_1(x)\rho_2(x) = 1$, we have $T_2(f)(\tau(x)) = \rho_2(x)S_2(f)(x)$ if and only if $S_2(f)(x) = \rho_1(x)\,T_2(f)(\tau(x))$. If $x = e_X$, we have $S_2(f)(e_X) = 0 = \rho_1(e_X)\,T_2(f)(\tau(e_X))$. Suppose $x \neq e_X$ and $S_2(f)(x) = 0$, then, given $\varepsilon > 0$, Lemma 2.1 (c) implies that there exists a peaking function $h \in \mathcal{P}_x(\mathrm{Lip}_0(X))$ such that $\|hS_2(f)\|_\infty < \varepsilon$. Choosing $k \in \mathrm{Lip}_0(X)$ such that $S_1(k) = h$, then, as $k \in S_1^{-1}[\mathcal{P}_x(\mathrm{Lip}_0(X))]$, (7) yields that $\rho_1(x) = T_1(k)(\tau(x))$. Hence

$$|\rho_1(x)\,T_2(f)(\tau(x))| = |T_1(k)(\tau(x))\,T_2(f)(\tau(x))| \le \|T_1(k)\,T_2(f)\|_\infty = \|S_1(k)\,S_2(f)\|_\infty = \|hS_2(f)\|_\infty < \varepsilon.$$

As ε was chosen arbitrarily, $\rho_1(x)\,T_2(f)(\tau(x)) = 0 = S_2(f)(x)$.

If $S_2(f)(x) \neq 0$, then, by Lemma 2.1 (b), there exists a peaking function $h \in \mathcal{P}_x(\mathrm{Lip}_0(X))$ such that $M(h) = M(hS_2(f)) = \{x\}$, and note that $\mathrm{Ran}_\pi(hS_2(f)) = \{S_2(f)(x)\}$. If $k \in \mathrm{Lip}_0(X)$ is such that $S_1(k) = h$ and $y \in M(T_1(k)\,T_2(f))$, then

$$|h(\psi(y))\,S_2(f)(\psi(y))| = |S_1(k)(\psi(y))\,S_2(f)(\psi(y))| = |T_1(k)(y)\,T_2(f)(y)| = \|T_1(k)\,T_2(f)\|_\infty = \|S_1(k)\,S_2(f)\|_\infty = \|hS_2(f)\|_\infty.$$

Since $M(hS_2(f)) = \{x\}$, we have that $\psi(y) = x$ and $y = \tau(x)$, hence $M(T_1(k)\,T_2(f)) = \{\tau(x)\}$. This implies that $\mathrm{Ran}_\pi(T_1(k)\,T_2(f)) = \{T_1(k)(\tau(x))\,T_2(f)(\tau(x))\}$.

By (6), $\mathrm{Ran}_\pi(S_1(k)\,S_2(f)) \cap \mathrm{Ran}_\pi(T_1(k)\,T_2(f)) \neq \varnothing$, so $S_2(f)(x) = T_1(k)(\tau(x))\,T_2(f)(\tau(x)) = \rho_1(x)\,T_2(f)(\tau(x))$. A similar argument shows $S_1(f)(x) = \rho_2(x)\,T_1(f)(\tau(x))$. As $\rho_1(x)\rho_2(x) = 1$ for all $x \in X$, $T_1(f)(\tau(x)) = \rho_1(x)S_1(f)(x)$ and $T_2(f)(\tau(x)) = \rho_2(x)S_2(f)(x)$. □

We now prove Main Theorem.

