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Projections and averages of isometries on Lipschitz spaces

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ABSTRACT

We characterize projections on spaces of Lipschitz functions expressed as the average of two and three linear surjective isometries. Generalized bi-circular projections are the only projections on these spaces given as the convex combination of two surjective isometries.

1. Introduction

Let (X, d) be a metric space and let $\mathbb{K}$ be the field of real or complex numbers. A function $f : X \to \mathbb{K}$ is said to be Lipschitz if

$$L(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < \infty.$$

The Lipschitz space $\mathrm{Lip}(X)$ is the Banach space of all $\mathbb{K}$-valued bounded Lipschitz functions f on X with the norm

$$\|f\| = \max\{L(f), \|f\|_\infty\},$$

where

$$\|f\|_\infty = \sup\{|f(x)|: x \in X\}.$$

The little Lipschitz space $\mathrm{lip}(X)$ is the closed subspace of $\mathrm{Lip}(X)$ consisting of those functions f such that

$$\lim_{\delta \to 0} \sup_{0 < d(x,y) < \delta} \frac{|f(x) - f(y)|}{d(x, y)} = 0.$$

The space $\mathrm{Lip}(X)$ separates the points of X but, in some cases, $\mathrm{lip}(X)$ may contain only constant functions. To avoid this pathology, we only consider the little Lipschitz spaces $\mathrm{lip}(X^\alpha)$ with $\alpha \in (0, 1)$, where $X^\alpha = (X, d^\alpha)$ and d^α is the metric on X defined by $d^\alpha(x, y) = d(x, y)^\alpha$ for all $x, y \in X$. It is easy to show that $\mathrm{Lip}(X)$ is contained in $\mathrm{lip}(X^\alpha)$ whenever $\alpha \in (0, 1)$.

Extensive study of surjective linear isometries between spaces of Lipschitz functions started with de Leeuw [5], Mayer-Wolf [6], Roy [7] and Vasavada [8]. In [9], Weaver proves that if X is a complete 1-connected metric space with diameter at

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most 2, then a map T is a linear isometry from $\mathrm{Lip}(X)$ onto itself if and only if T is of the form $T = \tau \cdot (f \circ \phi)$, where ϕ is an isometry from X onto itself and τ is a scalar of modulus 1. Moreover, this characterization also holds true for isometric isomorphisms of $\mathrm{lip}(X^\alpha)$ when X is, in addition, compact.

Unless otherwise stated, throughout this paper, X will denote a compact 1-connected metric space with diameter at most 2, α a real parameter in the interval $(0, 1]$, and $A_\alpha(X)$ will be either $\mathrm{Lip}(X)$ with $\alpha = 1$ or $\mathrm{lip}(X^\alpha)$ with $\alpha \in (0, 1)$.

In this paper 'isometry' on a Banach space refers to a linear surjective distance preserving map. We first gather the essential results on the isometries of $A_\alpha(X)$.

Theorem 1.1. *(See Theorem 2.6.7 and Proposition 3.3.7(a) in [9].) Let X be a compact 1-connected metric space with diameter at most 2. Then a map $T : A_\alpha(X) \to A_\alpha(X)$ is an isometry if and only if there exist a $\tau \in \mathbb{K}$ with $|\tau| = 1$ and a surjective isometry $\phi : X \to X$ such that*

$$T(f)(x) = \tau f\big(\phi(x)\big), \quad \forall f \in A_\alpha(X),\ \forall x \in X.$$

The notion of generalized bi-circular projection was introduced by Fosner, Ilisevic and Li in [4]. We recall that a linear projection P on a Banach space is said to be a generalized bi-circular projection if $P + \lambda(Id - P)$ is an isometry for some $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ and $\lambda \neq 1$. In [2, Proposition 3.7], it was shown that every generalized bi-circular projection of $\mathrm{lip}(X^\alpha)$ with X compact is the average of the identity with an isometric reflection. The same fact was stated there for other Banach spaces of Lipschitz functions, among them, $\mathrm{Lip}(X^\alpha)$ with X compact. The next theorem establishes the form of generalized bi-circular projections on $A_\alpha(X)$.

Theorem 1.2. *Let X be a compact 1-connected metric space with diameter at most 2. Then a map $P : A_\alpha(X) \to A_\alpha(X)$ is a generalized bi-circular projection if and only if there exist a number $\tau \in \{-1, 1\}$ and a surjective isometry $\phi : X \to X$ satisfying $\phi^2(x) = x$ for all $x \in X$ such that*

$$P(f)(x) = \frac{f(x) + \tau f(\phi(x))}{2}, \quad \forall f \in A_\alpha(X),\ \forall x \in X.$$

Proof. If P is the average of the identity with an isometric reflection on $A_\alpha(X)$, then it is immediate that P is a generalized bi-circular projection.

Conversely, let P be a generalized bi-circular projection on $A_\alpha(X)$. Suppose that $P + \lambda(\mathrm{Id} - P)$ is an isometry on $A_\alpha(X)$ for some $\lambda \in \mathbb{K}$ such that $|\lambda| = 1$ and $\lambda \neq 1$. Then, by Theorem 1.1,

$$\big[P + \lambda(\mathrm{Id} - P)\big](f)(x) = \tau f\big(\phi(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big)$$

for some $\tau \in \mathbb{K}$ with $|\tau| = 1$ and ϕ a surjective isometry of X. Therefore

$$P(f)(x) = \frac{1}{1-\lambda}\big[-\lambda f(x) + \tau f\big(\phi(x)\big)\big] \quad \big(f \in A_\alpha(X),\ x \in X\big).$$

Using that P is a projection, we derive the equation

$$\lambda f(x) - (\lambda + 1)\tau f\big(\phi(x)\big) + \tau^2 f\big(\phi^2(x)\big) = 0, \quad \forall f \in A_\alpha(X),\ \forall x \in X.$$

If $x \neq \phi(x)$ and $x \neq \phi^2(x)$ for some $x \in X$, we can take a function $f \in A_\alpha(X)$ such that $f(x) = 1$ and $f(\phi(x)) = f(\phi^2(x)) = 0$ (see Lemma 1.3). Thus, $\lambda = 0$, a contradiction. Hence $\phi(x) = x$ or $\phi^2(x) = x$. In either case, $\phi^2 = \mathrm{Id}$.

We now distinguish two cases. If $\phi \neq \mathrm{Id}$, let us take some $x_0 \in X$ such that $x_0 \neq \phi(x_0)$ and consider $f \in A_\alpha(X)$ such that $f(x_0) = 1$ and $f(\phi(x_0)) = 0$. Then we have

$$\lambda + \tau^2 = \lambda f(x_0) - (\lambda + 1)\tau f\big(\phi(x_0)\big) + \tau^2 f\big(\phi^2(x_0)\big) = 0,$$
$$\big(\lambda - (\lambda + 1)\tau + \tau^2\big)1_X = \lambda 1_X - (\lambda + 1)\tau 1_X + \tau^2 1_X = 0,$$

where 1_X is the function constantly 1 on X. Thus, $\lambda = -1$ and $\tau^2 = 1$. Then

$$P(f) = \frac{1}{2}\big[f + \tau \cdot (f \circ \phi)\big], \quad \forall f \in A_\alpha(X).$$

If $\phi = \mathrm{Id}$, using 1_X as above we obtain $\lambda - (\lambda + 1)\tau + \tau^2 = 0$. Hence $\tau = \lambda$ or $\tau = 1$. If $\tau = \lambda$, we have

$$P(f) = \frac{1}{1-\lambda}(-\lambda f + \lambda f) = 0 = \frac{1}{2}\big[f + (-1)(f \circ \phi)\big], \quad \forall f \in A_\alpha(X),$$

and if $\tau = 1$,

$$P(f) = f = \frac{1}{2}\big[f + (f \circ \phi)\big], \quad \forall f \in A_\alpha(X). \qquad \square$$

Hence every generalized bi-circular projection on $A_\alpha(X)$ can be expressed as the average of two isometries. In Section 2, we show that generalized bi-circular projections are the only linear projections on $A_\alpha(X)$ satisfying this property. In order to achieve this goal, we first characterize when the average of two isometries is a projection on $A_\alpha(X)$. Similar studies were obtained in [1,3] for such projections on the Banach spaces of continuous functions with values in the complex field or in a strictly convex Banach space. The methods used in the second section are expanded in Section 3 to study when the average of three isometries is a projection on $A_\alpha(X)$. The concept of n-circular projection permits us to state that the average P of two (three) isometries on $A_\alpha(X)$ is a projection if and only if P is either a trivial projection or a 2-circular projection (respectively, 3-circular projection). We close the paper with a question, some illustrative examples and some remarks.

