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Linear Biseparating Maps between Vector-valued Little Lipschitz Function Spaces

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Abstract In this paper we provide a complete description of linear biseparating maps between spaces $\mathrm{lip}_0(X^\alpha, E)$ of Banach-valued little Lipschitz functions vanishing at infinity on locally compact Hölder metric spaces $X^\alpha = (X, d_X^\alpha)$ with $0 < \alpha < 1$. Namely, it is proved that any linear bijection $T : \mathrm{lip}_0(X^\alpha, E) \to \mathrm{lip}_0(Y^\alpha, F)$ satisfying that $\|Tf(y)\|_F \|Tg(y)\|_F = 0$ for all $y \in Y$ if and only if $\|f(x)\|_E \|g(x)\|_E = 0$ for all $x \in X$, is a weighted composition operator of the form $Tf(y) = h(y)(f(\varphi(y)))$, where φ is a homeomorphism from Y onto X and h is a map from Y into the set of all linear bijections from E onto F. Moreover, T is continuous if and only if $h(y)$ is continuous for all $y \in Y$. In this case, φ becomes a locally Lipschitz homeomorphism and h a locally Lipschitz map from Y^α into the space of all continuous linear bijections from E onto F with the metric induced by the operator canonical norm. This enables us to study the automatic continuity of T and the existence of discontinuous linear biseparating maps.

Keywords linear biseparating map, little Lipschitz function, Banach–Stone theorem, automatic continuity

MR(2000) Subject Classification 46E40, 46E15; 47B33, 47B38

1 Introduction

Let (X, d_X) be a metric space, α a real number in $(0, 1]$ and $(E, \|\cdot\|_E)$ a real or complex Banach space. Let X^α denote the same set X endowed with the new metric d_X^α. We denote by $\mathrm{Lip}(X^\alpha, E)$ the Banach space of all functions $f : X \to E$ such that

$$p_\alpha(f) = \sup\{\|f(x) - f(y)\|_E / d_X(x, y)^\alpha : x, y \in X,\ x \neq y\}$$

and

$$\|f\|_\infty = \sup\{\|f(x)\|_E : x \in X\}$$

are finite, endowed with the sum norm $\|f\|_\alpha = p_\alpha(f) + \|f\|_\infty$.

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The *little Lipschitz space* $\mathrm{lip}(X^\alpha, E)$ denotes the closed subspace of functions f in $\mathrm{Lip}(X^\alpha, E)$ such that

$$\lim_{d_X(x,y)\to 0} \frac{\|f(x)-f(y)\|_E}{d_X(x,y)^\alpha} = 0,$$

that is, satisfying the following property:

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 : 0 < d_X(x,y) < \delta \Rightarrow \|f(x)-f(y)\|_E / d_X(x,y)^\alpha < \varepsilon.$$

If (X, d_X) is locally compact, let $\mathrm{lip}_0(X^\alpha, E)$ and $\mathrm{lip}_{00}(X^\alpha, E)$ stand for the spaces of functions in $\mathrm{lip}(X^\alpha, E)$ that vanish at infinity and that have compact support, respectively. We drop the letter E when E is the scalar field, and the superscript α when $\alpha = 1$. In order to avoid trivial cases, we make a blanket assumption that all metric spaces are nonempty and that all Banach spaces are nonzero.

The study of these spaces started with de Leeuw [1] and Sherbert [2, 3] for scalar-valued functions in the 60's, and with Johnson [4] for vector-valued functions in the 70's. From then on, a rich literature exists on this subject. A nice survey is Weaver's book on *Lipschitz Algebras* [5].

A linear map $T : \mathrm{lip}_0(X^\alpha, E) \to \mathrm{lip}_0(Y^\alpha, F)$ is *separating* if $\|Tf(y)\|_F \|Tg(y)\|_F = 0$ for all $y \in Y$ whenever $f, g \in \mathrm{lip}_0(X^\alpha, E)$ satisfy $\|f(x)\|_E \|g(x)\|_E = 0$ for all $x \in X$. T is *biseparating* if it is bijective and both T and T^{-1} are separating.

Separating maps between spaces of scalar-valued continuous functions on locally compact and compact spaces were first studied by Beckenstein, et al. [6], Font and Hernández [7], Jarosz [8] and Jeang and Wong [9], but the investigation into such maps has a long history in Functional Analysis in the context of rings, algebras, or vector lattices under the names of *Lamperti operators* [10] or *disjointness preserving operators* [11]. In recent years, quite a lot of attention has been given to linear biseparating maps on spaces of vector-valued continuous functions by Araujo [12–15], Araujo and Dubarbie [16], Araujo and Jarosz [17], Gau, et al. [18] and Hernández, et al. [19].

The study of linear separating maps between spaces of scalar-valued Lipschitz functions was initiated by Wu in [20]. Later, the first author of the current paper obtained in [21] the general form of all linear separating maps between little Lipschitz algebras $\mathrm{lip}(X^\alpha)$ for X compact and $0 < \alpha < 1$. Moreover, it was shown that any linear separating bijection between such algebras is biseparating and automatically continuous, and the problem of the existence of discontinuous linear separating functionals on $\mathrm{lip}(X^\alpha)$ was solved. Related to this problem, the paper [22] of Brown and Wong gives full account of the structure of unbounded separating linear functionals on spaces of continuous functions. As a natural generalization, we dealt in [23] with the question of describing the structure of all linear biseparating maps between spaces $\mathrm{lip}(X^\alpha, E)$ for X and α as above. Recently, a complete description of linear biseparating maps between spaces $\mathrm{Lip}(X, E)$ for complete metric spaces X has been given by Araujo and Dubarbie in [16] and the automatic continuity of such maps has been derived in some cases.