Proof of Main Theorem. The mappings T_1, T_2, S_1, and S_2 satisfy $\|T_1(f)\,T_2(g)\|_\infty = \|S_1(f)\,S_2(g)\|_\infty$ for all $f, g \in \mathrm{Lip}_0(X)$, thus we can apply all of the previous results. Let $\psi \colon Y \to X$ be the formal inverse of the mapping τ defined by (4). Note that $\psi(e_Y) = e_X$ and that Lemma 3.7 implies that ψ is a homeomorphism. Define $\varphi_1, \varphi_2 \colon Y \to \mathbb{K}$ by $\varphi_1 = \rho_1 \circ \psi$ and $\varphi_2 = \rho_2 \circ \psi$ – where ρ_1 and ρ_2 are the mappings defined by (7) and (8), respectively – then $\varphi_1(y)\varphi_2(y) = 1$ for all $y \in Y$, and Lemma 4.2 implies that $T_j(f)(y) = \varphi_j(y)S_j(f)(\psi(y))$, $j = 1, 2$, for all $f \in \mathrm{Lip}_0(X)$ and all $y \in Y$. Thus, it is only to show that ψ is a Lipschitz homeomorphism. Indeed, suppose that ψ is not Lipschitz, then Lemma 2.4 gives sequences $\{y_n\}$ and $\{z_n\}$ in Y that converge to a point $y \in Y$ and a function $f \in \mathrm{Lip}_0(X)$ such that $y_n \neq z_n$, $n < d_X(\psi(y_n), \psi(z_n))/d_Y(y_n, z_n)$, $f(\psi(y_n)) = d_X(\psi(y_n), \psi(z_n))$, and $f(\psi(z_n)) = 0$ for all $n \in \mathbb{N}$. Let $h \in \mathrm{Lip}_0(X)$ be such that $S_1(h) = f$, then

$$\begin{aligned} n|\varphi_1(y_n)| &< \frac{|\varphi_1(y_n)|\,d_X(\psi(y_n), \psi(z_n))}{d_Y(y_n, z_n)} = \frac{|\varphi_1(y_n)f(\psi(y_n)) - \varphi_1(z_n)f(\psi(z_n))|}{d_Y(y_n, z_n)} \\ &= \frac{|\varphi_1(y_n)S_1(h)(\psi(y_n)) - \varphi_1(z_n)S_1(h)(\psi(z_n))|}{d_Y(y_n, z_n)} = \frac{|T_1(h)(y_n) - T_1(h)(z_n)|}{d_Y(y_n, z_n)} \le L_{d_Y}(T_1(h)) \end{aligned}$$

for all $n \in \mathbb{N}$, so $\varphi_1(y_n) \to 0$. By a similar argument $\varphi_2(y_n) \to 0$, and, consequently, $\varphi_1(y_n)\varphi_2(y_n) \to 0$. However this is not possible since $\varphi_1(y_n)\varphi_2(y_n) = 1$ for all n. This contradiction shows that ψ is a Lipschitz function. An analogous argument shows that $\psi^{-1} = \tau$ is Lipschitz, which completes the proof. □

The importance of this result is that it connects the range structure of the functions in $\mathrm{Lip}_0(X)$ and $\mathrm{Lip}_0(Y)$ – via the generalized weak peripheral multiplicativity condition (6) – to the underlying topological structures of X and Y – via the homeomorphism ψ – and to the algebraic structures of $\mathrm{Lip}_0(X)$ and $\mathrm{Lip}_0(Y)$ – via the resulting characterization of T_j and S_j as generalized weighted composition operators (2).

4.1. Corollaries

In general, the mappings ρ_j defined by (7) and (8) need not be continuous, cf. [8, Example 2.1]. However, given a proper, open neighborhood U of e_X, then ρ_j is Lipschitz on $X \setminus U$.

Corollary 4.3.
Let U be a proper, open neighborhood of e_X and let $j \in \{1,2\}$. Then ρ_j is Lipschitz on $X \setminus U$.

Proof. Set $F = X \setminus U$, then the function

$$f(x) = \frac{d(x, e_X)}{d(x, e_X) + \mathrm{dist}(x, F)}$$

is Lipschitz, $f(e_X) = 0$, and $f[F] = \{1\}$. Thus $f \in \mathcal{P}_x(\mathrm{Lip}_0(X))$ for each $x \in F$. Let $h \in \mathrm{Lip}_0(X)$ be such that $S_j(h) = f$, then by definition $\rho_j(x) = T_j(h)(\tau(x))$ for all $x \in F$. By Main Theorem, τ is Lipschitz, and, as $T_j(h) \in \mathrm{Lip}_0(Y)$, it follows that ρ_j is Lipschitz on $F = X \setminus U$. □

When S_1 and S_2 are identity mappings, then T_1 and T_2 are weighted composition operators.

