ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 6, 2010

WEAKLY PERIPHERALLY MULTIPLICATIVE SURJECTIONS OF POINTED LIPSCHITZ ALGEBRAS

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ABSTRACT. Let (X, d) be a compact metric space with a distinguished base point e_X, and let $\mathrm{Lip}_0(X)$ be the Banach algebra of all scalar-valued Lipschitz functions f on X such that $f(e_X) = 0$, with the norm

$$L(f) = \sup \{|f(x) - f(y)| / d(x, y) : x, y \in X,\ x \neq y\}.$$

Let $\mathrm{Ran}_\pi(f) = \{f(x) : x \in X,\ |f(x)| = \|f\|_\infty\}$ denote the peripheral range of f. We prove that if $\Phi : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is a surjective map, not assumed to be linear, with the property that $\mathrm{Ran}_\pi(fg) \cap \mathrm{Ran}_\pi(\Phi(f)\Phi(g)) \neq \emptyset$ for all $f, g \in \mathrm{Lip}_0(X)$, then Φ is a weighted composition operator of the form

$$\Phi(f)(y) = \tau(y) f(\varphi(y)), \text{ for all } f \in \mathrm{Lip}_0(X), \text{ for all } y \in Y,$$

where τ is a function from Y into $\{-1, 1\}$ and φ is a Lipschitz homeomorphism from Y onto X such that $\varphi(e_Y) = e_X$.

1. Introduction. Given two function algebras $A(X)$ and $B(Y)$ on the compact Hausdorff spaces X and Y, respectively, a map $\Phi : A(X) \to B(Y)$ is said to be *multiplicatively range-preserving* provided that

$$(\Phi(f)\Phi(g))(Y) = (fg)(X), \text{ for all } f, g \in A(X).$$

Several papers on multiplicatively range-preserving surjective maps between function algebras have appeared in recent years [**3, 8, 9**].

If $C(X)$ is the Banach algebra of all complex-valued continuous functions on a compact Hausdorff space X, equipped with the supremum norm, and X is a first-countable space, Molnár [**8**, Theorem 5] proved that every multiplicatively range-preserving surjective map $\Phi : C(X) \to C(X)$ is a weighted composition operator of the form

$$\Phi(f)(x) = \tau(x) f(\varphi(x)), \text{ for all } f \in C(X), \text{ for all } x \in X,$$

The first and third authors were supported in part by JA project FQM2008-3737 and MEC project MTM2010-17687.
Received by the editors on January 30, 2008, and in revised form on May 15, 2008.
DOI:10.1216/RMJ-2010-40-6-1903

where $\tau : X \to \{-1, 1\}$ is a continuous function and $\varphi: X \to X$ is a homeomorphism. Molnár's theorem was extended by Rao and Roy for mappings from a uniform algebra to itself [**9**, Main Theorem], and by Hatori, Miura and Takagi for mappings between uniform algebras on arbitrary compact Hausdorff spaces [**3**, Theorem 1.1].

Hatori, Miura and Takagi generalized all the known results on multiplicatively range-preserving surjective maps by showing that every multiplicatively spectrum-preserving surjective map between unital semi-simple commutative Banach algebras $\Phi : A \to B$ has the form

$$\Phi(f)(y) = \tau(y) f(\varphi(y)), \text{ for all } f \in A, \text{ for all } y \in M_B,$$

where τ is an element in B with spectrum $\sigma(\tau) \subset \{-1, 1\}$ and φ is a homeomorphism from the maximal ideal space M_B of B onto M_A [**4**, Theorem 3.2]. Moreover, they pointed out that the technique of proof of their result depends essentially of the existence of the units of the Banach algebras, and it cannot be adopted directly to prove a similar result for non-unital commutative Banach algebras.

The peripheral range of a function $f \in A(X)$,

$$\mathrm{Ran}_\pi(f) = \{f(x): x \in X, \ |f(x)| = \|f\|_\infty\},$$

is the subset of values of $f(X)$ of maximum modulus. Luttman and Tonev [**7**] introduced a new point of view on this matter by describing the structure of *peripherally multiplicative* surjective maps between uniform algebras, i.e., maps that multiplicatively preserve only peripheral ranges, or equivalently the peripheral spectra, rather than the entire ranges of products. Most recently, the peripheral multiplicativity of maps on uniformly closed algebras of complex-valued continuous functions which vanish at infinity and on algebras of scalar-valued Lipschitz functions has been analyzed in [**2**] and in [**5**], respectively.

According to [**6**], a map $\Phi: A(X) \to B(Y)$ is said to be *weakly peripherally multiplicative* if

$$\mathrm{Ran}_\pi(fg) \cap \mathrm{Ran}_\pi(\Phi(f)\Phi(g)) \neq \varnothing, \text{ for all } f, g \in A(X).$$

This weaker condition was studied in [**6**], where it is proved that any weakly peripherally multiplicative map $\Phi: A(X) \to B(X)$ between uniform algebras, not assumed to be surjective, under the additional assumption that Φ preserves a special class of functions known as the peaking functions, must be a weighted composition operator of a form similar to those above. Surjective weakly peripherally multiplicative maps without the assumption on the peaking functions were not studied. Moreover, general results in unital semi-simple commutative Banach algebras do not exist in the literature for weakly peripherally multiplicative maps.

Motivated by this and also because, as it is mentioned in [**4**], it is interesting to study for which Banach algebras a theorem of Molnár holds; in this paper, we study weakly peripherally multiplicative surjective maps $\Phi\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$, where $\mathrm{Lip}_0(X)$ is a pointed Lipschitz algebra, a type of non-unital commutative algebra.

Our approach follows closely that of Rao and Roy in [**10**], which contains a version of Molnár's theorem for closed, point-separating subalgebras A of the Banach algebra $C_0(M_A)$ of complex-valued continuous functions which vanish at infinity on M_A, endowed with the supremum norm.

To present our main result, we first explain briefly the notations and terminology that we shall use. Let $X = (X, d)$ be a metric space with a distinguished base point $e_X \in X$; X is called a pointed metric space. Then $\mathrm{Lip}_0(X)$ is the Banach space of all scalar-valued Lipschitz functions f on X which vanish at e_X, with the norm

$$L(f) = \sup\left\{\left|f(x) - f(y)\right| / d(x, y) : x, y \in X,\ x \neq y\right\}.$$

If $f, g \in \mathrm{Lip}_0(X)$ are bounded, then $(fg)(e_X) = f(e_X)g(e_X) = 0$, and the straightforward inequality

$$|(fg)(x) - (fg)(y)| \leq \left(L(f)\left\|g\right\|_\infty + L(g)\left\|f\right\|_\infty\right) d(x, y), \quad \text{for all } x, y \in X,$$

implies that fg is Lipschitz, so that $fg \in \mathrm{Lip}_0(X)$. If X has finite diameter, every scalar-valued Lipschitz function f on X is bounded with $\left\|f\right\|_\infty \leq \mathrm{diam}\,(X)L(f)$, where $\mathrm{diam}\,(X)$ denotes the diameter of X, and therefore $\mathrm{Lip}_0(X)$ is an algebra. If X has infinite diameter, the function on X given by $f(z) = d(z, e_X)$ is in $\mathrm{Lip}_0(X)$, but f^2 is not Lipschitz, so that $\mathrm{Lip}_0(X)$ is not an algebra. The algebras $\mathrm{Lip}_0(X)$, with X a finite-diameter, pointed metric space have been the subject of considerable study (see, for example, [**11**] and its references).