The aim of this paper is to extend the results stated in [21, 23] to spaces $\mathrm{lip}_0(X^\alpha, E)$ when X is locally compact and $\alpha \in (0, 1)$. We will prove in Theorem 3.1 that all linear biseparating maps between spaces $\mathrm{lip}_0(X^\alpha, E)$ are, in a natural sense, weighted composition operators. More precisely, if $T : \text{lip}_0(X^\alpha, E) \to \text{lip}_0(Y^\alpha, F)$ is such a map, then there exist a homeomorphism φ from Y onto X and a map h from Y into the set $L^{-1}(E, F)$ of all linear bijections from E onto F such that the representation $Tf(y) = h(y)(f(\varphi(y)))$ holds for every $f \in \text{lip}_0(X^\alpha, E)$ and all $y \in Y$.

However, such maps T need not be continuous. Indeed, it is shown in Theorem 4.1 that T is continuous if and only if $h(y)$ is continuous for all $y \in Y$, in whose case the aforementioned representation of T can be considerably improved since φ becomes a locally Lipschitz homeomorphism from Y onto X and h a locally Lipschitz map from Y^α into the space $B^{-1}(E, F)$ of all continuous linear bijections from E onto F with the metric induced by the operator canonical norm.

From Theorem 4.1, we will deduce immediately in Corollary 4.2 that if the Banach spaces E, F are finite-dimensional, then any linear biseparating map T must be automatically continuous. The same conclusion will be obtained in Theorem 4.3 if X or Y has no isolated points with independence of the dimensions of E and F. In fact, we will prove in Corollary 4.4 that if X and Y are Lipschitz homeomorphic and E is infinite-dimensional, then there exists a discontinuous linear biseparating map from $\text{lip}_0(X^\alpha, E)$ to $\text{lip}_0(Y^\alpha, E)$ if and only if X or Y has isolated points. Research into automatic continuity properties and existence of discontinuous maps in Lipschitz algebras can be found in [24, 25] by Pavlović.

The arguments given in [21, 23] depend essentially on the concept of support map of a separating map. We will follow here a completely different approach which proves to be very fruitful in the work [18] of Gau, et al. to whom the introduction of this new view-point is due.

2 Preliminaries

Our notation is mainly standard. Let (X, d_X) be a metric space. For a subset A of X, $\overline{A}$ and $\text{int}(A)$ stand for the closure and the interior of A in X, respectively. For $x \in X$ and $r > 0$, $B(x, r)$ denotes the open ball of radius r centred at x, and $\overline{B}(x, r)$ the corresponding closed ball. Given a function f defined on X, we write $\text{Z}(f)$, $\text{coz}(f)$ and $\text{supp}(f)$ for the zero set of f, the cozero set of f and the closure in X of the cozero set of f, respectively. For each $x \in X$, we denote by δ_x the evaluation map at the point x.

The following lemma whose proof is straightforward summarizes some properties of the spaces of Lipschitz functions which we will use later on.

Lemma 2.1 *Let X be a metric space, $\alpha \in (0, 1)$ and E a Banach space. Then*

a) $\text{Lip}(X, E) \subseteq \text{lip}(X^\alpha, E)$.

If, in addition, X is locally compact, we have

b) *If $k \in \text{lip}_{00}(X^\alpha)$ and $f \in \text{lip}(X^\alpha, E)$, then $kf \in \text{lip}_{00}(X^\alpha, E)$.*

c) *If $k \in \text{lip}(X^\alpha)$ and $f \in \text{lip}_0(X^\alpha, E)$, then $kf \in \text{lip}_0(X^\alpha, E)$.*

d) $\text{lip}_0(X^\alpha, E)$ *is a Banach space containing* $\text{lip}_{00}(X^\alpha, E)$.

In our arguments, we will use also the following Lipschitz version of the classical Urysohn's lemma.

Lemma 2.2 *Let X be a locally compact metric space, $\alpha \in (0, 1)$ and E a Banach space.*

i) *Let K be a nonempty compact subset of X and let U be an open subset of X with $K \subseteq U$. Then there exist a relatively compact open set V with $K \subseteq V \subseteq \overline{V} \subseteq U$ and a function $f \in \mathrm{lip}_{00}(X^\alpha)$ with $\|f\|_\alpha \leq \max\{1/d_X(K, X\setminus V), 1\} + 1$ such that $0 \leq f \leq 1$, $f|_K = 1$, $\mathrm{coz}(f) = V$ and $f|_{X\setminus U} = 0$.*

ii) *Given $x \in X$ and a neighborhood U of x, there exist relatively compact neighborhoods V, W of x with $\overline{V} \subseteq W \subseteq \overline{W} \subseteq U$ and a function $k \in \mathrm{lip}_{00}(X^\alpha)$ with*

$$\|k\|_\alpha \leq \max\left\{1/d_X(\overline{V}, X\setminus W), 1\right\} + 1$$

such that $0 \leq k \leq 1$, $k|_{\overline{V}} = 1$, $\mathrm{coz}(k) = W$ and $k|_{X\setminus U} = 0$.

iii) *Given $x \in X$, a neighborhood U of x and $e \in E\backslash\{0\}$, there exists a $g \in \mathrm{lip}_{00}(X^\alpha, E)$ such that $g(x) = e$ and $g|_{X\setminus U} = 0$.*

Proof Let $K \subseteq X$ be nonempty compact and let $U \subseteq X$ be open with $K \subseteq U$. By [26, Theorem 6.79], there exists a relatively compact open set V such that $K \subseteq V \subseteq \overline{V} \subseteq U$. Define

$$f_{KV}(z) = d_X(z, X\setminus V)/[d_X(z, K) + d_X(z, X\setminus V)], \quad z \in X.$$

Take $f = f_{KV}$. f is well defined since K and $X\setminus V$ are disjoint closed sets. It is obvious that $0 \leq f \leq 1$ and $f|_K = 1$. Clearly, $\mathrm{coz}(f) = V$ and thus $\mathrm{supp}(f)$ is compact and $f|_{X\setminus U} = 0$. Notice that $d_X(K, X\setminus V) > 0$ since K is compact, $X\setminus V$ closed and $K \cap (X\setminus V) = \emptyset$. A trivial verification yields

$$|f(z) - f(y)| \leq d_X(z, y)/d_X(K, X\setminus V)$$

for any $z, y \in X$. Hence $f \in \mathrm{Lip}(X) \subseteq \mathrm{lip}(X^\alpha)$ by Lemma 2.1 a). Furthermore, if $d_X(z, y) \leq 1$, we have

$$|f(z) - f(y)| \leq d_X(z, y)^\alpha/d_X(K, X\setminus V),$$

and if $d_X(z, y) > 1$, since $0 \leq f \leq 1$, we get

$$|f(z) - f(y)| \leq 1 < d_X(z, y)^\alpha.$$

Hence $p_\alpha(f) \leq \max\{1/d_X(K, X\setminus V), 1\}$. Since $\|f\|_\infty = 1$, it follows that

$$\|f\|_\alpha \leq \max\{1/d_X(K, X\setminus V), 1\} + 1,$$

and so i) is proved.