We start with a preliminary lemma that will be used repeatedly throughout the paper.

Lemma 1.3. *Let X be a compact metric space, Y a closed subset of X, and a an element in $X \setminus Y$. The mapping $f : X \to [0,1]$ defined by*

$$f(x) = \max\left\{0, 1 - \frac{d(x,a)}{d(Y,a)}\right\}, \quad \forall x \in X,$$

belongs to $A_\alpha(X)$, $f(x) = 0$ for all $x \in Y$ and $f(a) = 1$.

2. Projections in the convex hull of two isometries

Let I_1 and I_2 be two isometries on $A_\alpha(X)$ defined by

$$I_k(f)(x) = \tau_k f\big(\phi_k(x)\big), \quad \forall f \in A_\alpha(X),\ \forall x \in X\ (k = 1, 2),$$

where $\tau_k \in \mathbb{K}$ with $|\tau_k| = 1$ and $\phi_k : X \to X$ is a surjective isometry.

Our initial focus is to find conditions on the constants τ_k, the functions ϕ_k and the parameter $0 < \lambda < 1$ under which $\lambda I_1 + (1-\lambda)I_2$ is a projection on $A_\alpha(X)$.

Proposition 2.1. *Let P be a projection on $A_\alpha(X)$ and $0 < \lambda < 1$. If $P = \lambda I_1 + (1-\lambda)I_2$, we have:*

i) *$\tau_1 = \tau_2 = 1$, or $\tau_1 = -\tau_2$ and $\lambda = 1/2$.*
ii) *If $\phi_1(x) \neq \phi_2(x)$, then either $\phi_1(x) = x$ or $\phi_2(x) = x$.*
iii) *If $x = \phi_1(x) \neq \phi_2(x)$, then $\phi_1(\phi_2(x)) = \phi_2(x)$, $\phi_2^2(x) = x$, $\lambda = 1/2$, $\tau_1 = 1$ and $\tau_2^2 = 1$.*
iv) *If $x = \phi_2(x) \neq \phi_1(x)$, then $\phi_2(\phi_1(x)) = \phi_1(x)$, $\phi_1^2(x) = x$, $\lambda = 1/2$, $\tau_2 = 1$ and $\tau_1^2 = 1$.*

Proof. We have

$$P(f)(x) = \lambda\tau_1 f\big(\phi_1(x)\big) + (1-\lambda)\tau_2 f\big(\phi_2(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big).$$

Since P is a projection on $A_\alpha(X)$, that is $P^2(f) = P(f)$ for all $f \in A_\alpha(X)$, then

$$\begin{aligned}&\lambda^2\tau_1^2 f\big(\phi_1^2(x)\big) + \lambda(1-\lambda)\tau_1\tau_2 f\big(\phi_2\big(\phi_1(x)\big)\big) + \lambda(1-\lambda)\tau_1\tau_2 f\big(\phi_1\big(\phi_2(x)\big)\big) + (1-\lambda)^2\tau_2^2 f\big(\phi_2^2(x)\big)\\ &\quad = \lambda\tau_1 f\big(\phi_1(x)\big) + (1-\lambda)\tau_2 f\big(\phi_2(x)\big),\end{aligned} \tag{1}$$

holds for every $f \in A_\alpha(X)$ and all $x \in X$. In particular, taking $f = 1_X$, we obtain

$$\big[\lambda\tau_1 + (1-\lambda)\tau_2\big]^2 = \lambda\tau_1 + (1-\lambda)\tau_2.$$

Hence $\lambda\tau_1 + (1-\lambda)\tau_2 = 0$ which gives $\lambda = 1/2$ and $\tau_1 = -\tau_2$, or $\lambda\tau_1 + (1-\lambda)\tau_2 = 1$ which implies $\tau_1 = \tau_2 = 1$. This proves i).

In order to prove ii), let $x \in X$ be such that $\phi_1(x) \neq \phi_2(x)$ and assume on the contrary that $\phi_1(x) \neq x$ and $\phi_2(x) \neq x$. We claim that $\phi_1^2(x) = \phi_2(x)$. Otherwise, we set $Y = \{\phi_1(x), \phi_1^2(x), \phi_2(\phi_1(x)), \phi_2^2(x)\}$ and $a = \phi_2(x)$ in Lemma 1.3. It then asserts the existence of a function $f : X \to [0,1]$ in $A_\alpha(X)$ that vanishes at all the points in Y and is equal to 1 at a. Hence Eq. (1) reduces to $\lambda f(\phi_1(\phi_2(x))) = 1$ and so $f(\phi_1(\phi_2(x))) > 1$. This contradiction proves our claim. It follows that $\phi_1(\phi_2(x)) \neq \phi_2(x)$, and another application of Lemma 1.3 with $Y = \{\phi_1(x), \phi_2(\phi_1(x)), \phi_1(\phi_2(x)), \phi_2^2(x)\}$ and $a = \phi_2(x)$ yields $\lambda^2 = 1 - \lambda$. Then $\lambda = (-1+\sqrt{5})/2$.

Similarly, we can show that $\phi_2^2(x) = \phi_1(x)$ and therefore $\phi_2(\phi_1(x)) \neq \phi_1(x)$. Considering now $Y = \{\phi_2(x), \phi_2(\phi_1(x)), \phi_1(\phi_2(x)), \phi_1^2(x)\}$, $a = \phi_1(x)$ and $f \in A_\alpha(X)$ as in Lemma 1.3, Eq. (1) becomes $(1-\lambda)^2 = \lambda$ and so $\lambda = (3+\sqrt{5})/2$ which is impossible. This proves ii).

We now prove iii). If $x = \phi_1(x) \neq \phi_2(x)$, Eq. (1) can be rewritten as

$$\begin{aligned}&\lambda^2\tau_1^2 f(x) + \lambda(1-\lambda)\tau_1\tau_2 f\big(\phi_2(x)\big) + \lambda(1-\lambda)\tau_1\tau_2 f\big(\phi_1\big(\phi_2(x)\big)\big) + (1-\lambda)^2\tau_2^2 f\big(\phi_2^2(x)\big)\\ &\quad = \lambda\tau_1 f(x) + (1-\lambda)\tau_2 f\big(\phi_2(x)\big)\end{aligned} \tag{2}$$

for every $f \in A_\alpha(X)$. If $\phi_1(\phi_2(x)) = x$ or $\phi_2^2(x) = \phi_2(x)$, we have $\phi_2(x) = x$, a contradiction. Hence $\phi_1(\phi_2(x)) \neq x$ and $\phi_2^2(x) \neq \phi_2(x)$.

We now show that $\phi_1(\phi_2(x)) = \phi_2(x)$. Otherwise, we consider $f \in A_\alpha(X)$ as in Lemma 1.3 with $Y = \{x, \phi_2^2(x), \phi_1(\phi_2(x))\}$ and $a = \phi_2(x)$. Then Eq. (2) reduces to $\lambda = 1$, which is impossible.

Similarly, we see that $\phi_2^2(x) = x$. If $\phi_2^2(x) \neq x$, we consider $f \in A_\alpha(X)$ as in Lemma 1.3 with $Y = \{\phi_2(x), \phi_2^2(x), \phi_1(\phi_2(x))\}$ and $a = x$. Eq. (2) gives $\lambda = 0$ or $\lambda = 1$, which is not possible.

Therefore $\phi_1(\phi_2(x)) = \phi_2(x)$ and $\phi_2^2(x) = x$. Then Eq. (2) is rewritten as

$$\lambda^2\tau_1^2 f(x) + \lambda(1-\lambda)\tau_1\tau_2 f\big(\phi_2(x)\big) + \lambda(1-\lambda)\tau_1\tau_2 f\big(\phi_2(x)\big) + (1-\lambda)^2\tau_2^2 f(x) \\ = \lambda\tau_1 f(x) + (1-\lambda)\tau_2 f\big(\phi_2(x)\big) \tag{3}$$

for all $f \in A_\alpha(X)$. In particular, taking $Y = \{x\}$, $a = \phi_2(x)$ and $f \in A_\alpha(X)$ as in Lemma 1.3, Eq. (3) becomes $2\lambda(1-\lambda)\tau_1\tau_2 = (1-\lambda)\tau_2$ which yields $\lambda = 1/2$ and $\tau_1 = 1$. Taking $f = 1_X$ in Eq. (3), it follows that $\tau_2^2 = 1$, and this completes the proof of iii). Similar arguments apply to prove iv). □

We now give a characterization of the operators $(I_1 + I_2)/2$ that are projections on $A_\alpha(X)$.