Corollary 4.4.
Let (X, d_X) and (Y, d_Y) be pointed compact metric spaces, and let $T_1, T_2\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be surjective mappings that satisfy

$$\mathrm{Ran}_\pi(T_1(f)\,T_2(g)) \cap \mathrm{Ran}_\pi(fg) \neq \varnothing \qquad \text{for all} \quad f, g \in \mathrm{Lip}_0(X).$$

Then there exist mappings $\varphi_1, \varphi_2\colon Y \to \mathbb{K}$ with $\varphi_1(y)\varphi_2(y) = 1$ for all $y \in Y$ and a base-point preserving Lipschitz homeomorphism $\psi\colon Y \to X$ such that

$$T_j(f)(y) = \varphi_j(y) f(\psi(y)) \qquad \text{for all} \quad f \in \mathrm{Lip}_0(X), \quad y \in Y, \quad j = 1, 2.$$

In particular,

1. *$\widetilde{T}(f) = \varphi_2 \cdot T_1(f) = f \circ \psi$ is a sup-norm-preserving algebra isomorphism.*
2. *If $T_1 = T_2 = T$, then $T(f) = \varphi \cdot (f \circ \psi)$ where φ is a mapping from Y to $\{-1, 1\}$.*

Given any $f \in \mathrm{Lip}_0(X)$, notice that the function $\overline{f}\colon X \to \mathbb{C}$ defined by $\overline{f}(x) = \overline{f(x)}$ is again Lipschitz. Another case of Main Theorem concerns the situation when T_1 is the conjugation of T_2, S_1 is conjugation, and S_2 is the identity mapping. This generalizes what was considered by Molnár in his seminal paper [13, Theorem 6].

Corollary 4.5.
Let (X, d_X) and (Y, d_Y) be pointed compact metric spaces, and let $T\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a surjective mapping that satisfies

$$\mathrm{Ran}_\pi(\overline{T(f)}\,T(g)) \cap \mathrm{Ran}_\pi(\overline{f}g) \neq \varnothing \qquad \text{for all} \quad f, g \in \mathrm{Lip}_0(X).$$

Then there exists a unimodular mapping $\varphi\colon Y \to K$ and a base-point preserving Lipschitz homeomorphism $\psi\colon Y \to X$ such that

$$T(f)(y) = \varphi(y) f(\psi(y)) \qquad \text{for all} \quad f \in \mathrm{Lip}_0(X), \quad y \in Y.$$

Next we describe the form of all jointly weakly peripherally multiplicative surjective maps between Lipschitz algebras $\mathrm{Lip}(X)$. Recall that every Lip space can be identified with a convenient Lip_0 space. Given a metric space (X,d) and a point $e_X \notin X$, set $X_0 = X \cup \{e_X\}$, and define on X_0 the metric $d_{X_0}\colon X_0 \times X_0 \to \mathbb{R}$ by

$$d_{X_0}(x,y) = \begin{cases} \min\{2, d_X(x,y)\} & \text{if} \quad x,y \in X, \\ 1 & \text{if} \quad x = e_X \text{ or } y = e_X \text{ (but not both)}, \\ 0 & \text{if} \quad x = y = e_X. \end{cases}$$

The mapping $T_X\colon \mathrm{Lip}(X,d) \to \mathrm{Lip}_0(X_0, d_{X_0})$ given by

$$T_X(f)(x) = f(x), \qquad x \in X, \qquad T_X(f)(e_X) = 0$$

is an isometric isomorphism. See [8, Lemma 3.3] for a proof.

Corollary 4.6.