Let $x \in X$ and let U be a neighborhood of x. We can first choose a relatively compact neighborhood V of x with $\overline{V} \subseteq U$ and afterwards a relatively compact open set W with $\overline{V} \subseteq W \subseteq \overline{W} \subseteq U$ (see [26, Theorems 6.78 and 6.79]). Taking $k = f_{\overline{V}W}$, we obtain ii). Take now $e \in E\backslash\{0\}$. Then iii) follows defining $g(z) = k(z)e$ for all $z \in X$ since $g \in \mathrm{lip}_{00}(X^\alpha, E)$ by Lemma 2.1 b). □

For each $x \in X$, define

$$I_x = \{f \in \mathrm{lip}_0(X^\alpha, E) : x \notin \mathrm{supp}(f)\},$$
$$M_x = \{f \in \mathrm{lip}_0(X^\alpha, E) : f(x) = 0\}.$$

Clearly, $\emptyset \neq I_x \subseteq M_x$. These sets are concerned in the ideal structure of $\mathrm{lip}(X^\alpha)$ for X compact and $0 < \alpha < 1$ (see [3]).

The key idea to prove the sought-after result on the representation of a linear biseparating map T from $\mathrm{lip}_0(X^\alpha, E)$ onto $\mathrm{lip}_0(Y^\alpha, F)$ will be the fact that T preserves the linear subspaces I_x's and M_x's. It will not be hard to show that T preserves the former, but to prove that T preserves the latter, the next result will be required. Its proof is similar to that of [23, Lemma 2.8].

Lemma 2.3 *Let X be a locally compact metric space, $\alpha \in (0,1)$ and E a Banach space. Let $x \in X$ and let $\{x_n\}$ be a sequence of distinct points in $X \setminus \{x\}$ converging to x. Then, for each $f \in M_x$, there exist a subsequence $\{x_{n_k}\}$, a sequence $\{s_k\}$ in $(0,1)$, a sequence $\{B(x_{n_k}, 2s_k)\}$ of pairwise disjoint relatively compact open balls in X and a sequence $\{g_k\}$ in $\mathrm{lip}_{00}(X^\alpha, E)$ such that $g_k(x) = f(x)$ for all $x \in B(x_{n_k}, s_k)$, $\mathrm{coz}(g_k) \subseteq B(x_{n_k}, 2s_k)$ and $\|g_k\|_\alpha \le 10/k^2$ for all $k \in \mathbb{N}$.*

We also will need the following fact.

Lemma 2.4 *Let $x, y \in X$. If $I_x \subseteq I_y$, then $x = y$.*

Proof Assume $x \neq y$. Let U be a neighborhood of y such that $x \in X \setminus \overline{U}$. Given $e \in E \setminus \{0\}$, Lemma 2.2 iii) provides a $g \in \mathrm{lip}_{00}(X^\alpha, E)$ such that $g(y) = e$ and $g|_{X\setminus U} = 0$. Hence $g \in I_x \setminus I_y$ and I_x is not contained in I_y. □

3 Banach–Stone Type Representation

After this preparation, we are in a position to prove the main result of this paper.

Theorem 3.1 *Let X, Y be locally compact metric spaces, let α be in $(0,1)$ and let E, F be Banach spaces. Every biseparating linear bijection $T : \mathrm{lip}_0(X^\alpha, E) \to \mathrm{lip}_0(Y^\alpha, F)$ is a weighted composition operator of the form $Tf(y) = h(y)(f(\varphi(y)))$ for every $f \in \mathrm{lip}_0(X^\alpha, E)$ and all $y \in Y$, where $h(y)$ is a linear bijection from E onto F for each y in Y, and φ is a homeomorphism from Y onto X.*

Proof We have divided the proof into a series of claims.

Claim 1 For each $y \in Y$, $\bigcap_{g \in I_y} \mathrm{Z}(T^{-1}g) = \emptyset$ if and only if $T(\mathrm{lip}_{00}(X^\alpha, E)) \subseteq I_y$.