Let X and Y be metric spaces. Recall that a map $\varphi : X \to Y$ is a Lipschitz homeomorphism if φ is bijective and both φ and φ^{-1} are Lipschitz. When X and Y are both pointed, we say that φ preserves base point if $\varphi(e_X) = e_Y$. Our main result reads as follows:

Theorem 1.1. *Let X and Y be pointed compact metric spaces, and let $\Phi\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be a surjective map, not assumed to be linear, that is weakly peripherally multiplicative, i.e. that satisfies*

$$\mathrm{Ran}_\pi(fg) \cap \mathrm{Ran}_\pi(\Phi(f)\Phi(g)) \neq \varnothing \tag{1}$$

for all $f, g \in \mathrm{Lip}_0(X)$. Then there exists a unique function $\tau : Y \to \{-1, 1\}$ with $\tau(e_Y) = 1$ and a unique base point-preserving Lipschitz homeomorphism $\varphi : Y \to X$ such that Φ is of the form

$$\Phi(f)(y) = \tau(y) f(\varphi(y))$$

for all $f \in \mathrm{Lip}_0(X)$ and $y \in Y$.

Under the condition of compactness on the pointed metric spaces X and Y, Theorem 1.1 determines all surjections from $\mathrm{Lip}_0(X)$ onto $\mathrm{Lip}_0(Y)$ that satisfy (1), since every map Φ of the form $\Phi(f) = \tau \cdot (f \circ \varphi)$ for all $f \in \mathrm{Lip}_0(X)$, with τ, φ being as in the statement above, automatically satisfies (1).

Moreover, we prove that Φ is a topological isomorphism and that Φ^2 is multiplicative (Propositions 2.10 and 2.11).

According to [**11**, Corollary 4.2.9], any algebra isomorphism $\Phi : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is a composition operator $\Phi(f)(y) = f(\varphi(y))$ for all $f \in \mathrm{Lip}_0(X)$ and $y \in Y$, where $\varphi : Y \to X$ is a base point-preserving Lipschitz homeomorphism. We provide sufficient conditions for a weakly peripherally multiplicative surjective map between algebras $\mathrm{Lip}_0(X)$ to be an algebra isomorphism (Corollaries 3.1 and 3.2).

Finally, given a metric space (X, d), let $\mathrm{Lip}\,(X)$ denote the Banach space of all bounded scalar-valued Lipschitz functions f on X, with $\|f\| = \max\{\|f\|_\infty, L(f)\}$. Every space $\mathrm{Lip}\,(X)$ can be naturally identified with a space $\mathrm{Lip}_0(X_0)$ for a suitable pointed metric space X_0 (see Lemma 3.3). Using this fact, we characterize weakly peripherally multiplicative surjections between algebras of the type $\mathrm{Lip}\,(X)$ (Corollary 3.4) and give an extension of [**5**, Theorem 3.1].

2. A proof of Theorem 1.1. The key point of the proof of Theorem 1.1 is the fact that the spaces $\mathrm{Lip}_0(X)$ are, similar to uniform algebras, rich enough in peaking functions.

Definition 2.1. Let X be a pointed metric space. A function f in $\mathrm{Lip}_0(X)$ is said to be a peaking function if $\mathrm{Ran}_\pi(f) = \{1\}$. We denote by $P(X)$ the set of all peaking functions of $\mathrm{Lip}_0(X)$ and $P_x(X) = \{f \in P(X) : f(x) = 1\}$ for $x \in X$.

Lemma 2.1. *Let X be a pointed metric space.*

i) *For each $x \in X$, the function f_x defined by*

$$f_x(z) = d(z,x) - d(e_X, x), \text{ for all } z \in X,$$

is in $\mathrm{Lip}_0(X)$ with $L(f_x) = 1$. The family $\{f_x : x \in X\}$ separates the points of X.

ii) *For $x \in X$ and $\delta > 0$, the function $h_{x,\delta} : X \to [0,1]$ given by*

$$h_{x,\delta}(z) = \max\{0, 1 - d(z,x)/\delta\}$$

is Lipschitz with $L(h_{x,\delta}) \leq 1/\delta$, $h_{x,\delta}(x) = 1$ and $h_{x,\delta}(z) < 1$ for all $z \neq x$ with $h_{x,\delta}(z) = 0$ if $d(z,x) \geq \delta$. Moreover, $h_{x,\delta}(e_X) = 0$ if and only if $d(x, e_X) \geq \delta$.

iii) *For any $x \in X$ and $f \in \mathrm{Lip}_0(X)$ with $f(x) \neq 0$, there exists a function $h \in P_x(X)$ such that $|h(z)| < 1$ and $|f(z)h(z)| < |f(x)|$ for all $z \neq x$. In particular, $\mathrm{Ran}_\pi(fh) = \{f(x)\}$.*

Proof. The proofs of i) and ii) are straightforward. To prove iii) we may assume that $|f(x)| = 1$ for, if not, we replace f by f/λ with $\lambda = |f(x)|$. Let g be the function from X into $[0,1]$ given by

$$g(z) = \begin{cases} 1 & \text{if } |f(z)| < 1, \\ 2 - |f(z)| & \text{if } 1 \leq |f(z)| \leq 2, \\ 0 & \text{if } |f(z)| > 2. \end{cases}$$

It is easily seen that g is Lipschitz with $g(x) = 1$ and $|fg| \leq 1$. Choose $\delta \in (0, d(x, e_X)]$, and put $h = gh_{x,\delta}$. A trivial verification shows that h satisfies the properties required in the statement. □

Assertion ii) shows that the spaces $\mathrm{Lip}_0(X)$ are endowed with an abundance of peaking functions.