Let (X,d_X) and (Y,d_Y) be compact metric spaces. Assume that $T_1,T_2\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ and $S_1,S_2\colon \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ are surjective mappings satisfying

$$\mathrm{Ran}_\pi(T_1(f)T_2(g)) \cap \mathrm{Ran}_\pi(S_1(f)S_2(g)) \neq \varnothing \qquad \text{for all} \quad f,g \in \mathrm{Lip}(X).$$

Then there exist Lipschitz functions $\varphi_1,\varphi_2\colon Y \to \mathbb{K}$ with $\varphi_1(y)\varphi_2(y) = 1$ for all $y \in Y$ and a Lipschitz homeomorphism $\psi\colon Y \to X$ such that

$$T_j(f)(y) = \varphi_j(y)S_j(f)(\psi(y)) \qquad \text{for all} \quad f \in \mathrm{Lip}(X), \quad y \in Y, \quad j = 1,2.$$

In particular, if S_1 and S_2 are identity functions on $\mathrm{Lip}(X)$, then

$$T_j(f)(y) = \varphi_j(y)f(\psi(y)) \qquad \text{for all} \quad f \in \mathrm{Lip}(X), \quad y \in Y, \quad j = 1,2,$$

$T_1(f)T_2(1) = T_1(1)T_2(f)$ for all $f \in \mathrm{Lip}(X)$, and $T\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ defined by $T(f) = T_1(f)T_2(1) = f\circ\psi$ is an algebra isomorphism.

Proof. It is clear that $\widehat{T}_j = T_Y T_j T_X^{-1}$, $j = 1,2$, from $\mathrm{Lip}_0(X_0, d_{X_0})$ to $\mathrm{Lip}_0(Y_0, d_{Y_0})$ and $\widehat{S}_j = T_X S_j T_X^{-1}$, $j = 1,2$, from $\mathrm{Lip}_0(X_0, d_{X_0})$ to $\mathrm{Lip}_0(X_0, d_{X_0})$ are surjective mappings satisfying

$$\mathrm{Ran}_\pi(\widehat{T}_1(f)\widehat{T}_2(g)) \cap \mathrm{Ran}_\pi(\widehat{S}_1(f)\widehat{S}_2(g)) \neq \varnothing \qquad \text{for all} \quad f,g \in \mathrm{Lip}_0(X_0, d_{X_0}).$$

By Main Theorem, there exist mappings $\widehat{\varphi}_1,\widehat{\varphi}_2\colon Y_0 \to \mathbb{K}$ with $\widehat{\varphi}_1(y)\widehat{\varphi}_2(y) = 1$ for all $y \in Y_0$ and a base-point preserving Lipschitz homeomorphism $\widehat{\psi}\colon Y_0 \to X_0$ such that

$$\widehat{T}_j(f)(y) = \widehat{\varphi}_j(y)\widehat{S}_j(f)(\widehat{\psi}(y)) \qquad \text{for all} \quad f \in \mathrm{Lip}_0(X_0, d_{X_0}), \quad y \in Y_0, \quad j = 1,2.$$

Let $\varphi_j = \widehat{\varphi}_j\restriction_Y$ for $j = 1,2$ and $\psi = \widehat{\psi}\restriction_Y$. Then $\varphi_1(y)\varphi_2(y) = 1$ for all $y \in Y$ and $\psi\colon Y \to X$ is a Lipschitz homeomorphism such that

$$\varphi_j(y)S_j(f)(\psi(y)) = \widehat{\varphi}_j(y)T_X(S_j(f))(\widehat{\psi}(y)) = \widehat{\varphi}_j(y)\widehat{S}_j(T_X(f))(\widehat{\psi}(y)) = \widehat{T}_j(T_X(f))(y) = T_Y(T_j(f))(y) = T_j(f)(y)$$

for all $f \in \mathrm{Lip}(X)$, all $y \in Y$ and $j = 1,2$. Finally, using that S_1 and S_2 are surjective and that the function constantly 1 on X is in $\mathrm{Lip}(X)$, we conclude that φ_1 and φ_2 are in $\mathrm{Lip}(Y)$. $\square$

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