Fix $y \in Y$ and suppose $\bigcap_{g \in I_y} \mathrm{Z}(T^{-1}g) = \emptyset$. Then, for any $x \in X$, there exist $g_x \in I_y$ and a neighborhood U_x of x such that $T^{-1}g_x$ never vanishes on U_x. Let $f \in \mathrm{lip}_{00}(X^\alpha, E)$ and $g = Tf \in \mathrm{lip}_0(Y^\alpha, F)$. Since $\{U_x : x \in X\}$ is an open covering of the compact set $\mathrm{supp}(f)$, we have $\mathrm{supp}(f) \subseteq \bigcup_{i=1}^n U_{x_i}$ for some $n \in \mathbb{N}$. Clearly, $U = \mathrm{int}\left(\bigcap_{i=1}^n \mathrm{Z}(g_{x_i})\right)$ is a neighborhood of y. According to Lemma 2.2 ii), there exist a neighborhood W of y and $k \in \mathrm{lip}_{00}(Y^\alpha)$ with $\mathrm{coz}(k) = W$. Notice that $kg \in \mathrm{lip}_{00}(Y^\alpha, F)$ by Lemma 2.1 b). We see that $\|(kg)(z)\|_F \|g_{x_i}(z)\|_F = 0$ for all $i \in \{1, \ldots, n\}$ and $z \in Y$. This implies that $\left\|T^{-1}(kg)(x)\right\|_E \left\|T^{-1}g_{x_i}(x)\right\|_E = 0$ for all $i \in \{1, \ldots, n\}$ and $x \in X$ since T^{-1} is separating. It follows that $\left\|T^{-1}(kg)(x)\right\|_E \|f(x)\|_E = 0$ for all $x \in X$ since $f(x) = 0$ if $x \notin \bigcup_{i=1}^n U_{x_i}$, and $T^{-1}(kg)(x) = 0$ if $x \in U_{x_i}$ for some $i \in \{1, \ldots, n\}$ because $T^{-1}g_{x_i}(x) \neq 0$. Thus $\left\|T^{-1}(kg)(x)\right\|_E \left\|T^{-1}g(x)\right\|_E = 0$ for all $x \in X$. Then $|k(z)| \|g(z)\|_F^2 = 0$ for all $z \in Y$ since T is separating. Hence $g|_W = 0$ and thus $g \in I_y$. This proves that $T(\mathrm{lip}_{00}(X^\alpha, E)) \subseteq I_y$.

Conversely, assume that $T(\mathrm{lip}_{00}(X^\alpha, E)) \subseteq I_y$ and let $x \in \bigcap_{g \in I_y} \mathrm{Z}(T^{-1}g)$ be given. Then

$f(x) = T^{-1}(Tf)(x) = 0$ for all $f \in \mathrm{lip}_{00}(X^\alpha, E)$, but this contradicts Lemma 2.2 iii).

Now we define $X_1 = \bigcup_{g \in \mathrm{lip}_{00}(Y^\alpha, F)} \mathrm{coz}(T^{-1}g)$ and $Y_1 = \bigcup_{f \in \mathrm{lip}_{00}(X^\alpha, E)} \mathrm{coz}(Tf)$.

Claim 2 The sets X_1 and Y_1 are dense in X and Y, respectively.

Assume that X_1 is nondense in X. Then there exist an $x \in X$ and a neighborhood U of x such that $U \cap X_1 = \emptyset$. Consequently, for any $g \in \mathrm{lip}_{00}(Y^\alpha, F)$, we have $T^{-1}g(z) = 0$ for all $z \in U$. According to Lemma 2.2 iii), we can take an $f \in \mathrm{lip}_{00}(X^\alpha, E)$ such that $f(x) \neq 0$ and $f|_{X \setminus U} = 0$. Let $k = Tf \in \mathrm{lip}_0(Y^\alpha, F)$. Clearly, $\|f(z)\|_E \left\|T^{-1}g(z)\right\|_E = 0$ for all $z \in X$. This gives $\|k(y)\|_F \|g(y)\|_F = 0$ for all $y \in Y$, since T is separating. Now, fix $y \in Y$ and, using Lemma 2.2 iii), choose a $g \in \mathrm{lip}_{00}(Y^\alpha, F)$ for which $g(y) \neq 0$. We thus get $k(y) = 0$ and, consequently, $k = 0$. Since T is linear and injective, it follows that $f = 0$, which is impossible. Therefore X_1 is dense in X. Similarly, it is proved that so is Y_1 in Y.

Claim 3 If $x \in \bigcap_{g \in I_y} \mathrm{Z}(T^{-1}g)$, then $T^{-1}I_y \subseteq I_x$ and $x \in X_1$.

To prove that $T^{-1}I_y \subseteq I_x$, let $g \in I_y$ and assume that $T^{-1}g \notin I_x$. First we observe that if f is any function in $\mathrm{lip}_0(Y^\alpha, F)$ such that $\|f(z)\|_F \|g(z)\|_F = 0$ for all $z \in Y$, then $T^{-1}f(x) = 0$. This is true, because if $T^{-1}f(x) \neq 0$, then $T^{-1}f$ is nonvanishing in a neighborhood of x and, since T^{-1} is separating, it follows that $T^{-1}g$ vanishes in the said neighborhood, a contradiction. Note now that $U = \mathrm{int}(\mathrm{Z}(g))$ is a neighborhood of y and, according to Lemma 2.2 ii), there exists a neighborhood V of y and a $k \in \mathrm{lip}_{00}(Y^\alpha)$ such that $k|_V = 1$ and $k|_{Y \setminus U} = 0$. Fix $f \in \mathrm{lip}_0(Y^\alpha, F)$. It is immediate that $f = kf + (1-k)f$ with kf and $(1-k)f$ in $\mathrm{lip}_0(Y^\alpha, F)$ by Lemma 2.1 c). Since $(1-k)|_V = 0$, we have $(1-k)f \in I_y$, and thus $T^{-1}((1-k)f)(x) = 0$ since $x \in \bigcap_{g \in I_y} \mathrm{Z}(T^{-1}g)$. On the other hand, we have $(kf)|_{\mathrm{coz}(g)} = 0$ since $\mathrm{coz}(g) \subseteq Y \setminus U$. Hence $\|(kf)(z)\|_F \|g(z)\|_F = 0$ for all $z \in Y$. Then the observation above implies $T^{-1}(kf)(x) = 0$. It follows that $T^{-1}f(x) = T^{-1}(kf)(x) + T^{-1}((1-k)f)(x) = 0$. Since f is arbitrary, the surjectivity of T^{-1} yields $h(x) = 0$ for all $h \in \mathrm{lip}_0(X^\alpha, E)$, a contradiction. This proves that $T^{-1}I_y \subseteq I_x$.