Proposition 2.2. *The operator $(I_1 + I_2)/2$ is a projection on $A_\alpha(X)$ if and only if one of the following statements holds*:

(1) $\tau_1 = \tau_2 = 1$ *and every* $x \in X$ *satisfies*:
   (a) $x = \phi_1(x) = \phi_2(x)$, *or*
   (b) $x = \phi_1(x) \neq \phi_2(x)$, $\phi_1(\phi_2(x)) = \phi_2(x)$ *and* $\phi_2^2(x) = x$, *or*
   (c) $x = \phi_2(x) \neq \phi_1(x)$, $\phi_2(\phi_1(x)) = \phi_1(x)$ *and* $\phi_1^2(x) = x$.
(2) $\tau_1 = -\tau_2$ *and* $\phi_1(x) = \phi_2(x)$ *for every* $x \in X$, *that is* $((I_1 + I_2)/2)(f)(x) = 0$, *for all* $f \in A_\alpha(X)$.
(3) $\tau_1 = 1$, $\tau_2 = -1$ *and every* $x \in X$ *satisfies*:
   (a) $\phi_1(x) = \phi_2(x)$, *or*
   (b) $x = \phi_1(x) \neq \phi_2(x)$, $\phi_1(\phi_2(x)) = \phi_2(x)$ *and* $\phi_2^2(x) = x$.
(4) $\tau_1 = -1$, $\tau_2 = 1$ *and every* $x \in X$ *satisfies*:
   (a) $\phi_1(x) = \phi_2(x)$, *or*
   (b) $x = \phi_2(x) \neq \phi_1(x)$, $\phi_2(\phi_1(x)) = \phi_1(x)$ *and* $\phi_1^2(x) = x$.

Proof. Recall that $(I_1 + I_2)/2$ is a projection on $A_\alpha(X)$ if and only if

$$\tau_1^2 f\big(\phi_1^2(x)\big) + \tau_1\tau_2 f\big(\phi_2\big(\phi_1(x)\big)\big) + \tau_1\tau_2 f\big(\phi_1\big(\phi_2(x)\big)\big) + \tau_2^2 f\big(\phi_2^2(x)\big) = 2\big[\tau_1 f\big(\phi_1(x)\big) + \tau_2 f\big(\phi_2(x)\big)\big], \tag{4}$$

for every $f \in A_\alpha(X)$ and all $x \in X$.

It is straightforward to check that Eq. (4) holds for each of the cases (1) through (4) in the statement of the proposition. Conversely, assume that $(I_1 + I_2)/2$ is a projection. Then $\tau_1 = \tau_2 = 1$ or $\tau_1 = -\tau_2$ by Proposition 2.1i).

Let us assume first $\tau_1 = \tau_2 = 1$. Hence Eq. (4) reduces to

$$f\big(\phi_1^2(x)\big) + f\big(\phi_2\big(\phi_1(x)\big)\big) + f\big(\phi_1\big(\phi_2(x)\big)\big) + f\big(\phi_2^2(x)\big) = 2\big[f\big(\phi_1(x)\big) + f\big(\phi_2(x)\big)\big] \tag{5}$$

for every $f \in A_\alpha(X)$ and $x \in X$. Let $x \in X$. If $\phi_1(x) = \phi_2(x)$, Eq. (5) becomes

$$f\big(\phi_1^2(x)\big) + f\big(\phi_2^2(x)\big) = 2f\big(\phi_1(x)\big)$$

for every $f \in A_\alpha(X)$. In particular, taking

$$f(z) = d\big(z, \phi_1(x)\big), \quad \forall z \in X,$$

we get $d(\phi_1^2(x), \phi_1(x)) + d(\phi_2^2(x), \phi_1(x)) = 0$. This gives $\phi_1(x) = x$ and so $x = \phi_1(x) = \phi_2(x)$, as in the condition (1)(a). Assume now $\phi_1(x) \neq \phi_2(x)$. According to the statements iii) and iv) in Proposition 2.1, x satisfies either the condition (1)(b) or the condition (1)(c). Therefore, statement (1) holds.

Suppose now $\tau_1 = -\tau_2$. If $\phi_1 = \phi_2$, we have the statement (2). Otherwise, let $x \in X$ be such that $\phi_1(x) \neq \phi_2(x)$. Then $\phi_1(x) = x$ or $\phi_2(x) = x$ by Proposition 2.1ii). If the former holds, then Proposition 2.1iii) implies that $\tau_1 = 1$, $\tau_2 = -1$ and x satisfies the condition (3)(b). Moreover, if such x exists then the condition (3)(b) also holds for every $y \in X$ such that $\phi_1(y) \neq \phi_2(y)$. We observe that given $y \in X$ such that $\phi_1(y) \neq \phi_2(y) = y$, then $\tau_2 = 1$ by Proposition 2.1 iv). This contradicts our assumption $\tau_1 = -\tau_2$. If $\phi_2(x) = x$, then Proposition 2.1iv) implies that $\tau_2 = 1 = -\tau_1$, and x satisfies (4)(b). Similar reasoning shows that every $y \in X$ such that $\phi_1(y) \neq \phi_2(y)$ also satisfies the statement claimed in (4)(b). This completes the proof of the proposition. □

We are ready to prove that the only projections on $A_\alpha(X)$ that can be represented as the average of two isometries are generalized bi-circular projections.

Theorem 2.3. *A projection on $A_\alpha(X)$ is the average of two surjective isometries if and only if it is a generalized bi-circular projection.*

Proof. A generalized bi-circular projection on $A_\alpha(X)$ is the average of the identity and an involutive isometry by Theorem 1.2.

Conversely, assume that $(I_1 + I_2)/2$ is a projection on $A_\alpha(X)$ where I_1 and I_2 are isometries on $A_\alpha(X)$, of the form

$$I_k(f)(x) = \tau_k f\big(\phi_k(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big) \quad (k = 1, 2),$$

where $\tau_k \in \mathbb{K}$ with $|\tau_k| = 1$ and $\phi_k : X \to X$ is a surjective isometry.

In view of Proposition 2.2, we can consider four cases. Taking into account Theorem 1.2, our goal is to find in each one of these cases a number $\tau \in \{-1, 1\}$ and a surjective isometry $\phi : X \to X$ satisfying $\phi^2(x) = x$ and

$$\tau_1 f\big(\phi_1(x)\big) + \tau_2 f\big(\phi_2(x)\big) = f(x) + \tau f\big(\phi(x)\big) \tag{6}$$

for every $f \in A_\alpha(X)$ and all $x \in X$.

According to Proposition 2.1, the sets X_0, X_1 and X_2 given by

$$X_0 = \big\{x \in X:\ \phi_1(x) = \phi_2(x)\big\},$$
$$X_1 = \big\{x \in X:\ x = \phi_1(x) \neq \phi_2(x),\ \phi_1\big(\phi_2(x)\big) = \phi_2(x),\ \phi_2^2(x) = x\big\}$$

and

$$X_2 = \big\{x \in X:\ x = \phi_2(x) \neq \phi_1(x),\ \phi_2\big(\phi_1(x)\big) = \phi_1(x),\ \phi_1^2(x) = x\big\}$$

constitute a partition of X. Define now the function

$$\phi(x) = \begin{cases} x & \text{if } x \in X_0, \\ \phi_2(x) & \text{if } x \in X_1, \\ \phi_1(x) & \text{if } x \in X_2. \end{cases}$$

It is easy to show that $x \in X_1$ ($x \in X_2$) if and only if $\phi_2(x) \in X_1$ (respectively, $\phi_1(x) \in X_2$). Using this, we show that ϕ is involutive. Indeed, if $x \in X_0$, we have $\phi^2(x) = \phi(x) = x$; if $x \in X_1$, then $\phi^2(x) = \phi(\phi_2(x)) = \phi_2^2(x) = x$; and if $x \in X_2$ we conclude that $\phi^2(x) = \phi(\phi_1(x)) = \phi_1^2(x) = x$. Notice that ϕ is surjective since it is involutive.