Following an argument similar to that of [**7**, Lemma 2], we prove the following:

Lemma 2.2. *Let X be a pointed metric space and, let $f, g \in \mathrm{Lip}_0(X)$. If $\|fh\|_\infty \leq \|gh\|_\infty$ for all $h \in P(X)$, then $|f| \leq |g|$.*

Proof. Suppose that $|f(x)| > |g(x)|$ for some point $x \in X$. Observe that $x \neq e_X$. Take $\varepsilon > 0$ such that $|f(x)| > \varepsilon > |g(x)|$. Since g is continuous at x, there is a $\delta \in (0, d(x, e_X)]$ such that $|g(z)| < \varepsilon$ when $d(z, x) < \delta$. It is easily seen that $|gh_{x,\delta}| < \varepsilon$ and $\|fh_{x,\delta}\|_\infty \geq |f(x)h_{x,\delta}(x)| = |f(x)| > \varepsilon$. Then $\|gh_{x,\delta}\|_\infty \leq \varepsilon < \|fh_{x,\delta}\|_\infty$, which proves the lemma. $\square$

Definition 2.2. A map $\Phi : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is said to be

a) *uniform norm-multiplicative* if $\|\Phi(f)\Phi(g)\|_\infty = \|fg\|_\infty$ for any $f, g \in \mathrm{Lip}_0(X)$;

b) *uniform norm-preserving* if $\|\Phi(f)\|_\infty = \|f\|_\infty$ for any $f \in \mathrm{Lip}_0(X)$;

c) *monotone increasing in modulus* if $|f| \leq |g|$ implies $|\Phi(f)| \leq |\Phi(g)|$ for any $f, g \in \mathrm{Lip}_0(X)$ (see [**1**]);

d) *monotone increasing in modulus in both directions* if for any $f, g \in \mathrm{Lip}_0(X)$, $|f| \leq |g|$ if and only if $|\Phi(f)| \leq |\Phi(g)|$.

Given the preliminaries above, we prove Theorem 1.1 through a sequence of propositions that follow closely the steps in the proof of the corresponding result in [**6**].

From now on, we shall assume that $\Phi: \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is a surjective map satisfying (1) for all $f, g \in \mathrm{Lip}_0(X)$.

Proposition 2.3. *Φ is uniform norm-multiplicative and, as a consequence, it is uniform norm-preserving.*

Proof. Let $f, g \in \mathrm{Lip}_0(X)$. Then (1) implies $\|fg\|_\infty = \|\Phi(f)\Phi(g)\|_\infty$.

Moreover, taking $g = f$,

$$\|f\|_\infty^2 = \|f^2\|_\infty = \|\Phi(f)^2\|_\infty = \|\Phi(f)\|_\infty^2.$$

Thus, $\|f\|_\infty = \|\Phi(f)\|_\infty$. □

Proposition 2.4. *Φ is monotone increasing in modulus in both directions.*

Proof. If $|f| \leq |g|$, then $\|fh\|_\infty \leq \|gh\|_\infty$ for all $h \in \mathrm{Lip}_0(X)$. By the surjectivity of Φ, for each $k \in P(Y)$ there is an $h \in \mathrm{Lip}_0(X)$ such that $k = \Phi(h)$. Since Φ is uniform norm-multiplicative, we obtain

$$\|\Phi(f)\Phi(h)\|_\infty = \|fh\|_\infty \leq \|gh\|_\infty = \|\Phi(g)\Phi(h)\|_\infty,$$

and thus $\|\Phi(f)k\|_\infty \leq \|\Phi(g)k\|_\infty$. Since k is arbitrary in $P(Y)$, we infer that $|\Phi(f)| \leq |\Phi(g)|$ by Lemma 2.2.

Similarly, if $|\Phi(f)| \leq |\Phi(g)|$, then $\|\Phi(f)k\|_\infty \leq \|\Phi(g)k\|_\infty$ for all $k \in \mathrm{Lip}_0(Y)$. For each $h \in P(X)$, then we have

$$\|fh\|_\infty = \|\Phi(f)\Phi(h)\|_\infty \leq \|\Phi(g)\Phi(h)\|_\infty = \|gh\|_\infty,$$

which implies $|f| \leq |g|$. □

For each $x \in X \setminus \{e_X\}$, consider the set

$$F_x(X) = \{f \in \mathrm{Lip}_0(X) : |f(x)| = \|f\|_\infty = 1\}.$$

Analogously, for any $y \in Y \setminus \{e_Y\}$, let

$$F_y(Y) = \{g \in \mathrm{Lip}_0(Y) : |g(y)| = \|g\|_\infty = 1\}.$$

These sets are nonempty by Lemma 2.1 ii).

Proposition 2.5. *For each $x \in X \setminus \{e_X\}$, there exists a unique $y \in Y \setminus \{e_Y\}$ such that $\Phi(F_x(X)) = F_y(Y)$.*

Proof. Fix $x_0 \in X \setminus \{e_X\}$. For any $f \in F_{x_0}(X)$, define

$$P(f) = \{y \in Y \setminus \{e_Y\} : |\Phi(f)(y)| = \|\Phi(f)\|_\infty = 1\}.$$

First we observe that $P(f)$ is nonempty, since Φ is uniform norm-preserving and Y is compact. We claim that there exists a $y_0 \in Y \setminus \{e_Y\}$ such that $\Phi(F_{x_0}(X)) \subset F_{y_0}(Y)$. Note that this holds if $\cap_{f\in F_{x_0}(X)} P(f) \neq \varnothing$. In order to verify this, it is sufficient to show that the family $\{P(f) : f \in F_{x_0}(X)\}$ has the finite intersection property, since each $P(f)$ is a closed subset of the compact Hausdorff space Y. Pick $f_1, \dots , f_n \in F_{x_0}(X)$. Clearly $g = f_1 \cdots f_n \in F_{x_0}(X)$ and $|g| \leq |f_i|$ for all $i \in \{1, \dots , n\}$. Since Φ is uniform norm-preserving, we have $\|\Phi(f_i)\|_\infty = 1$ for all $i \in \{1, \dots , n\}$ and $\|\Phi(g)\|_\infty = 1$. Then $|\Phi(g)(y_0)| = 1$ for some $y_0 \in Y \setminus \{e_Y\}$. By Proposition 2.4, it follows that $|\Phi(g)| \leq |\Phi(f_i)|$ for all $i \in \{1, \dots , n\}$, hence $|\Phi(f_i)(y_0)| = 1$ for all $i \in \{1, \dots , n\}$, and so we have that $y_0 \in \cap_{i=1}^n P(f_i)$.

Similarly, we prove now that $F_{y_0}(Y) \subset \Phi(F_{x_1}(X))$ for some $x_1 \in X \setminus \{e_X\}$. For any $g \in F_{y_0}(Y)$, define

$$P(g) = \left\{x \in X \setminus \{e_X\}: |h(x)| = 1, \text{ for all } h \in \Phi^{-1}(\{g\})\right\}.$$

Let $g_1, \dots , g_n \in F_{y_0}(Y)$, and choose $h_1, \dots , h_n \in \mathrm{Lip}_0(X)$ such that $\Phi(h_j) = g_j$. Let $g = g_1 \cdots g_n$, and choose $h \in \mathrm{Lip}_0(X)$ such that $\Phi(h) = g$. Then $\|g\|_\infty = \|g_j\|_\infty = 1$ for all $j \in \{1, \dots , n\}$. Since Φ is uniform norm-preserving, $\|h_j\|_\infty = \|\Phi(h_j)\|_\infty = \|g_j\|_\infty = 1$. Similarly, $\|h\|_\infty = \|g\|_\infty = 1$. By Proposition 2.4, $|\Phi(h)| \leq |\Phi(h_j)|$ implies $|h| \leq |h_j|$. Since $\|h\|_\infty = 1$, it follows that there exists an $x \in X \setminus \{e_X\}$ such that $|h_j(x)| = 1$ for all $j \in \{1, \dots , n\}$. Therefore, $\{P(g): g \in F_{y_0}(Y)\}$ has the finite intersection property, and there exists an $x_1 \in \cap_{g\in F_{y_0}(Y)} P(g)$. Thus, $\Phi^{-1}(F_{y_0}(Y)) \subset F_{x_1}(X)$, and $F_{y_0}(Y) \subset \Phi(F_{x_1}(X))$.