We now show that $x \in X_1$. For any $g \in \mathrm{lip}_0(Y^\alpha, F)$, $kg, (1-k)g \in \mathrm{lip}_0(Y^\alpha, F)$ by Lemma 2.1 c). It is clear that $T^{-1}g(x) = T^{-1}(kg)(x) + T^{-1}((1-k)g)(x)$. Since $(1-k)g \in I_y$, we have $T^{-1}((1-k)g)(x) = 0$. Hence there must exist a $g \in \mathrm{lip}_0(Y^\alpha, F)$ such that $T^{-1}(kg)(x) \neq 0$; otherwise we would have $T^{-1}g(x) = 0$ for all $g \in \mathrm{lip}_0(Y^\alpha, F)$, which gives a contradiction. Hence $x \in X_1$.

Since T is separating as T^{-1}, we can apply the argument above to T and obtain the following

Claim 4 If $y \in \bigcap_{f \in I_x} \mathrm{Z}(Tf)$, then $TI_x \subseteq I_y$ and $y \in Y_1$.

Claim 5 For each $y \in Y_1$, there exists an $x \in X_1$ such that $T^{-1}I_y \subseteq I_x$.

Fix $y \in Y_1$. We first show that $\bigcap_{g \in I_y} \mathrm{Z}(T^{-1}g)$ is nonempty. If it were not true, we would have $T(\mathrm{lip}_{00}(X^\alpha, E)) \subseteq I_y$ by Claim 1. Hence $Tf(y) = 0$ for any $f \in \mathrm{lip}_{00}(X^\alpha, E)$, but this contradicts that $y \in Y_1$. Pick an $x \in \bigcap_{g \in I_y} \mathrm{Z}(T^{-1}g)$. Applying Claim 3, we see that $x \in X_1$ and $T^{-1}I_y \subseteq I_x$.

Claim 6 For each $y \in Y_1$, there corresponds a unique $x \in X_1$ such that $TI_x = I_y$.

Fix $y \in Y_1$. By Claim 5, there exists an $x \in X_1$ such that $T^{-1}I_y \subseteq I_x$. Since T^{-1} has a similar property to that of T, we can apply Claim 5 to T^{-1} and find a $y' \in Y_1$ such that

$TI_x \subseteq I_{y'}$. It follows that $I_y = T(T^{-1}I_y) \subseteq TI_x \subseteq I_{y'}$. This relation implies that $y = y'$ by Lemma 2.4. Then it follows that $TI_x = I_y$.

To show the uniqueness of x, suppose that $TI_{x'} = I_y$ for an $x' \in X_1$. We see that $I_x = T^{-1}I_y = I_{x'}$ and conclude that $x = x'$ by Lemma 2.4.

By Claim 6, we can define a map $\varphi : Y_1 \to X_1$ satisfying that $TI_{\varphi(y)} = I_y \, (y \in Y_1)$. Since T^{-1} is separating as T, we can apply the argument above to T^{-1} and obtain a map $\phi : X_1 \to Y_1$ such that $T^{-1}I_{\phi(x)} = I_x \, (x \in X_1)$.

Claim 7 The map $\varphi : Y_1 \to X_1$ is a homeomorphism.

First we see that for each $y \in Y_1$, $I_{\phi(\varphi(y))} = T(T^{-1}I_{\phi(\varphi(y))}) = TI_{\varphi(y)} = I_y$. Applying Lemma 2.4, we have $\phi(\varphi(y)) = y$ for every $y \in Y_1$. Thus $\phi \circ \varphi$ is the identity map on Y_1. In a similar way, we obtain that $\varphi \circ \phi$ is the identity map on X_1. These facts imply that φ is a one-to-one map of Y_1 onto X_1 and $\varphi^{-1} = \phi$.

Now we prove that φ is continuous. Pick $y \in Y_1$ and let $\{y_n\}$ be a sequence in Y_1 converging to y. Suppose that $\{\varphi(y_n)\}$ does not converge to $\varphi(y)$. Then there exist a neighborhood U of $\varphi(y)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ satisfying that $\varphi(y_{n_k}) \in X \setminus \overline{U}$ for all $k \in \mathbb{N}$.

Next we observe that $Tf(y) = 0$ for any $f \in \mathrm{lip}_0(X^\alpha, E)$ such that $f|_{X \setminus U} = 0$. Indeed, since $f|_{X \setminus \overline{U}} = 0$ and $X \setminus \overline{U}$ is a neighborhood of $\varphi(y_{n_k})$, we infer that $f \in I_{\varphi(y_{n_k})}$; hence $Tf \in I_{y_{n_k}}$ by the definition of φ; in particular, $Tf(y_{n_k}) = 0$ for all $k \in \mathbb{N}$ and we conclude that $Tf(y) = 0$ by the continuity of Tf.

According to Lemma 2.2 ii), there are a neighborhood V of $\varphi(y)$ and a $k \in \mathrm{lip}_{00}(X^\alpha)$ such that $k|_V = 1$ and $k|_{X \setminus U} = 0$. It is evident that $f = kf + (1-k)f$ for any $f \in \mathrm{lip}_0(X^\alpha, E)$. Since $(kf)|_{X \setminus U} = 0$, we have $T(kf)(y) = 0$ by the observation above. On the other hand, we have $(1-k)f \in I_{\varphi(y)}$ since $(1-k)|_V = 0$. By the definition of φ, $T((1-k)f) \in I_y$ and thus $T((1-k)f)(y) = 0$. It follows that $Tf(y) = T(kf)(y) + T((1-k)f)(y) = 0$. Since T is onto, we have $g(y) = 0$ for all $g \in \mathrm{lip}_0(Y^\alpha, F)$, a contradiction. Hence φ is continuous.

We can apply similar argument to $\phi = \varphi^{-1}$ and see that φ^{-1} is also continuous. Hence φ is a homeomorphism of Y_1 onto X_1.

Claim 8 For any $y \in Y_1$, $TM_{\varphi(y)} = M_y$.