We now check that ϕ is an isometry. Let $x, y \in X$. For $x \in X_0$ and $y \in X_1$, we have

$$d\big(\phi(x), \phi(y)\big) = d\big(x, \phi_2(y)\big) = d\big(\phi_1(x), \phi_1\big(\phi_2(y)\big)\big) = d\big(\phi_2(x), \phi_2(y)\big) = d(x, y);$$

for $x \in X_0$ and $y \in X_2$,

$$d\big(\phi(x), \phi(y)\big) = d\big(x, \phi_1(y)\big) = d\big(\phi_2(x), \phi_2\big(\phi_1(y)\big)\big) = d\big(\phi_1(x), \phi_1(y)\big) = d(x, y);$$

and, finally, for $x \in X_1$ and $y \in X_2$,

$$d\big(\phi(x), \phi(y)\big) = d\big(\phi_2(x), \phi_1(y)\big) = d\big(\phi_2^2(x), \phi_2\big(\phi_1(y)\big)\big) = d\big(x, \phi_1(y)\big) = d\big(\phi_1(x), \phi_1^2(y)\big) = d(x, y).$$

Notice that taking $f = 1_X$ in Eq. (6), we obtain $\tau = \tau_1 + \tau_2 - 1$. Defining $\tau = 1$ in the case given in the statement (1) of Proposition 2.2 and $\tau = -1$ in the other three cases, it is easy to check that Eq. (6) is satisfied for every $f \in A_\alpha(X)$ and $x \in X$. This completes the proof of the theorem. □

3. Projections in the convex hull of three isometries

In this section we investigate whether the convex hull of three isometries contains any projections. We consider the isometries on $A_\alpha(X)$,

$$I_k(f)(x) = \tau_k f\big(\phi_k(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big) \quad (k = 1, 2, 3),$$

with τ_k unimodular scalars and ϕ_k surjective isometries on X. Throughout this section we set $Q = (I_1 + I_2 + I_3)/3$, this defines an operator on $A_\alpha(X)$. The operator Q is a projection on $A_\alpha(X)$ if and only if

$$\sum_{i,j=1}^{3} \tau_i \tau_j f\big(\phi_j\big(\phi_i(x)\big)\big) = 3 \sum_{k=1}^{3} \tau_k f\big(\phi_k(x)\big), \tag{7}$$

for every $x \in X$ and $f \in A_\alpha(X)$. Taking $f = 1_X$ in Eq. (7), we obtain $\sum_{i,j=1}^{3} \tau_i \tau_j = 3 \sum_{k=1}^{3} \tau_k$, that is

$$(\tau_1 + \tau_2 + \tau_3)^2 = 3(\tau_1 + \tau_2 + \tau_3).$$

Hence $\tau_1 + \tau_2 + \tau_3 = 3$ or $\tau_1 + \tau_2 + \tau_3 = 0$. From these equalities we easily derive the following lemma.

Lemma 3.1. *If Q is a projection, then $\tau_1 = \tau_2 = \tau_3 = 1$ or there exists a permutation of $(1,2,3)$, (l, j, k), such that $\tau_j = e^{2\pi i/3}\tau_l$ and $\tau_k = e^{4\pi i/3}\tau_l$.*

We observe that each triplet $\{\tau_1, \tau_2, \tau_3\}$ as given in the second case of the previous lemma can be referred to as an orbit of the action of the group of the 3rd roots of unity on S^1.

Given an arbitrary point $x \in X$, we define the set

$$S_x = \{\phi_1(x), \phi_2(x), \phi_3(x)\}.$$

We denote by $\operatorname{card}(S_x)$, the cardinality of S_x. Clearly, one of the following holds:

1. $\operatorname{card}(S_x) = 1$, that is $\phi_1(x) = \phi_2(x) = \phi_3(x)$.
2. $\operatorname{card}(S_x) = 2$, that is S_x consists of two elements, as for example $\phi_1(x) = \phi_2(x) \neq \phi_3(x)$.
3. $\operatorname{card}(S_x) = 3$, that is $\phi_1(x) \neq \phi_2(x) \neq \phi_3(x) \neq \phi_1(x)$.

Lemma 3.2. *If Q is a projection on $A_\alpha(X)$, then for every $x \in X$, $\operatorname{card}(S_x)$ is either equal to 1 or equal to 3.*

Proof. We assume that there exists $x \in X$ such that S_x consists of two elements, say $\phi_1(x) = \phi_2(x) \neq \phi_3(x)$. We present the proof for the lemma in this case but the remaining two possibilities follow similarly. Eq. (7) now takes the form

$$\begin{aligned}(\tau_1+\tau_2)\big[\tau_1 f\big(\phi_1^2(x)\big) + \tau_2 f\big(\phi_2^2(x)\big) + \tau_3 f\big(\phi_3\big(\phi_1(x)\big)\big)\big] + \tau_3\big[\tau_1 f\big(\phi_1\big(\phi_3(x)\big)\big) + \tau_2 f\big(\phi_2\big(\phi_3(x)\big)\big) + \tau_3 f\big(\phi_3^2(x)\big)\big]\\ = 3(\tau_1+\tau_2) f\big(\phi_1(x)\big) + 3\tau_3 f\big(\phi_3(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big).\end{aligned} \tag{8}$$

We claim that $\tau_1 + \tau_2 \neq 0$, otherwise Eq. (8) reduces to

$$\tau_1 f\big(\phi_1\big(\phi_3(x)\big)\big) + \tau_2 f\big(\phi_2\big(\phi_3(x)\big)\big) + \tau_3 f\big(\phi_3^2(x)\big) = 3 f\big(\phi_3(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big).$$

In particular, for $f = 1_X$, we have $\tau_1 + \tau_2 + \tau_3 = 3$ and so $\tau_1 = \tau_2 = \tau_3 = 1$. This contradicts our assumption that $\tau_1 + \tau_2 = 0$ and shows that $\tau_1 + \tau_2 \neq 0$.

We now consider the following three possibilities:

i. $x \neq \phi_1(x) = \phi_2(x) \neq \phi_3(x) \neq x$.
ii. $x \neq \phi_1(x) = \phi_2(x) \neq \phi_3(x) = x$.
iii. $x = \phi_1(x) = \phi_2(x) \neq \phi_3(x) \neq x$.

i. $x \neq \phi_1(x) \neq \phi_3(x) \neq x$. Considering now $Y = \{\phi_3(x), \phi_1(\phi_3(x)), \phi_2(\phi_3(x)), \phi_1^2(x), \phi_2^2(x)\}$, $a = \phi_1(x)$ and $f \in A_\alpha(X)$ as in Lemma 1.3, Eq. (8) becomes

$$(\tau_1+\tau_2)\tau_3 f\big(\phi_3\big(\phi_1(x)\big)\big) + \tau_3^2 f\big(\phi_3^2(x)\big) = 3(\tau_1+\tau_2).$$

We observe that $\phi_3(\phi_1(x))$ and $\phi_3^2(x)$ can't both be equal to $\phi_1(x)$ since $\phi_1(x) \neq \phi_3(x)$. If they are both different from $\phi_1(x)$, then we select f satisfying the same conditions as the last function with the additional constraint that it also vanishes at $\phi_3(\phi_1(x))$ and $\phi_3^2(x)$. This leads to a contradiction, since $\tau_1 + \tau_2 \neq 0$. If $\phi_3^2(x) \neq \phi_1(x)$ and $\phi_3(\phi_1(x)) = \phi_1(x)$, an appropriate choice of f implies that $\tau_3 = 3$, which is impossible. The only possibility left is $\phi_3^2(x) = \phi_1(x)$ and $\phi_3(\phi_1(x)) \neq \phi_1(x)$. In such case f can be chosen equal to zero on $\phi_3(\phi_1(x))$ and equal to 1 on $\phi_3^2(x)$. This implies that $\tau_3^2 = 3(\tau_1 + \tau_2)$ and Eq. (8) reduces to

$$\begin{aligned}(\tau_1+\tau_2)\big[\tau_1 f\big(\phi_1^2(x)\big) + \tau_2 f\big(\phi_2^2(x)\big) + \tau_3 f\big(\phi_3\big(\phi_1(x)\big)\big)\big] + \tau_3\big[\tau_1 f\big(\phi_1\big(\phi_3(x)\big)\big) + \tau_2 f\big(\phi_2\big(\phi_3(x)\big)\big)\big]\\ = 3\tau_3 f\big(\phi_3(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big)\end{aligned}$$

or equivalently

$$\begin{aligned}\tau_3^2\big[\tau_1 f\big(\phi_1^2(x)\big) + \tau_2 f\big(\phi_2^2(x)\big) + \tau_3 f\big(\phi_3\big(\phi_1(x)\big)\big)\big] + 3\tau_3\big[\tau_1 f\big(\phi_1\big(\phi_3(x)\big)\big) + \tau_2 f\big(\phi_2\big(\phi_3(x)\big)\big)\big]\\ = 9\tau_3 f\big(\phi_3(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big).\end{aligned}$$

In particular for $f = 1_X$, we have $\tau_3(\tau_1 + \tau_2 + \tau_3) + \tau_3^2 = 9$ and this is impossible.