The preceding paragraphs show that there exist $y_0 \in Y \setminus \{e_Y\}$ and $x_1 \in X \setminus \{e_X\}$ such that $\Phi(F_{x_0}(X)) \subset F_{y_0}(Y) \subset \Phi(F_{x_1}(X))$. To see that $\Phi(F_{x_0}(X)) = F_{y_0}(Y)$, we prove that $x_0 = x_1$. Suppose $x_0 \neq x_1$ and choose $f_0 \in \mathrm{Lip}_0(X)$ such that $f_0(x_0) = 1$ and $|f_0(x)| < 1$ for all $x \neq x_0$. Since $\Phi(f_0) \in \Phi(F_{x_1}(X))$, there exists an $f_1 \in F_{x_1}(X)$ such that $\Phi(f_1) = \Phi(f_0)$. Thus, $|\Phi(f_0)(y)| = |\Phi(f_1)(y)|$ for all $y \in Y$ but $|f_0(x_1)| < |f_1(x_1)| = 1$, which contradicts Proposition 2.4. Thus, $x_0 = x_1$, and $\Phi(F_{x_0}(X)) = F_{y_0}(Y)$. To see that y_0 is unique, note that, if $\Phi(F_{x_0}(X)) = F_{y_1}(Y)$ for some $y_1 \in Y \setminus \{e_Y\}$, then $F_{y_0}(Y) = F_{y_1}(Y)$, which immediately implies that $y_0 = y_1$. $\square$

Proposition 2.5 justifies the following:

Definition 2.3. Let $\psi: X \to Y$ be the map which sends each point $x \in X \setminus \{e_X\}$ to the unique point $\psi(x) \in Y \setminus \{e_Y\}$ such that $\Phi(F_x(X)) = F_{\psi(x)}(Y)$, and $\psi(e_X) = e_Y$.

Proposition 2.6. *ψ is bijective.*

Proof. If $x \in X\setminus\{e_X\}$, then $\psi(x) \neq e_Y = \psi(e_X)$. If $x_1, x_2 \in X\setminus\{e_X\}$ are such that $\psi(x_1) = \psi(x_2)$, then $\Phi(F_{x_1}(X)) = \Phi(F_{x_2}(X))$, which was shown above to imply that $x_1 = x_2$. Hence, ψ is injective. To see that ψ is surjective, let $y_0 \in Y \setminus \{e_Y\}$ be given. We must show that there exists an $x_0 \in X\setminus\{e_X\}$ such that $\Phi(F_{x_0}(X)) = F_{y_0}(Y)$. The arguments above imply that there exist $x_0 \in X\setminus\{e_X\}$ and $y_1 \in Y\setminus\{e_Y\}$ such that $F_{y_0}(Y) \subset \Phi(F_{x_0}(X)) \subset F_{y_1}(Y)$, which immediately yields $y_0 = y_1$, and therefore $F_{y_0}(Y) = \Phi(F_{x_0}(X))$. □

Proposition 2.7. *For all $f \in \mathrm{Lip}_0(X)$ and $x \in X$, $|f(x)| = |\Phi(f)(\psi(x))|$.*

Proof. If $x = e_X$, then $f(x) = 0$ and $\Phi(f)(\psi(x)) = \Phi(f)(e_Y) = 0$, so $|f(x)| = 0 = |\Phi(f)(\psi(x))|$.

Fix $x \neq e_X$, and assume $f(x) \neq 0$. Then, by Lemma 2.1 iii), there exists an $h \in P_x(X)$ such that $\|fh\|_\infty = |f(x)|$. Since $\|\Phi(f)\Phi(h)\|_\infty = \|fh\|_\infty$ by Proposition 2.3, it follows that $|\Phi(f)(\psi(x))\Phi(h)(\psi(x))| \leq |f(x)|$. Since $h \in F_x(X)$, $|\Phi(h)(\psi(x))| = 1$, so $|\Phi(f)(\psi(x))| \leq |f(x)|$. The reverse inequality is proven similarly.

Now suppose $f(x) = 0$, and let $\varepsilon > 0$ be given. Then there is a $\delta \in (0, d(x, e_X)]$ such that $|f(z)| < \varepsilon/2$ if $d(z, x) < \delta$. Consider $h_{x,\delta} \in \mathrm{Lip}_0(X)$. Given $z \in X$, we have

$$d(z,x) < \delta \Longrightarrow |f(z)h_{x,\delta}(z)| \leq |f(z)| < \varepsilon/2,$$
$$d(z,x) \geq \delta \Longrightarrow |f(z)h_{x,\delta}(z)| = 0 < \varepsilon/2.$$

Hence, $\|fh_{x,\delta}\|_\infty < \varepsilon$. Therefore, just as above, since $\|\Phi(f)\Phi(h_{x,\delta})\|_\infty = \|fh_{x,\delta}\|_\infty$ by Proposition 2.3, it follows that

$$|\Phi(f)(\psi(x))| = |\Phi(f)(\psi(x))\Phi(h_{x,\delta})(\psi(x))| < \varepsilon.$$

Since ε was arbitrary, $\Phi(f)(\psi(x)) = 0$. Thus, $|\Phi(f)(\psi(x))| = 0 = |f(x)|$, which proves the result. □

Proposition 2.8. *For each* $x \in X \setminus \{e_X\}$, $\Phi(h)(\psi(x)) = 1$ *for all* $h \in P_x(X)$, *or* $\Phi(h)(\psi(x)) = -1$ *for all* $h \in P_x(X)$.

Proof. Let $x_0 \in X \setminus \{e_X\}$ and $h \in P(X)$ be such that $h(x_0) = 1$. We claim that $\Phi(h)(\psi(x_0)) = 1$ or $\Phi(h)(\psi(x_0)) = -1$. First observe that $|\Phi(h)(\psi(x_0))| = 1$ by Proposition 2.7. By Lemma 2.1 iii) there exists a $k \in P(Y)$ such that $k(\psi(x_0)) = 1$, $|k(y)| < 1$ for all $y \neq \psi(x_0)$, and $\mathrm{Ran}_\pi(\Phi(h)k) = \{\Phi(h)(\psi(x_0))\}$. If $f \in \Phi^{-1}(\{k\})$, then $\Phi(h)(\psi(x_0)) \in \mathrm{Ran}_\pi(fh)$ by (1). Given $x \in X$, taking into account that $|k(\psi(x))| = |\Phi(f)(\psi(x))| = |f(x)|$, it is clear that $|f(x)| = 1$ if and only if $x = x_0$. Thus $\Phi(h)(\psi(x_0)) \in \mathrm{Ran}_\pi(fh) = \{f(x_0)\} = \mathrm{Ran}_\pi(f)$. Therefore, $\{f(x_0)^2\} = \mathrm{Ran}_\pi(f^2)$ and $\{1\} = \mathrm{Ran}_\pi(k^2)$ together imply that $f(x_0)^2 = 1$ by (1). Thus, $\Phi(h)(\psi(x_0))^2 = f(x_0)^2 = 1$, which is the claim.