Fix $y \in Y_1$ and put $x = \varphi(y) \in X_1$. First we show that $TM_x \subseteq M_y$. Let $f \in M_x$ and assume $Tf(y) \neq 0$. If x is an isolated point of X_1, then $f \in I_x$ and, by the definition of φ, $Tf \in I_y$, a contradiction. Assume now that x is not isolated in X_1. Then we may find a sequence $\{x_n\}$ of distinct points in $X_1 \setminus \{x\}$ converging to x. Let $y_n \in Y_1$ with $\varphi(y_n) = x_n$ for all $n \in \mathbb{N}$. By the continuity of φ^{-1}, $\{y_n\}$ converges to y. Since $\{Tf(y_n)\}$ converges to $Tf(y) \neq 0$, passing to a subsequence if necessary, we may assume that $\|Tf(y_n)\|_F > (1/2)\|Tf(y)\|_F$ for all $n \in \mathbb{N}$.

Since $f \in M_x$, by Lemma 2.3 there exist a subsequence $\{x_{n_k}\}$, a sequence $\{s_k\}$ in $(0,1)$, a sequence $\{B(x_{n_k}, 2s_k)\}$ of pairwise disjoint relatively compact open balls in X and a sequence $\{g_k\}$ in $\mathrm{lip}_{00}(X^\alpha, E)$ such that $g_k = f$ on $B(x_{n_k}, s_k)$, $\mathrm{coz}(g_k) \subseteq B(x_{n_k}, 2s_k)$ and $\|g_k\|_\alpha \leq 10/k^2$ for all $k \in \mathbb{N}$. As $\mathrm{lip}_0(X^\alpha, E)$ is a Banach space by Lemma 2.1 d) and $\left\|k^{1/2}g_k\right\|_\alpha \leq 10/k^{3/2}$ for all $k \in \mathbb{N}$, let $g \in \mathrm{lip}_0(X^\alpha, E)$ be the function defined by $g = \sum_{k=1}^{+\infty} k^{1/2}g_k$. For each $k \in \mathbb{N}$, it is clear that $g = k^{1/2}f$ on $B(x_{n_k}, s_k)$ since the sets $\mathrm{coz}(g_k)$ are pairwise disjoint, and therefore

$g - k^{1/2} f \in I_{x_{n_k}}$. It follows that $Tg(y_{n_k}) = k^{1/2} Tf(y_{n_k})$, and thus

$$\|Tg(y_{n_k})\|_F = k^{1/2} \|Tf(y_{n_k})\|_F > (1/2)k^{1/2} \|Tf(y)\|_F, \quad \forall\, k \in \mathbb{N}.$$

This implies that Tg is unbounded, a contradiction. In this way it is proved that $TM_{\varphi(y)} \subseteq M_y$.

Applying the same argument to T^{-1}, we have $T^{-1}M_{\phi(\varphi(y))} \subseteq M_{\varphi(y)}$. Hence $T^{-1}M_y \subseteq M_{\varphi(y)}$ since $\phi(\varphi(y)) = y$, and thus $M_y \subseteq TM_{\varphi(y)}$.

Claim 9 For each $y \in Y_1$, there exists a linear bijection $h(y) : E \to F$ such that $Tf(y) = h(y)(f(\varphi(y))$ for all $f \in \mathrm{lip}_0(X^\alpha, E)$.

Given $y \in Y_1$, we know that $\ker \delta_{\varphi(y)} = \ker(\delta_y \circ T)$ by Claim 8. Consequently, there is a linear bijection $h(y) : E \to F$ such that $\delta_y \circ T = h(y) \circ \delta_{\varphi(y)}$. In other words, $Tf(y) = h(y)(f(\varphi(y)))$ for all $f \in \mathrm{lip}_0(X^\alpha, E)$.

Claim 10 Let $y \in Y \setminus Y_1$ and let $\{y_n\}$ be a sequence in Y_1 convergent to y. Let $x_n = \varphi(y_n)$ for all $n \in \mathbb{N}$. Then $\{x_n\} \to \infty$.

First observe that such a sequence $\{y_n\}$ exists by Claim 2. Assume that $\{x_n\}$ does not converge to ∞. Then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to a point $x \in X$. Let $f \in I_x$. Since $\{x_{n_k}\} \to x$, we have $f \in I_{x_{n_k}}$ for k large enough. Thus, $Tf \in I_{y_{n_k}}$ by the definition of φ, and $Tf(y) = 0$ by the continuity of Tf. Hence $y \in \bigcap_{f \in I_x} \mathrm{Z}(Tf)$. Then $y \in Y_1$ by Claim 4, which is impossible.

Claim 11 Let $\{y_n\}$ be a sequence of distinct points in Y_1. Then $\limsup \|h(y_n)\| < +\infty$.

Let $x_n = \varphi(y_n)$ for all $n \in \mathbb{N}$. Since φ is injective, $\{x_n\}$ is a sequence of distinct points in X_1. As X is a regular Hausdorff space, taking a subsequence if necessary we can find a sequence $\{U_n\}$ of disjoint pairwise neighborhoods of points x_n. According to Lemma 2.2 ii), there exist sequences $\{V_n\}, \{W_n\}$ of relatively compact neighborhoods of points x_n with $\overline{V}_n \subseteq W_n \subseteq \overline{W}_n \subseteq U_n$ and a sequence $\{k_n\}$ in $\mathrm{lip}_{00}(X^\alpha)$ with $\|k_n\|_\alpha \le r_n := \max\{1/d_X(\overline{V}_n, X \setminus W_n), 1\} + 1$ such that $k_n(x_n) = 1$ and $\mathrm{coz}(k_n) = W_n$.

Assume to the contrary that $\limsup \|h(y_n)\| = +\infty$. Taking a subsequence if necessary, we can assume that $\|h(y_n)\| \ge r_n n^4$ for all $n \in \mathbb{N}$. Then there exists a sequence $\{e_n\}$ in E with $\|e_n\|_E = 1$ such that $\|h(y_n)(e_n)\|_F \ge r_n n^3$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define $f_n(z) = (1/r_n n^2) k_n(z) e_n$ for all $z \in X$. Clearly, $f_n \in \mathrm{lip}_{00}(X^\alpha, E)$ and $f_n(x_n) = (1/r_n n^2) e_n$. We have $Tf_n(y_n) = h(y_n)(f_n(x_n)) = (1/r_n n^2) h(y_n)(e_n)$ by using Claim 9. It follows that $\|Tf_n(y_n)\|_F = (1/r_n n^2) \|h(y_n)(e_n)\|_F \ge n$. Moreover, since $\|f_n\|_\alpha = (1/r_n n^2) \|k_n\|_\alpha \|e_n\|_E \le 1/n^2$ for all $n \in \mathbb{N}$, we can define the map $f := \sum_{n=1}^{+\infty} f_n \in \mathrm{lip}_0(X^\alpha, E)$.