ii. $x \neq \phi_1(x) \neq \phi_3(x) = x$. Eq. (8) can be written as:

$$(\tau_1+\tau_2)\big[\tau_1 f\big(\phi_1^2(x)\big) + \tau_2 f\big(\phi_2^2(x)\big) + \tau_3 f\big(\phi_3\big(\phi_1(x)\big)\big)\big] = (3-\tau_3)\big[(\tau_1+\tau_2) f\big(\phi_1(x)\big) + \tau_3 f\big(\phi_3(x)\big)\big]$$

for every $x \in X$ and $f \in A_\alpha(X)$. Lemma 1.3 asserts the existence of a function $f \in A_\alpha(X)$ with range the interval $[0,1]$ and such that $f(\phi_1(x)) = 1$, $f(\phi_3(x)) = f(\phi_2^2(x)) = f(\phi_1^2(x)) = 0$. Therefore $\tau_3 f(\phi_3(\phi_1(x))) = 3 - \tau_3$ and this is impossible since $|3 - \tau_3| \geqslant 2$.

iii. $x = \phi_1(x) \neq \phi_3(x) \neq x$. Under these assumptions Eq. (8) can be rewritten as:

$$(\tau_1+\tau_2)^2 f\big(\phi_1(x)\big) + (\tau_1+\tau_2)\tau_3 f\big(\phi_3(x)\big) + \tau_3\big[\tau_1 f\big(\phi_1\big(\phi_3(x)\big)\big) + \tau_2 f\big(\phi_2\big(\phi_3(x)\big)\big) + \tau_3 f\big(\phi_3^2(x)\big)\big]$$
$$= 3(\tau_1+\tau_2) f\big(\phi_1(x)\big) + 3\tau_3 f\big(\phi_3(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big). \tag{9}$$

If $\phi_3(x) \neq \phi_1(\phi_3(x))$ and $\phi_3(x) \neq \phi_2(\phi_3(x))$, then there exists a Lipschitz function f with range in the interval $[0,1]$ and satisfying the conditions $f(\phi_1(x)) = f(\phi_1(\phi_3(x))) = f(\phi_2(\phi_3(x))) = 0$ and $f(\phi_3(x)) = 1$. Eq. (9) becomes $(\tau_1+\tau_2) + \tau_3 f(\phi_3^2(x)) = 3$. This implies that $\phi_3^2(x) = \phi_3(x)$ which contradicts our assumptions. Therefore $\phi_3(x) = \phi_1(\phi_3(x))$ or $\phi_3(x) = \phi_2(\phi_3(x))$. If we assume that $\phi_3(x) = \phi_1(\phi_3(x)) = \phi_2(\phi_3(x))$, then we set f satisfying $f(x) = f(\phi_3^2(x)) = 0$ and $f(\phi_3(x)) = 1$. This implies that $\tau_1 + \tau_2 = 3/2$. On the other hand, by considering $1_X - f$ we get $\tau_3^2 = 9/4$ which is impossible. We have two cases left to analyze. We first assume that $\phi_3(x) = \phi_1(\phi_3(x)) \neq \phi_2(\phi_3(x))$. Eq. (9) reduces to

$$(\tau_1+\tau_2)^2 f(x) + (2\tau_1+\tau_2)\tau_3 f\big(\phi_3(x)\big) + \tau_2\tau_3 f\big(\phi_2\big(\phi_3(x)\big)\big) + \tau_3^2 f\big(\phi_3^2(x)\big)$$
$$= 3(\tau_1+\tau_2) f(x) + 3\tau_3 f\big(\phi_3(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big). \tag{10}$$

We select a Lipschitz function $f : X \to [0,1]$ such that $f(x) = f(\phi_2(\phi_3(x)) = f(\phi_3^2(x)) = 0$ and $f(\phi_3(x)) = 1$. Then we have $2\tau_1 + \tau_2 = 3$ and $\tau_1 = \tau_2 = 1$. Therefore Eq. (10) becomes

$$\tau_3 f\big(\phi_2\big(\phi_3(x)\big)\big) + \tau_3^2 f\big(\phi_3^2(x)\big) = 2f(x) \quad \big(f \in A_\alpha(X),\ x \in X\big).$$

In particular, for a Lipschitz function with range the interval $[0,1]$ with $f(x) = 1$ and $f(\phi_2(\phi_3(x))) = 0$ we have $\tau_3^2 f(\phi_3^2(x)) = 2$. This is clearly impossible. A similar approach also shows that $\phi_3(x) = \phi_2(\phi_3(x)) \neq \phi_1(\phi_3(x))$ leads to a contradiction. □

Lemma 3.3. *Let $x \in X$ be such that $\phi_1(x) = \phi_2(x) = \phi_3(x)$ and $\tau_1 = \tau_2 = \tau_3 = 1$. If Q is a projection, then $x = \phi_1(x) = \phi_2(x) = \phi_3(x)$.*

Proof. Eq. (7) can be rewritten as follows:

$$f\big(\phi_1^2(x)\big) + f\big(\phi_2^2(x)\big) + f\big(\phi_3^2(x)\big) = 3f\big(\phi_1(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big).$$

In particular, taking

$$f(z) = d\big(z, \phi_1(x)\big), \quad \forall z \in X,$$

gives

$$d\big(\phi_1^2(x), \phi_1(x)\big) + d\big(\phi_2^2(x), \phi_1(x)\big) + d\big(\phi_3^2(x), \phi_1(x)\big) = 0$$

which implies $d(\phi_1^2(x), \phi_1(x)) = 0$ and so $\phi_1(x) = x$. □

Lemma 3.4. *Let $x \in X$ be such that $\phi_1(x) \neq \phi_2(x) \neq \phi_3(x) \neq \phi_1(x)$. If Q is a projection, then there exists $k \in \{1,2,3\}$ such that $\phi_k(x) = x$.*

Proof. Suppose that $\phi_k(x) \neq x$ for all $k \in \{1,2,3\}$. Therefore $\phi_j(\phi_k(x)) \neq \phi_j(x)$ for all $j,k \in \{1,2,3\}$. Using Lemma 1.3, we have a function $f \in A_\alpha(X)$ such that $f(\phi_1(x)) = 1$ and $f(\phi_1(\phi_k(x)) = f(\phi_j(x)) = 0$ for all $k \in \{1,2,3\}$ and $j \in \{2,3\}$. Eq. (7) becomes

$$\sum_{k=1,\, j=2}^{3} \tau_k\tau_j f\big(\phi_j\big(\phi_k(x)\big)\big) = 3\tau_1.$$

This implies that at least three points in the set

$$\big\{\phi_2\big(\phi_1(x)\big), \phi_3\big(\phi_1(x)\big), \phi_2^2(x), \phi_3\big(\phi_2(x)\big), \phi_2\big(\phi_3(x)\big), \phi_3^2(x)\big\}$$

must be equal to $\phi_1(x)$. This contradiction proves the statement. □

Lemma 3.5. *Let $x \in X$ be such that $\phi_1(x) \neq \phi_2(x) \neq \phi_3(x) \neq \phi_1(x)$. If Q is a projection, then there exists (l, j, k), a permutation of $(1,2,3)$, such that one of the following holds:*

1. $x = \phi_l(x) = \phi_j(\phi_k(x)) = \phi_k(\phi_j(x))$, $\phi_j(x) = \phi_k^2(x) = \phi_l(\phi_j(x))$, $\phi_k(x) = \phi_j^2(x) = \phi_l(\phi_k(x))$ *and* $\tau_1 = \tau_2 = \tau_3 = 1$, *or* $\tau_l = 1$, $\tau_j = e^{2\pi i/3}$ *and* $\tau_k = e^{4\pi i/3}$.
2. $x = \phi_l(x) = \phi_k^2(x) = \phi_j^2(x)$, $\phi_l(\phi_j(x)) = \phi_j(\phi_k(x)) = \phi_k(x)$, $\phi_l(\phi_k(x)) = \phi_k(\phi_j(x)) = \phi_j(x)$ *and* $\tau_1 = \tau_2 = \tau_3 = 1$.

Proof. From Lemma 3.4 and without loss of generality, we may assume that $\phi_1(x) = x$. Another choice for $f \in A_\alpha(X)$ with $f(x) = 1$ and $f(\phi_2(x)) = f(\phi_3(x)) = 0$, also implies that there must exist at least two points in the set

$$\{\phi_1(\phi_2(x)), \phi_2^2(x), \phi_3(\phi_2(x)), \phi_1(\phi_3(x)), \phi_2(\phi_3(x)), \phi_3^2(x)\}$$

that are equal to x. This implies the following list of possibilities.

(i) $x = \phi_2^2(x) = \phi_3(\phi_2(x))$,
(ii) $x = \phi_2^2(x) = \phi_3^2(x)$,
(iii) $x = \phi_3(\phi_2(x)) = \phi_2(\phi_3(x))$,
(iv) $x = \phi_3^2(x) = \phi_2(\phi_3(x))$.