To see that $\Phi(h)(\psi(x_0))$ is independent of the chosen function h, let $h_1, h_2 \in P(X)$ be such that $h_1(x_0) = h_2(x_0) = 1$. Let $g = \Phi(h_1)\Phi(h_2)$. Then it must be shown that $g(\psi(x_0)) = 1$. Now $|g(\psi(x_0))| = 1$ by Proposition 2.7, so Lemma 2.1 iii) provides $k \in P(Y)$ such that $k(\psi(x_0)) = 1$, $|k(y)| < 1$ for all $y \neq \psi(x_0)$ and $\mathrm{Ran}_\pi(gk) = \{g(\psi(x_0))\}$. Choose $f_1 \in \Phi^{-1}(\{k\Phi(h_1)\})$ and $f_2 \in \Phi^{-1}(\{k\Phi(h_2)\})$. Then $g(\psi(x_0)) \in \mathrm{Ran}_\pi(h_1 f_2) \cap \mathrm{Ran}_\pi(h_2 f_1)$ by (1). Using Proposition 2.7, we have that $|f_1(x)| = |f_2(x)| = 1$ if and only if $x = x_0$. Therefore, $g(\psi(x_0)) = h_1(x_0)f_2(x_0) = h_2(x_0)f_1(x_0)$, which implies that $g(\psi(x_0)) = f_1(x_0) = f_2(x_0)$. Therefore, $\{g(\psi(x_0))^2\} = \mathrm{Ran}_\pi(f_1 f_2) \subset \mathrm{Ran}_\pi(\Phi(f_1)\Phi(f_2)) = \mathrm{Ran}_\pi(gk^2)$. Since $|k^2(y)| < 1$ for $y \neq \psi(x_0)$, it follows that $g(\psi(x_0))^2 \in \mathrm{Ran}_\pi(gk^2) = \{g(\psi(x_0))\}$, which proves that $g(\psi(x_0)) = 1$. □

In view of Proposition 2.8, we can consider:

Definition 2.4. Let $\rho: X \to \{-1, 1\}$ be the function defined by $\rho(e_X) = 1$ and $\rho(x) = \Phi(h)(\psi(x))$ if $x \neq e_X$, where h is any function of $P_x(X)$.

Proposition 2.9. *For all* $f \in \mathrm{Lip}_0(X)$ *and* $x \in X$, $\Phi(f)(\psi(x)) = \rho(x)f(x)$.

Proof. Let $x_0 \in X$. If $f(x_0) = 0$, then $\Phi(f)(\psi(x_0)) = 0$ by Proposition 2.7, and thus $f(x_0) = \rho(x_0)\Phi(f)(\psi(x_0))$.

If $f(x_0) \neq 0$, by Lemma 2.1 iii) there exists an $h \in P(X)$ such that $h(x_0) = 1$, $|h(x)| < 1$ for $x \neq x_0$, and $\mathrm{Ran}_\pi(fh) = \{f(x_0)\}$. By (1), $f(x_0) \in \mathrm{Ran}_\pi(\Phi(f)\Phi(h))$. Using that Φ is uniform norm-multiplicative and Proposition 2.7, given $x \in X$ we have that $|f(x)h(x)| = \|fh\|_\infty$ if and only if $|\Phi(f)(\psi(x))\Phi(h)(\psi(x))| = \|\Phi(f)\Phi(h)\|_\infty$. Since $|f(x)h(x)| = \|fh\|_\infty$ only when $x = x_0$, we have

$$\mathrm{Ran}_\pi(\Phi(f)\Phi(h)) = \{\Phi(f)(\psi(x_0))\Phi(h)(\psi(x_0))\}.$$

Thus, $f(x_0) = \Phi(f)(\psi(x_0))\Phi(h)(\psi(x_0)) = \rho(x_0)\Phi(f)(\psi(x_0))$. □

Proposition 2.10. *Φ is a topological isomorphism from* $\mathrm{Lip}_0(X)$ *onto* $\mathrm{Lip}_0(Y)$.

Proof. Let $f, g \in \mathrm{Lip}_0(X)$. Then $\Phi(f)(\psi(x)) = \Phi(g)(\psi(x))$ for all $x \in X$ implies $\rho(x)f(x) = \rho(x)g(x)$ for all $x \in X$ by Proposition 2.9, which implies that $f = g$. Thus, Φ is injective.

Similarly, if $\alpha \in \mathbf{K}$, where $\mathbf{K}$ is either the real or complex field, then since ψ is surjective by Proposition 2.6,

$$\begin{aligned}\Phi(f+g)(\psi(x)) &= \rho(x)(f+g)(x) = \rho(x)f(x) + \rho(x)g(x)\\ &= \Phi(f)(\psi(x)) + \Phi(g)(\psi(x)),\end{aligned}$$

and

$$\Phi(\alpha f)(\psi(x)) = \rho(x)(\alpha f)(x) = \alpha\rho(x)f(x) = \alpha\Phi(f)(\psi(x))$$

together show that Φ is additive and homogeneous, respectively. Since $\Phi\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is bijective, we can consider its inverse Φ^{-1} of $\mathrm{Lip}_0(Y)$ onto $\mathrm{Lip}_0(X)$, which clearly satisfies

$$\mathrm{Ran}_\pi(\Phi^{-1}(g)\Phi^{-1}(k)) \cap \mathrm{Ran}_\pi(gk) \neq \varnothing \quad (g, k \in \mathrm{Lip}_0(Y)).$$

Since Φ is a linear bijection between Banach spaces, if Φ is continuous, then it is a topological isomorphism by the inverse mapping theorem. To prove the continuity of Φ, observe that the formula

$\|f\|_0 = L(\Phi^{-1}(f))\text{diam}\,(X)$ for all $f \in \text{Lip}_0(Y)$ defines a complete norm on $\text{Lip}_0(Y)$ satisfying

$$\|f\|_\infty = \left\|\Phi^{-1}(f)\right\|_\infty \leq L(\Phi^{-1}(f))\,\text{diam}\,(X) = \|f\|_0\,, \quad \text{for all } f \in \text{Lip}_0(Y).$$

In the above inequality we have used that $\|f\|_\infty \leq L(f)\,\text{diam}\,(X)$ for all $f \in \text{Lip}_0(X)$, which is deduced immediately from

$$|f(z)| = |f(z) - f(e_X)| \leq L(f)d(z, e_X) \leq L(f)\,\text{diam}\,(X), \text{ for all } z \in X.$$

First we show that the identity I on $\text{Lip}_0(Y)$ is continuous from $(\text{Lip}_0(Y), \|\cdot\|_0)$ onto $\text{Lip}_0(Y)$. Let $\{f_n\}$ be a sequence in $\text{Lip}_0(Y)$ such that $\lim_{n\to\infty} \|f_n\|_0 = 0$, and suppose that $\lim_{n\to\infty} L(f_n - f) = 0$ for some function $f \in \text{Lip}_0(Y)$. It follows that

$$\|f\|_\infty \leq \|f - f_n\|_\infty + \|f_n\|_\infty \leq L(f_n - f)\,\text{diam}\,(Y) + \|f_n\|_0\,, \quad \text{for all } n \in \mathbf{N},$$

and, taking limits, it follows that $f = 0$. Thus, I has a closed graph and is therefore continuous by the closed graph theorem.