On the other hand, for each $m \in \mathbb{N}$, as the sets $\mathrm{coz}(f_n) = W_n$ are pairwise disjoint, we see that $f_n|_{W_m} = 0$ for all $n \ne m$, which implies $(\sum_{n=1, n\ne m}^{+\infty} f_n)|_{W_m} = 0$, and hence $\sum_{n=1, n\ne m}^{+\infty} f_n \in I_{x_m}$. Then, by the definition of φ, $T(\sum_{n=1, n\ne m}^{+\infty} f_n) \in I_{y_m}$ and, in particular, $T(\sum_{n=1, n\ne m}^{+\infty} f_n)(y_m) = 0$. It follows that

$$\|Tf(y_m)\|_F = \left\| Tf_m(y_m) + T\biggl(\sum_{n=1, n\ne m}^{+\infty} f_n\biggr)(y_m) \right\|_F = \|Tf_m(y_m)\|_F \ge m.$$

Hence Tf is unbounded, which is a contradiction.

Claim 12 $Y_1 = Y$ and $X_1 = X$.

We first show that $Y_1 = Y$. Notice that $Y = Y_1 \cup Y_1'$ by Claim 2, where Y_1' is the derived set of Y_1 in Y. Take $y \in Y$. If $y \in Y_1$, we have finished. If $y \in Y_1'$, there exists a sequence $\{y_n\}$ of distinct points in $Y_1 \setminus \{y\}$ convergent to y. Assume, to the contrary, that $y \notin Y_1$. Then Claim 10 gives $\{\varphi(y_n)\} \to \infty$. Fix $f \in \mathrm{lip}_0(X^\alpha, E)$. Clearly, $\lim \|f(\varphi(y_n)\|_E = 0$. Moreover, $\limsup \|h(y_n)\| < +\infty$ by Claim 11, and this says, in particular, that only finitely many $h(y_n)$ can have infinite norms. In consequence, we have $\|h(y_n)(f(\varphi(y_n))\|_F \leq \|h(y_n)\| \, \|f(\varphi(y_n)\|_E$ for all but finitely many $n \in \mathbb{N}$, and hence

$$\limsup \|h(y_n)(f(\varphi(y_n))\|_F \leq (\limsup \|h(y_n)\|) \, (\limsup \|f(\varphi(y_n)\|_E) \, .$$

On the other hand, we get $\|Tf(y)\|_F = \lim \|Tf(y_n)\|_F = \lim \|h(y_n)(f(\varphi(y_n))\|_F$. It follows that $Tf(y) = 0$. Since f is arbitrary in $\mathrm{lip}_0(X^\alpha, E)$ and T is bijective, it follows that $g(y) = 0$ for all $g \in \mathrm{lip}_0(Y^\alpha, F)$ which yields a contradiction and so $Y_1 = Y$. Finally, by using Claims 2 and 7 we have

$$X_1 = \varphi(Y_1) = \varphi(Y) = \varphi(\overline{Y}_1) = \overline{\varphi(Y_1)} = \overline{X}_1 = X.$$

This concludes the proof of Theorem 3.1. □

4 Continuity

In this section, we will study the continuity of linear biseparating maps defined between spaces $\mathrm{lip}_0(X^\alpha, E)$.

Let us recall that a map between metric spaces $f : X \to Y$ is *locally Lipschitz* if each point of X has a neighborhood on which f is Lipschitz. If f is bijective and both f and f^{-1} are locally Lipschitz (Lipschitz), f is said to be a *locally Lipschitz homeomorphism* (respectively, *Lipschitz homeomorphism*). In [27, Theorem 2.1], Scanlon showed that $f : X \to Y$ is locally Lipschitz if and only if f is Lipschitz on each compact subset of X.

Given two Banach spaces E, F, let $B(E, F)$ denote the space of all continuous linear operators $S : E \to F$. We can consider different topologies in $B(E, F)$. It is well known that the *uniform operator topology* (UOT) in $B(E, F)$ is the metric topology induced by the operator canonical norm:

$$\|S\| = \sup \{\|S(e)\|_F : e \in E, \ \|e\|_E \leq 1\} \, .$$

Let us recall that the *strong operator topology* (SOT) in $B(E, F)$ is the topology defined by the basic set of neighborhoods:

$$N(S; A, \varepsilon) = \{R \in B(E, F) : \|(R - S)(e)\|_F < \varepsilon, \ \forall \, e \in A\},$$

where $A \subseteq E$ finite and $\varepsilon > 0$ are arbitrary. Given a topological space Y, it is easy to see that a map h from Y into $(B(E, F), \mathrm{SOT})$ is continuous if and only if for each $e \in E$, the map $y \mapsto h(y)(e)$ from Y to F is continuous. It is evident that the uniform operator topology is stronger than the strong operator topology. In consequence, every continuous map from Y into $(B(E, F), \mathrm{UOT})$ is also continuous as map of Y to $(B(E, F), \mathrm{SOT})$. For a comprehensive study of these topologies, we refer to the book [28] by Dunford and Schwartz. In what follows, we will denote by $L^{-1}(E,F)$ and $B^{-1}(E,F)$ the sets of all linear bijections and continuous linear bijections from E onto F, respectively.