The symmetry of the equations involved imply that case (iv) follows from a similar argument to the one presented for case (i), by just permuting the indices 2 and 3.

We proceed to show that case (i) leads to an absurd. We select a function $f \in A_\alpha(X)$ so that $f(x) = f(\phi_2(x)) = f(\phi_3^2(x)) = 0$ and $f(\phi_3(x)) = 1$. Therefore we have

$$\tau_2\tau_1 f(\phi_1(\phi_2(x))) + \tau_3[\tau_1 f(\phi_1(\phi_3(x))) + \tau_2 f(\phi_2(\phi_3(x)))] = (3-\tau_1)\tau_3.$$

This implies that at least two points in the set $\{\phi_1(\phi_2(x)), \phi_1(\phi_3(x)), \phi_2(\phi_3(x))\}$ must be equal to $\phi_3(x)$. Since $\phi_1(\phi_2(x)) \neq \phi_1(\phi_3(x))$, we have the following two possibilities: $\phi_3(x) = \phi_1(\phi_2(x)) = \phi_2(\phi_3(x))$ (or $\phi_3(x) = \phi_1(\phi_3(x)) = \phi_2(\phi_3(x))$). Both cases lead to a contradiction following a similar approach. In fact, if $\phi_3(x) = \phi_1(\phi_2(x)) = \phi_2(\phi_3(x))$, we clearly have

$$\phi_1(\phi_2(x)) = \phi_3(x) \neq \phi_2(\phi_2(x)) = \phi_3(\phi_2(x)) = x.$$

Therefore the set $S_{\phi_2(x)}$ has cardinality two which contradicts Lemma 3.2.

We consider case (ii), that is $x = \phi_2^2(x) = \phi_3^2(x)$. We recall that Q is a projection if and only if Eq. (7) holds. In this case, (7) reduces to

$$\begin{aligned} &\tau_1^2 f(x) + \tau_1\tau_2 f(\phi_1(\phi_2(x))) + \tau_1\tau_3 f(\phi_1(\phi_3(x))) + \tau_2^2 f(x) \\ &\quad + \tau_2\tau_3 f(\phi_2(\phi_3(x))) + \tau_1\tau_3 f(\phi_3(x)) + \tau_3\tau_2 f(\phi_3(\phi_2(x))) + \tau_3^2 f(x) \\ &= 3[\tau_1 f(x) + \tau_2 f(\phi_2(x)) + \tau_3 f(\phi_3(x))], \quad (f \in A_\alpha(X),\ x \in X). \end{aligned} \tag{11}$$

We select a function f_0 such that $f_0(x) = f_0(\phi_2(x)) = f_0(\phi_3(\phi_2(x))) = 0$ and $f_0(\phi_3(x)) = 1$. Therefore

$$\tau_1\tau_2 f_0(\phi_1(\phi_2(x))) + \tau_1\tau_3 f_0(\phi_1(\phi_3(x))) + \tau_2\tau_3 f_0(\phi_2(\phi_3(x))) + \tau_1\tau_3 = 3\tau_3. \tag{12}$$

We conclude that at least two elements in $\{\phi_1(\phi_2(x)), \phi_1(\phi_3(x)), \phi_2(\phi_3(x))\}$ must be equal to $\phi_3(x)$. Therefore we have two cases to analyze: 1. $\phi_1(\phi_3(x)) = \phi_2(\phi_3(x)) = \phi_3(x)$ and 2. $\phi_1(\phi_2(x)) = \phi_2(\phi_3(x)) = \phi_3(x)$.

We now examine case 1. $\phi_1(\phi_3(x)) = \phi_2(\phi_3(x)) = \phi_3(x) (\neq \phi_1(\phi_2(x)))$. The function f_0 selected above may be chosen satisfying the additional condition: $f_0(\phi_1(\phi_2(x))) = 0$. Then the equality (12) becomes $\tau_1\tau_3 + \tau_2\tau_3 + \tau_1\tau_3 = 3\tau_3$. This implies $\tau_1 = \tau_2 = \tau_3 = 1$ (see Lemma 3.1). Hence (11) yields $f(\phi_1(\phi_2(x))) + f(\phi_3(\phi_2(x))) = 2f(\phi_2(x))$. This implies that $\phi_1(\phi_2(x)) = \phi_3(\phi_2(x)) = \phi_2(x)$, then the cardinality of $S_{\phi_2(x)}$ is equal to 2, contradicting Lemma 3.2.

Now we consider case 2. $\phi_1(\phi_2(x)) = \phi_2(\phi_3(x)) = \phi_3(x) (\neq \phi_1(\phi_3(x)))$. As done in case 1, we select f_0 with the additional constraint that also vanishes at $\phi_1(\phi_3(x))$. It then follows that $\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 = 3\tau_3$, implying that $\tau_1 = \tau_2 = \tau_3 = 1$. Eq. (11) now yields $f(\phi_1(\phi_3(x))) + f(\phi_3(\phi_2(x))) = 2f(\phi_2(x))$ implying that $\phi_1(\phi_3(x)) = \phi_3(\phi_2(x)) = \phi_2(x)$, as stated in the statement (2).

We now consider case (iii), that is $x = \phi_3(\phi_2(x)) = \phi_2(\phi_3(x))$. As previously done, a choice of a Lipschitz function f such that $f(x) = f(\phi_3(x)) = f(\phi_2^2(x)) = 0$ and $f(\phi_2(x)) = 1$ implies that at least two points in the set $\{\phi_1(\phi_2(x)), \phi_1(\phi_3(x)), \phi_3^2(x)\}$ must be equal to $\phi_2(x)$. This determines the following possibilities: $\phi_2(x) = \phi_1(\phi_2(x)) = \phi_3^2(x)$ or $\phi_2(x) = \phi_1(\phi_3(x)) = \phi_3^2(x)$. An application of Lemma 1.3 yields a Lipschitz function f so that $f(x) = f(\phi_2(x)) = 0$ and $f(\phi_3(x)) = 1$. This leads to the equations:

$$\tau_2^2 f(\phi_2^2(x)) + \tau_3\tau_1 f(\phi_1(\phi_3(x))) = (3-\tau_1)\tau_3$$

or

$$\tau_2\tau_1 f(\phi_1(\phi_2(x))) + \tau_2^2 f(\phi_2^2(x)) = (3-\tau_1)\tau_3,$$

respectively. Therefore $\phi_3(x) = \phi_2^2(x) = \phi_1(\phi_3(x))$ or $\phi_3(x) = \phi_2^2(x) = \phi_1(\phi_2(x))$. We show that the equalities:

$$\phi_1(x) = \phi_2(\phi_3(x)) = \phi_3(\phi_2(x)), \qquad \phi_3(x) = \phi_2^2(x) = \phi_1(\phi_2(x)), \qquad \phi_2(x) = \phi_1(\phi_3(x)) = \phi_3^2(x)$$

cannot occur. Since $\phi_1(\phi_2(x)) = \phi_2^2(x)$, then the cardinality of $S_{\phi_2(x)}$ must be equal to 1 as shown in Lemma 3.2, hence we would have

$$\phi_1(x) = \phi_3(\phi_2(x)) = \phi_1(\phi_2(x)) = \phi_3(x)$$

contradicting our initial assumption. Therefore we must have $\phi_2(x) = \phi_1(\phi_2(x)) = \phi_3^2(x)$ and $\phi_3(x) = \phi_2^2(x) = \phi_1(\phi_3(x))$, which implies that $\phi_2^3(x) = \phi_3^3(x) = x$.

Thus we get

$$x = \phi_1(x) = \phi_2\big(\phi_3(x)\big) = \phi_3\big(\phi_2(x)\big), \qquad \phi_2(x) = \phi_3^2(x) = \phi_1\big(\phi_2(x)\big), \qquad \phi_3(x) = \phi_2^2(x) = \phi_1\big(\phi_3(x)\big).$$

Then Eq. (7) becomes

$$\begin{aligned} &\tau_1^2 f\big(\phi_1(x)\big) + \tau_1\tau_2 f\big(\phi_2(x)\big) + \tau_1\tau_3 f\big(\phi_3(x)\big) + \tau_2\tau_1 f\big(\phi_2(x)\big) + \tau_2^2 f\big(\phi_3(x)\big) \\ &\quad + \tau_2\tau_3 f\big(\phi_1(x)\big) + \tau_3\tau_1 f\big(\phi_3(x)\big) + \tau_3\tau_2 f\big(\phi_1(x)\big) + \tau_3^2 f\big(\phi_2(x)\big) \\ &= 3\tau_1 f\big(\phi_1(x)\big) + 3\tau_2 f\big(\phi_2(x)\big) + 3\tau_3 f\big(\phi_3(x)\big), \end{aligned}$$

for all $f \in A_\alpha(X)$. In particular for f, a function in $A_\alpha(X)$, such that $f(\phi_1(x)) = 1$ and $f(\phi_2(x)) = f(\phi_3(x)) = 0$, we obtain $\tau_1^2 + 2\tau_2\tau_3 = 3\tau_1$. An easy computation gives $\tau_1 = 1$. Then, applying Lemma 3.1, we can assert that $\tau_2 = \tau_3 = 1$, $\tau_2 = e^{2\pi i/3}$ and $\tau_3 = e^{4\pi i/3}$, or $\tau_2 = e^{4\pi i/3}$ and $\tau_3 = e^{2\pi i/3}$, as stated in the statement (1). □

Remark 3.6. It is straightforward to show that the conditions stated in Lemma 3.5 are sufficient for Q to be a projection.