On the other hand, Φ is continuous from $\text{Lip}_0(X)$ onto $(\text{Lip}_0(Y), \|\cdot\|_0)$ since

$$\|\Phi(f)\|_0 = L(f)\text{diam}\,(X), \text{ for all } f \in \text{Lip}_0(X).$$

From the above we conclude that Φ is a continuous map from $\text{Lip}_0(X)$ onto $\text{Lip}_0(Y)$. □

Let X and Y be pointed metric spaces and Φ a map from $\text{Lip}_0(X)$ into $\text{Lip}_0(Y)$. Then Φ is called *multiplicative* if $\Phi(fg) = \Phi(f)\Phi(g)$ for every $f, g \in \text{Lip}_0(X)$.

Proposition 2.11. $\Phi^2 : \text{Lip}_0(X) \to \text{Lip}_0(Y)$ *is multiplicative.*

Proof. Let $f, g \in \text{Lip}_0(X)$. For every $x \in X$, since $\rho(x) = 1$ or $\rho(x) = -1$, we have $\Phi^2(f)(\psi(x)) = f^2(x)$ by Proposition 2.9. It follows that $\Phi^2(fg)(\psi(x)) = \Phi^2(f)(\psi(x))\Phi^2(g)(\psi(x))$ for all $x \in X$. Since ψ is surjective by Proposition 2.6, we infer that $\Phi^2(fg) = \Phi^2(f)\Phi^2(g)$. □

Proposition 2.12. *There exists a unique function* $\tau : Y \to \{-1, 1\}$ *with* $\tau(e_Y) = 1$ *and a unique Lipschitz homeomorphism* $\varphi : Y \to X$ *with* $\varphi(e_Y) = e_X$ *such that*

$$\Phi(f)(y) = \tau(y) f(\varphi(y)), \text{ for all } f \in Lip_0(X), \text{ for all } y \in Y.$$

Proof. Let $\psi : X \to Y$ and $\rho : X \to \{-1, 1\}$ be the maps given in Definitions 2.3 and 2.4, respectively. By Proposition 2.6, we can define $\varphi = \psi^{-1}$ and $\tau = \rho \circ \psi^{-1}$. According to Proposition 2.9, we have

$$\Phi(f)(y) = \tau(y) f(\varphi(y)), \text{ for all } f \in \mathrm{Lip}_0(X), \text{ for all } y \in Y.$$

Clearly φ is bijective, $\varphi(e_Y) = e_X$ and $\tau(e_Y) = 1$. Next we show that φ is Lipschitz. For each $y \in Y$, consider

$$f_{\varphi(y)}(z) = d(z, \varphi(y)) - d(e_X, \varphi(y)), \text{ for all } z \in X.$$

By Lemma 2.1 i), $f_{\varphi(y)} \in \mathrm{Lip}_0(X)$ with $L(f_{\varphi(y)}) = 1$. Let $y, z \in Y$. If $\tau(y) \neq \tau(z)$, we have

$$\begin{aligned} &|\tau(y) f_{e_X}(\varphi(y)) - \tau(z) f_{e_X}(\varphi(z))| \\ &\qquad = f_{e_X}(\varphi(y)) + f_{e_X}(\varphi(z)) \\ &\qquad = d(\varphi(y), e_X) + d(\varphi(z), e_X) \geq d(\varphi(y), \varphi(z)), \end{aligned}$$

and therefore

$$\begin{aligned} d(\varphi(y), \varphi(z)) &\leq |\Phi(f_{e_X})(y) - \Phi(f_{e_X})(z)| \\ &\leq L(\Phi(f_{e_X}))\, d(y, z) \\ &\leq \|\Phi\|\, L(f_{e_X})\, d(y, z) = \|\Phi\|\ d(y, z). \end{aligned}$$

If $\tau(y) = \tau(z)$, in the same manner we can see that

$$\begin{aligned} d(\varphi(y), \varphi(z)) &= |-d(e_X, \varphi(y)) - [d(\varphi(z), \varphi(y)) - d(e_X, \varphi(y))]| \\ &= \left| f_{\varphi(y)}(\varphi(y)) - f_{\varphi(y)}(\varphi(z)) \right| \\ &= \left| \tau(y) f_{\varphi(y)}(\varphi(y)) - \tau(z) f_{\varphi(y)}(\varphi(z)) \right| \\ &= \left| \Phi(f_{\varphi(y)})(y) - \Phi(f_{\varphi(y)})(z) \right| \leq \|\Phi\|\ d(y, z). \end{aligned}$$

Thus, φ is Lipschitz.

Since Φ^{-1} is a linear bijection from $\mathrm{Lip}_0(Y)$ onto $\mathrm{Lip}_0(X)$ satisfying

$$\mathrm{Ran}_\pi(\Phi^{-1}(g)\Phi^{-1}(k)) \cap \mathrm{Ran}_\pi(gk) \neq \varnothing$$

for all $g, k \in \mathrm{Lip}_0(Y)$, the argument above gives a function $\tau' : X \to \{-1, 1\}$ with $\tau'(e_X) = 1$, and a Lipschitz bijection $\varphi' : X \to Y$ with $\varphi'(e_X) = e_Y$ such that

$$\Phi^{-1}(f)(x) = \tau'(x) f(\varphi'(x)), \text{ for all } f \in \mathrm{Lip}_0(Y), \text{ for all } x \in X.$$

Clearly $\psi(e_X) = e_Y = \varphi'(e_X)$ and, given $x \in X \setminus \{e_X\}$, we have

$$f(\psi(x)) = \Phi(\Phi^{-1}(f))(\psi(x)) = \rho(x)\Phi^{-1}(f)(x) = \rho(x)\tau'(x)f(\varphi'(x))$$

for all $f \in \mathrm{Lip}_0(Y)$. Hence, $(f(\psi(x)))^2 = (f(\varphi'(x)))^2$ for all $f \in \mathrm{Lip}_0(Y)$, which implies $\psi(x) = \varphi'(x)$. Hence, $\psi = \varphi'$, so ψ is also Lipschitz.