The next proposition summarizes the results obtained in the previous lemmas.

Proposition 3.7. *Let I_k be surjective isometries on $A_\alpha(X)$, given by*

$$I_k(f)(x) = \tau_k f\big(\phi_k(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big) \quad (k = 1, 2, 3),$$

with each τ_k a unimodular scalar and ϕ_k a surjective isometry on X, and let Q be the average of I_1, I_2 and I_3. Then Q is a projection on $A_\alpha(X)$ if and only if one of the following statements holds:

(1) $\tau_1 = \tau_2 = \tau_3 = 1$ *and every $x \in X$ satisfies*:
  (a) $x = \phi_1(x) = \phi_2(x) = \phi_3(x)$, *or*
  (b) $\phi_1(x) \neq \phi_2(x) \neq \phi_3(x) \neq \phi_1(x)$, $x = \phi_l(x) = \phi_j(\phi_k(x)) = \phi_k(\phi_j(x))$, $\phi_j(x) = \phi_k^2(x) = \phi_l(\phi_j(x))$ *and* $\phi_k(x) = \phi_j^2(x) = \phi_l(\phi_k(x))$, *where (l, j, k) is a permutation of* $(1, 2, 3)$, *or*
  (c) $\phi_1(x) \neq \phi_2(x) \neq \phi_3(x) \neq \phi_1(x)$, $x = \phi_l(x) = \phi_k^2(x) = \phi_j^2(x)$, $\phi_l(\phi_j(x)) = \phi_j(\phi_k(x)) = \phi_k(x)$, *and* $\phi_l(\phi_k(x)) = \phi_k(\phi_j(x)) = \phi_j(x)$, *where (l, j, k) is a permutation of* $(1, 2, 3)$.

(2) $\tau_j = e^{2\pi i/3}\tau_l$ *and* $\tau_k = e^{4\pi i/3}\tau_l$, *where (l, j, k) is a permutation of $(1, 2, 3)$, and $\phi_1(x) = \phi_2(x) = \phi_3(x)$ for every $x \in X$. In this case, $Q = 0$.*

(3) $\tau_l = 1$, $\tau_j = e^{2\pi i/3}$ *and* $\tau_k = e^{4\pi i/3}$, *where (l, j, k) is a permutation of $(1, 2, 3)$, and every $x \in X$ satisfies*:
  (a) $\phi_1(x) = \phi_2(x) = \phi_3(x)$, *or*
  (b) $\phi_1(x) \neq \phi_2(x) \neq \phi_3(x) \neq \phi_1(x)$, $x = \phi_l(x) = \phi_j(\phi_k(x)) = \phi_k(\phi_j(x))$, $\phi_j(x) = \phi_k^2(x) = \phi_l(\phi_j(x))$ *and* $\phi_k(x) = \phi_j^2(x) = \phi_l(\phi_k(x))$.

Now, we are in a position to characterize those projections given by the average of three surjective isometries on $A_\alpha(X)$.

Theorem 3.8. *Let I_k be surjective isometries on $A_\alpha(X)$, given by*

$$I_k(f)(x) = \tau_k f\big(\phi_k(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big) \quad (k = 1, 2, 3),$$

with each τ_k a unimodular scalar and ϕ_k a surjective isometry on X, and let Q be the average of I_1, I_2 and I_3. Then Q is a projection on $A_\alpha(X)$ if and only if there exist a scalar $\tau \in \mathbb{K}$ with $\tau^3 = 1$ and a surjective isometry ϕ on X with $\phi^3 = \mathrm{Id}$ such that

$$Q(f)(x) = \frac{f(x) + \tau f(\phi(x)) + \tau^2 f(\phi^2(x))}{3},$$

for every $f \in A_\alpha(X)$ and $x \in X$.

Proof. Since the sufficiency is clear, we prove only the necessity. Assume that $Q = (I_1 + I_2 + I_3)/3$ is a projection on $A_\alpha(X)$. Proposition 3.7 implies that X is partitioned into the following sets:

$$\begin{aligned} X_0 &= \big\{x \in X\colon\ \phi_1(x) = \phi_2(x) = \phi_3(x)\big\}, \\ X_l &= \big\{x \notin X_0\colon\ x = \phi_l(x) \neq \phi_j(x) \neq \phi_k(x) \neq x,\ \phi_l(x) = \phi_j\big(\phi_k(x)\big) = \phi_k\big(\phi_j(x)\big), \\ &\qquad \phi_j(x) = \phi_k^2(x) = \phi_l\big(\phi_j(x)\big),\ \phi_k(x) = \phi_j^2(x) = \phi_l\big(\phi_k(x)\big)\big\} \\ Y_l &= \big\{x \notin X_0\colon\ x = \phi_l(x) = \phi_j^2(x) = \phi_k^2(x)\ \phi_j(x) = \phi_l\big(\phi_k(x)\big) = \phi_k\big(\phi_j(x)\big), \phi_k(x) = \phi_l\big(\phi_j(x)\big) = \phi_j\big(\phi_k(x)\big)\big\} \end{aligned}$$

for $l = 1, 2, 3$ and (l, j, k) a permutation of $(1, 2, 3)$. For simplicity of exposition we assume that these sets are nonempty. We define ϕ as follows:

$$\phi(x) = \begin{cases} x & \text{if } x \in X_0, \\ \phi_3(x) & \text{if } x \in X_1 \cup Y_1, \\ \phi_1(x) & \text{if } x \in X_2 \cup Y_2, \\ \phi_2(x) & \text{if } x \in X_3 \cup Y_3. \end{cases}$$

We observe that $\phi_3(Y_1) \subseteq Y_2$, $\phi_1(Y_2) \subseteq Y_3$, and $\phi_2(Y_3) \subseteq Y_1$. Furthermore $\phi_j(X_i) \subseteq X_i$ for all i and j.

We check that ϕ is an isometry. We consider a few sample cases. The remaining cases follow from similar strategies.

1. If $x_1 \in X_1$ and $x_2 \in X_2$, then
$$\begin{aligned} d\big(\phi(x_1), \phi(x_2)\big) &= d\big(\phi_3(x_1), \phi_1(x_2)\big) = d\big(\phi_3(x_1), \phi_3^2(x_2)\big) \\ &= d\big(x_1, \phi_3(x_2)\big) = d\big(\phi_1(x_1), \phi_3(x_2)\big) \\ &= d\big(\phi_2\big(\phi_3(x_1)\big), \phi_2\big(\phi_3(x_2)\big)\big) = d(x_1, x_2). \end{aligned}$$
2. If $x_1 \in X_1$ and $y_2 \in Y_2$, then
$$d\big(\phi(x_1), \phi(y_2)\big) = d\big(\phi_3(x_1), \phi_1(y_2)\big) = d\big(\phi_3\big(\phi_1(x_1)\big), \phi_3\big(\phi_1(y_2)\big)\big) = d(x_1, y_2).$$
3. If $y_1 \in Y_1$ and $y_3 \in Y_3$, then
$$d\big(\phi(y_1), \phi(y_3)\big) = d\big(\phi_3(y_1), \phi_2(y_3)\big) = d\big(\phi_3\big(\phi_1(y_1)\big), \phi_3\big(\phi_1(y_3)\big)\big) = d(y_1, y_3).$$

We now show that $\phi^3 = \mathrm{Id}$ which also implies that ϕ is surjective. If $x \in X_0$, it is clear that $\phi^3(x) = x$; while that if $x \in X_l$, a simple verification shows that $\phi_j(x), \phi_k(x) \in X_l$ and hence $\phi^3(x) = \phi_j^3(x) = \phi_k^3(x) = x$. If $x \in Y_1$ then

$$\phi^3(x) = \phi^2\big(\phi_3(x)\big) = \phi(\phi_1\big(\phi_3(x)\big) = \phi_2\big(\phi_1\big(\phi_3(x)\big)\big) = \phi_2^2(x) = x.$$

Similar reasoning applies for $x \in Y_2$ or $x \in Y_3$.

Taking $\tau = 1$ when the statement (1) of Proposition 3.7 holds; $\tau = e^{2\pi i/3}$ when the statement (2) holds in which case $Q = 0$; and $\tau = e^{2\pi i/3}$ or $\tau = e^{4\pi i/3}$, depending on the permutation (l, j, k), when the statement (3) is satisfied, straightforward computations show that the equation

$$\tau_1 f\big(\phi_1(x)\big) + \tau_2 f\big(\phi_2(x)\big) + \tau_3 f\big(\phi_3(x)\big) = f(x) + \tau f\big(\phi(x)\big) + \tau^2 f\big(\phi^2(x)\big),$$

holds true for all $f \in A_\alpha(X)$ and $x \in X$. This completes the proof. □

4. Concluding remarks

The statement of Theorem 3.8 motivates the following definition.

Definition 4.1. Let $n \in \mathbb{N}$ be with $n \geqslant 2$. A bounded operator Q on $A_\alpha(X)$ is called a n-circular projection if and only if there exists a scalar $\tau \in \mathbb{K}$ such that $\tau^n = 1$ and a surjective isometry ϕ on X such that $\phi^n = \mathrm{Id}$ and $\phi^k \neq \mathrm{Id}$ for all $k = 1, \ldots, n-1$ satisfying

$$Q(f)(x) = \frac{\sum_{k=0}^{n-1} \tau^k f(\phi^k(x))}{n},$$

for every $f \in A_\alpha(X)$ and $x \in X$. We take $\phi^0 = \mathrm{Id}$.

Theorems 2.3 and 3.8 can be restate as in the following theorem. We refer to a projection as being trivial if it is equal to either the zero or the identity operators.

Theorem 4.2. *Let X be a compact 1-connected metric space with diameter at most 2 and $A_\alpha(X)$ be $\mathrm{Lip}(X)$ or $\mathrm{lip}(X^\alpha)$ with $\alpha \in (0, 1)$.*

1. *The average of two surjective isometries on $A_\alpha(X)$ is a projection if and only if it is either a trivial projection or a 2-circular projection.*
2. *The average of three surjective isometries on $A_\alpha(X)$ is a projection if and only if it is either a trivial projection or a 3-circular projection.*

The preceding results suggest that, under certain constraints, the average of n surjective isometries is a nontrivial projection if and only if it is an n-circular projection, so we ask.

Question 4.3. *Let X be a compact 1-connected metric space with diameter at most 2 and $n \geqslant 2$. Is the average of n pairwise distinct surjective isometries on $A_\alpha(X)$ a projection if and only if it is either a trivial projection or a n-circular projection?*

Next we describe some examples of n-circular projections on $A_\alpha(X)$, with X the circle (S^1), the sphere (S^2), or the torus (T^2). It might be of interest to point out that there are no n-circular projections with $n > 2$ on $A_\alpha([0,1])$. It is due to the nonexistence of homeomorphisms of $[0,1]$ with period $n \geqslant 3$.

Example 4.4. We set ϕ to be a period n rotation on S^1, $\phi(e^{i\theta}) = e^{i(\theta+2\pi/n)}$, and define

$$P(f)(x) = \sum_{k=1}^{n} \frac{f(\phi^k(x))}{n},$$

for all $f \in A_\alpha(S^1)$ and $x \in S^1$. This construction easily extends to S^2 by parameterizing S^2 as the set of all points of the form $(\sqrt{1-z^2}e^{i\theta}, z)$ with $z \in [-1,1]$ and $\theta \in [0, 2\pi)$. Then define an isometry ϕ as follows:

$$\phi\big(\sqrt{1-z^2}e^{i\theta}, z\big) = \big(\sqrt{1-z^2}e^{i(\theta+\frac{2\pi}{n})}, z\big).$$

If $X = T^2$, since $T^2 = S^1 \times S^1$ we construct examples of period n isometries on T^2.

We close with two remarks motivated by the results of this paper.

Remark 4.5. Let X be a compact 1-connected metric space with diameter at most 2. We observe that 3-circular projections on $A_\alpha(X)$ cannot be represented as the average of two surjective isometries on $A_\alpha(X)$. Let's assume otherwise. Then we can write

$$\frac{f(x) + \tau f(\phi(x)) + \tau^2 f(\phi^2(x))}{3} = \frac{\alpha_1 f(\psi_1(x)) + \alpha_2 f(\psi_2(x))}{2} \quad \big(f \in A_\alpha(X),\ x \in X\big), \tag{13}$$

where $\tau \in \mathbb{K}$ with $\tau^3 = 1$, ϕ is a surjective isometry on X such that $\phi \neq \mathrm{Id} \neq \phi^2$ and $\phi^3 = \mathrm{Id}$, $\alpha_1, \alpha_2 \in \mathbb{K}$ with $|\alpha_1| = |\alpha_2| = 1$ and ψ_1 and ψ_2 are surjective isometries on X. In particular, for $f = 1_X$, Eq. (13) becomes $(1+\tau+\tau^2)/3 = (\alpha_1+\alpha_2)/2$. If $\tau = 1$, then $\alpha_1 = \alpha_2 = 1$. If $\tau \neq 1$, then $\tau \in \{e^{2\pi i/3}, e^{4\pi i/3}\}$, hence $1 + \tau + \tau^2 = 0$ and so $\alpha_1 + \alpha_2 = 0$.

First we assume that $\tau = \alpha_1 = \alpha_2 = 1$. Since there exists $x \in X$ such that $\operatorname{card}\{x, \phi(x), \phi^2(x)\} = 3$ we select $f \in A_\alpha(X)$ with range the interval $[0,1]$ such that $f(x) = 1$ and $f(\phi(x)) = f(\phi^2(x)) = 0$. Hence Eq. (13) implies that

$$2 = 3\big(f\big(\psi_1(x)\big) + f\big(\psi_2(x)\big)\big).$$

Hence there must exist $k \in \{1,2\}$ so that $\psi_k(x) = x$ which leads to a contradiction.

Now we assume that $\tau \neq 1$ and consequently $\alpha_1 + \alpha_2 = 0$. As above we select $x \in X$ so that $\operatorname{card}\{x, \phi(x), \phi^2(x)\} = 3$. We show that $\{x, \phi(x), \phi^2(x)\}$ must intersect $\{\psi_1(x), \psi_2(x)\}$. If these two sets were disjoint, then there exists a function $f \in A_\alpha(X)$ satisfying $f(x) = 1$ and $f(z) = 0$ for all $z \in \{\psi_1(x), \psi_2(x), \phi(x), \phi^2(x)\}$. This leads to an absurd. Without loss of generality, we can assume that $\psi_1(x) = \phi^j(x)$ for some $j \in \{0,1,2\}$ and hence $\psi_1(x) \notin \{\phi^k(x)\colon\ k = 0,1,2,\ k \neq j\}$. We now set $f \in A_\alpha(X)$ such that $f(\psi_1(x)) = 1$ and $f(\phi^k(x)) = 0$ for all $k \in \{0,1,2\} \setminus \{j\}$. If $\psi_1(x) = \psi_2(x)$, Eq. (13) becomes $\tau^j/3 = (\alpha_1+\alpha_2)/2$, hence $\tau^j/3 = 0$, a contradiction. If $\psi_1(x) \neq \psi_2(x)$, we can also assume that $f(\psi_2(x)) = 0$, and now Eq. (13) gives $\tau^j/3 = \alpha_1/2$, another contradiction. This absurd proves the claim.

Remark 4.6. We recall that a projection P is bi-contractive if $\|P\| \leqslant 1$ and $\|I - P\| \leqslant 1$. It is known that generalized bi-circular projections are bi-contractive (see [4]). We note that 3-circular projections are not necessarily bi-contractive. In fact, let $X = \{a, b, c\}$ be equipped with the metric $d(a,b) = d(b,c) = d(a,c) = 2$. Consider $P = (\mathrm{Id} + R + R^2)/3$ with $R(f) = f \circ \phi$ and ϕ a period 3 isometry on X ($\phi(a) = b$, $\phi(b) = c$ and $\phi(c) = a$). Then $\mathrm{Id} - P = (2\mathrm{Id} - R - R^2)/3$. We consider f on $A_\alpha(X)$ such that $f(\phi(a)) = f(\phi^2(a)) = -1$ and $f(a) = 1$. We observe that $\|f\| = 6/5$ and $\|(\mathrm{Id} - P)(f)\| = 23/15$, hence $\|\mathrm{Id} - P\| > 1$.

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