Finally, we prove the uniqueness of τ and φ. Let us suppose that

$$\Phi(f)(y) = \tau'(y) f(\varphi'(y)), \text{ for all } f \in \mathrm{Lip}_0(X), \text{ for all } y \in Y,$$

where τ' is a function of Y into $\{-1, 1\}$ such that $\tau'(e_Y) = 1$ and $\varphi' : Y \to X$ is a Lipschitz homeomorphism satisfying that $\varphi'(e_Y) = e_X$. Then

$$\tau'(y) f(\varphi'(y)) = \tau(y) f(\varphi(y)), \text{ for all } f \in \mathrm{Lip}_0(X), \text{ for all } y \in Y.$$

Fix $y \in Y \setminus \{e_Y\}$. Since $\varphi(y) \neq e_X \neq \varphi'(y)$ by the injectivity of φ and φ', we can define the function

$$f(z) = d(z, e_X)/(d(z, \{\varphi(y), \varphi'(y)\}) + d(z, e_X)), \text{ for all } z \in X.$$

Clearly, $f \in \mathrm{Lip}_0(X)$ and $f(\varphi(y)) = f(\varphi'(y)) = 1$. Hence, $\tau'(y) = \tau(y)$ and, since $\tau'(e_Y) = 1 = \tau(e_Y)$, we conclude that $\tau' = \tau$. It follows that $f(\varphi'(y)) = f(\varphi(y))$ for all $f \in \mathrm{Lip}_0(X)$, which implies $\varphi'(y) = \varphi(y)$ and $\varphi'(e_Y) = e_X = \varphi(e_Y)$, so $\varphi' = \varphi$. $\square$

This concludes the proof of Theorem 1.1.

Remark 2.1. The function τ in Proposition 2.12 is Lipschitz if e_X is an isolated point of X (or equivalently if e_Y is an isolated point of Y). Indeed, let $y, y' \in Y \setminus \{e_Y\}$. Define

$$f(z) = d(z, e_X)/(d(z, \{\varphi(y), \varphi(y')\}) + d(z, e_X)), \text{ for all } z \in X.$$

Notice that $d(e_X, X \setminus \{e_X\}) > 0$ since e_X is isolated in X. Clearly $f \in \mathrm{Lip}_0(X)$, $L(f) \leq 1/d(e_X, X \setminus \{e_X\})$ and $f(\varphi(y)) = f(\varphi(y')) = 1$. We have

$$\begin{aligned}
|\tau(y) - \tau(y')| &= |\tau(y)f(\varphi(y)) - \tau(y')f(\varphi(y'))| \\
&= |\Phi(f)(y) - \Phi(f)(y')| \\
&\leq L(\Phi(f))d(y, y') \\
&\leq \|\Phi\| \, L(f)d(y, y') \\
&\leq \|\Phi\| \, (1/d(e_X, X \setminus \{e_X\}))d(y, y'),
\end{aligned}$$

and moreover $|\tau(y) - \tau(e_Y)| \leq 2(1/d(e_Y, Y \setminus \{e_Y\})d(y, e_Y)$. So τ is Lipschitz.

The next example shows that function τ in Proposition 2.12 is not necessarily continuous.

Example 2.1. Equip $[0, 2]$ with the usual metric and with base point 1. Define the function $\tau : [0, 2] \to \{-1, 1\}$ by $\tau = \chi_{[0,1]} - \chi_{]1,2]}$, where $\chi_{[0,1]}$ and $\chi_{]1,2]}$ are characteristic functions of the sets $[0, 1]$ and $]1, 2]$, respectively. For any $f \in \mathrm{Lip}_0([0, 2])$, define $\Phi(f) = \tau f$. For any $a, b \in [0, 1]$ or $a, b \in]1, 2]$, we have

$$|\Phi(f)(a) - \Phi(f)(b)| = |f(a) - f(b)| \leq L(f) \, |a - b| \, ,$$

and for any $a \in [0, 1]$ and $b \in]1, 2]$,

$$\begin{aligned}
|\Phi(f)(a) - \Phi(f)(b)| &\leq |\Phi(f)(a)| + |\Phi(f)(b)| \\
&= |f(a) - f(1)| + |f(1) - f(b)| \\
&\leq L(f) \, |a - 1| + L(f) \, |1 - b| = L(f) \, |a - b| \, .
\end{aligned}$$

Thus, $\Phi(f)$ is Lipschitz. Moreover, $\Phi(f)(1) = \tau(1)f(1) = 0$, and so $\Phi(f) \in \mathrm{Lip}_0([0, 2])$. Hence, the map $\Phi : \mathrm{Lip}_0([0, 2]) \to \mathrm{Lip}_0([0, 2])$ is well defined. An easy verification shows that Φ is surjective and weakly peripherally multiplicative.

3. Some consequences. Recall that any algebra isomorphism $\Phi\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is a composition operator $\Phi(f)(y) = f(\varphi(y))$ for all $f \in \mathrm{Lip}_0(X)$ and $y \in Y$, where $\varphi\colon Y \to X$ is a base point-preserving Lipschitz homeomorphism. First we apply Theorem 1.1 to study when a weakly peripherally multiplicative surjective map between algebras $\mathrm{Lip}_0(X)$ is an algebra isomorphism.

Corollary 3.1. *Let X and Y be pointed compact metric spaces. A surjective map $\Phi\colon Lip_0(X) \to Lip_0(Y)$ is multiplicative and weakly peripherally multiplicative if and only if there exists a base point-preserving Lipschitz homeomorphism $\varphi\colon Y \to X$ such that $\Phi(f)(y) = f(\varphi(y))$ for all $f \in Lip_0(X)$ and $y \in Y$.*

Proof. It is straightforward to check that every map Φ of the form

$$\Phi(f)(y) = f(\varphi(y)), \text{ for all } f \in \mathrm{Lip}_0(X), \text{ for all } y \in Y,$$

where $\varphi : Y \to X$ is a base point-preserving homeomorphism, is a multiplicative surjection of $\mathrm{Lip}_0(X)$ onto $\mathrm{Lip}_0(Y)$ satisfying (1).

Now, suppose that $\Phi\colon \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is surjective and multiplicative and satisfies (1). Then, by Theorem 1.1, there exist a function $\tau\colon Y \to \{-1, 1\}$ with $\tau(e_Y) = 1$ and a base point-preserving Lipschitz homeomorphism $\varphi : Y \to X$ such that the equation $\Phi(f)(y) = \tau(y)f(\varphi(y))$ holds for all $f \in \mathrm{Lip}_0(X)$ and all $y \in Y$. The multiplicativity of Φ implies

$$\tau(y)(fg)(\varphi(y)) = \tau(y)^2 f(\varphi(y))g(\varphi(y))$$

for every $f, g \in \mathrm{Lip}_0(X)$ and $y \in Y$. Hence,

$$\tau(y)(\tau(y) - 1)f(\varphi(y))^2 = 0, \text{ for all } f \in \mathrm{Lip}_0(X), \text{ for all } y \in Y.$$

For each $y \in Y \setminus \{e_Y\}$, let $f \in \mathrm{Lip}_0(X)$ with $f(\varphi(y)) = 1$. Then $\tau(y)(\tau(y) - 1) = 0$ and so $\tau(y) = 1$. Therefore,

$$\Phi(f)(y) = f(\varphi(y)), \text{ for all } f \in \mathrm{Lip}_0(X), \text{ for all } y \in Y \setminus \{e_Y\}.$$

Moreover, $\varphi(e_Y) = e_X$ implies $\Phi(f)(e_Y) = 0 = f(\varphi(e_Y))$, which completes the proof. $\square$

Next we see another condition under which any weakly peripherally multiplicative surjection between algebras of type $\mathrm{Lip}_0(X)$ is an algebra isomorphism. The following corollary is a version for algebras $\mathrm{Lip}_0(X)$ of the main theorem in [**6**].

Corollary 3.2. *Let X and Y be pointed compact metric spaces. A surjective map $\Phi : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ is weakly peripherally multiplicative and maps peaking functions of $\mathrm{Lip}_0(X)$ to peaking functions of $\mathrm{Lip}_0(Y)$ if and only if there exists a base point-preserving Lipschitz homeomorphism $\varphi : Y \to X$ such that $\Phi(f)(y) = f(\varphi(y))$ for all $f \in \mathrm{Lip}_0(X)$ and $y \in Y$.*

Proof. We only prove the "only if" part. By Theorem 1.1, we have

$$\Phi(f)(y) = \tau(y)f(\varphi(y)), \text{ for all } f \in \mathrm{Lip}_0(X), \text{ for all } y \in Y,$$

where τ is a signum function on Y and $\varphi : Y \to X$ is a base point-preserving Lipschitz homeomorphism. Fix $y \in Y$ and let $h \in P(X)$ such that $h(\varphi(y)) = 1$. Then $\tau(y) = \Phi(h)(y)$. Since $\Phi(h) \in P(Y)$ and $|\Phi(h)(y)| = |\tau(y)| = 1$, it follows that $\Phi(h)(y) = 1$, and thus $\tau(y) = 1$. $\square$

Finally, we shall apply Theorem 1.1 to describe the form of all weakly peripherally multiplicative surjective maps between algebras of type $\mathrm{Lip}(X)$. For it we shall need the following lemma:

Lemma 3.3. *Let (X, d_X) be a metric space, let $e_X \notin X$ and let X_0 be the set $X \cup \{e_X\}$ with the metric d_{X_0} defined by*

$$d_{X_0}(x,y) = \begin{cases} \min\{2, d_X(x,y)\} & \text{if } x, y \in X, \\ 0 & \text{if } x = e_X = y, \\ 1 & \text{if } x \in X \text{ and } y = e_X, \text{ or } x = e_X \text{ and } y \in X. \end{cases}$$

Then $\mathrm{Lip}\,(X)$ *is isometrically isomorphic to* $\mathrm{Lip}_0(X_0)$. *Namely, the map* $T_X : \mathrm{Lip}\,(X) \to \mathrm{Lip}_0(X_0)$ *given by*

$$T_X(f)(x) = \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x = e_X, \end{cases}$$

is an isometric isomorphism.

Proof. Clearly T_X is bijective and linear. Let $x, y \in X$, then

$$\frac{|T_X(f)(x) - T_X(f)(e_X)|}{d_{X_0}(x, e_X)} = \frac{|f(x) - 0|}{1} = |f(x)| \leq \|f\|_\infty .$$

Moreover, if $d_X(x, y) \leq 2$, we have

$$\frac{|T_X(f)(x) - T_X(f)(y)|}{d_{X_0}(x, y)} = \frac{|f(x) - f(y)|}{d_X(x, y)} \leq L(f),$$

and if $d_X(x, y) > 2$,

$$\frac{|T_X(f)(x) - T_X(f)(y)|}{d_{X_0}(x, y)} = \frac{|f(x) - f(y)|}{2} \leq \|f\|_\infty .$$

Therefore,

$$L(T_X(f)) \leq \max\{L(f), \|f\|_\infty\}.$$

To obtain the converse inequality observe that

$$\frac{|f(x) - f(y)|}{d_X(x, y)} \leq \frac{|T_X(f)(x) - T_X(f)(y)|}{d_{X_0}(x, y)} \leq L(T_X(f))$$

and

$$|f(x)| = \frac{|f(x) - 0|}{1} = \frac{|T_X(f)(x) - T_X(f)(e_X)|}{d_{X_0}(x, e_X)} \leq L(T_X(f)).$$

for all $x, y \in X$. □

Corollary 3.4. *Let* X *and* Y *be compact metric spaces. A surjective map* $\Phi : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ *is weakly peripherally multiplicative if and only if there exist a Lipschitz function* $\tau : Y \to \{-1, 1\}$ *and a Lipschitz* *homeomorphism* $\varphi : Y \to X$ *such that* $\Phi(f)(y) = \tau(y)f(\varphi(y))$ *for all* $f \in \mathrm{Lip}(X)$ *and* $y \in Y$.

Proof. Suppose that $\Phi\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is a weakly peripherally multiplicative surjection. Following the notation of Lemma 3.3, it is clear that $\Phi_0 = T_Y \Phi T_X^{-1}$ is a surjective map from $\mathrm{Lip}_0(X_0)$ onto $\mathrm{Lip}_0(Y_0)$ such that $\mathrm{Ran}_\pi(fg) \cap \mathrm{Ran}_\pi(\Phi_0(f)\Phi_0(g)) \neq \varnothing$ for all $f, g \in \mathrm{Lip}_0(X_0)$. By Theorem 1.1 and Remark 2.1, there exists a Lipschitz function $\tau_0\colon Y_0 \to \{-1, 1\}$ with $\tau_0(e_Y) = 1$, and a Lipschitz homeomorphism $\varphi_0\colon Y_0 \to X_0$ with $\varphi_0(e_Y) = e_X$ satisfying

$$\Phi_0(f)(y) = \tau_0(y)f(\varphi_0(y)),$$
$$\text{for all } f \in \mathrm{Lip}_0(X_0), \text{ for all } y \in Y_0.$$

Let $\tau = \tau_0|_Y$ and $\varphi = \varphi_0|_Y$. Then τ is a Lipschitz function from Y into $\{-1, 1\}$ and φ is a Lipschitz homeomorphism from Y onto X such that

$$\begin{aligned}\tau(y)f(\varphi(y)) &= \tau_0(y)f(\varphi_0(y)) \\ &= \tau_0(y)T_X(f)(\varphi_0(y)) = \Phi_0(T_X(f))(y) \\ &= (T_Y \Phi T_X^{-1})(T_X(f))(y) \\ &= T_Y(\Phi(f))(y) = \Phi(f)(y)\end{aligned}$$

for all $f \in \mathrm{Lip}(X)$ and $y \in Y$. ◻

Acknowledgments. The authors thank the referee for valuable comments to clarify and improve the manuscript.

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