Theorem 4.1 *Let $T : \mathrm{lip}_0(X^\alpha, E) \to \mathrm{lip}_0(Y^\alpha, F)$ be a linear biseparating map, and let $h : Y \to L^{-1}(E,F)$ and $\varphi : Y \to X$ be as in Theorem 3.1. Then T is continuous if and only if $h(y)$ is continuous for all $y \in Y$. In this case, h is a locally Lipschitz map from (Y, d_Y^α) into $B^{-1}(E,F)$ with the metric induced by the operator canonical norm and φ is a locally Lipschitz homeomorphism.*

Proof Assume T is continuous. Let $y \in Y$ and let U be a neighborhood of $\varphi(y)$. By Lemma 2.2 ii), we can take relatively compact neighborhoods V, W of $\varphi(y)$ with $\overline{V} \subseteq W \subseteq \overline{W} \subseteq U$ and a $k \in \mathrm{lip}_{00}(X^\alpha)$ with $\|k\|_\alpha \le a := \max\left\{1/d_X(\overline{V}, X \setminus W), 1\right\} + 1$ such that $k(\varphi(y)) = 1$. Suppose that $h(y)$ is not continuous. Then there exists a sequence $\{e_n\}$ in E with $\|e_n\|_E = 1$ such that $\|h(y)(e_n)\|_F \ge an$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define $f_n : X \to E$ by $f_n(z) = k(z)e_n$. Notice that $f_n \in \mathrm{lip}_{00}(X^\alpha, E)$, $f_n(\varphi(y)) = e_n$ and $\|f_n\|_\alpha \le a$. Then, for every $n \in \mathbb{N}$, we have

$$an \le \|h(y)(e_n)\|_F = \|h(y)(f_n(\varphi(y)))\|_F = \|Tf_n(y)\|_F \le \|Tf_n\|_\alpha \le a\,\|T\|.$$

Hence $n \le \|T\|$ for all $n \in \mathbb{N}$, a contradiction. Therefore $h(y)$ is continuous.

Conversely, assume that $h(y)$ is continuous for all $y \in Y$. To prove the continuity of T, we will use the closed graph theorem. Let $\{f_n\}$ be a sequence in $\mathrm{lip}_0(X^\alpha, E)$ such that $\{\|f_n\|_\alpha\} \to 0$ and $\{\|Tf_n - g\|_\alpha\} \to 0$ for some $g \in \mathrm{lip}_0(Y^\alpha, F)$. We must show that $g = 0$. Pick $y \in Y$. Since $\{\|Tf_n - g\|_\alpha\} \to 0$, it is clear that $\{\|Tf_n(y)\|_F\} \to \|g(y)\|_F$. Moreover, we have

$$\|Tf_n(y)\|_F = \|h(y)(f_n(\varphi(y)))\|_F \le \|h(y)\|\,\|f_n\|_\alpha$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get $g(y) = 0$. This finishes the proof of the first part of the theorem.

Suppose now that $T : \mathrm{lip}_0(X^\alpha, E) \to \mathrm{lip}_0(Y^\alpha, F)$ is continuous. We have seen above that $h : Y \to L^{-1}(E,F)$ takes values in $B^{-1}(E,F)$. We claim that h is locally Lipschitz from (Y, d_Y^α) into $B^{-1}(E,F)$ with the metric:

$$d(R,S) = \sup\left\{\|(R-S)(e)\|_F : e \in E,\ \ \|e\|_E \le 1\right\}.$$

To prove this, it suffices to show that $h|_K$ is Lipschitz for each compact $K \subseteq Y$. Let $U \subseteq X$ be open with $\varphi(K) \subseteq U$. By Lemma 2.2 i), there exist a relatively compact open set V with $\varphi(K) \subseteq V \subseteq \overline{V} \subseteq U$ and a $f \in \mathrm{lip}_{00}(X^\alpha)$ with $\|f\|_\alpha \le b := \max\{1/d_X(\varphi(K), X \setminus V), 1\} + 1$ such that $f|_{\varphi(K)} = 1$.

For each $e \in E$ with $\|e\|_E \le 1$, define $f_e : X \to E$ by $f_e(z) = f(z)e$. Clearly, $f_e \in \mathrm{lip}_{00}(X^\alpha, E)$, $f_e|_{\varphi(K)} = e$ and $\|f_e\|_\alpha \le b$. Given $y, z \in K$, we have

$$\begin{aligned}\|h(y)(e) - h(z)(e)\|_F &= \|h(y)(f_e(\varphi(y))) - h(z)(f_e(\varphi(z)))\|_F \\ &= \|Tf_e(y) - Tf_e(z)\|_F \le \|Tf_e\|_\alpha\, d_Y(y,z)^\alpha \le b\,\|T\|\, d_Y(y,z)^\alpha.\end{aligned}$$

Hence $d(h(y), h(z)) \le b\,\|T\|\, d_Y(y,z)^\alpha$, and this proves our claim. As a consequence, h is continuous from Y into $(B^{-1}(E,F), \mathrm{UOT})$. Then, according to comments preceding the theorem, the map $y \mapsto h(y)(e)$ from Y to F is continuous for each $e \in E$.

Finally, let us verify that $\varphi : Y \to X$ is a locally Lipschitz homeomorphism. We first show that φ is locally Lipschitz. Fix $y \in Y$ and $e \in E$ with $\|e\|_E = 1$. As $z \mapsto h(z)(e)$ from Y into F is continuous and $h(y)(e) \neq 0$, using the local compactness of Y we can find two relatively compact neighborhoods W, G of y with $\overline{W} \subseteq G \subseteq Y$ such that

$$\|h(z)(e) - h(y)(e)\|_F \leq (1/2)\|h(y)(e)\|_F, \quad \forall\, z \in \overline{W}. \tag{4.1}$$

Notice that $\varphi(\overline{W}) \subseteq \varphi(G) \subseteq X$ with $\varphi(\overline{W})$ compact and $\varphi(G)$ open. Then we can construct as in Lemma 2.2 a $k \in \mathrm{lip}_{00}(X^\alpha)$ with $\|k\|_\alpha \leq 1 + \max\left\{1/d_X(\varphi(\overline{W}), X \setminus \varphi(G)), 1\right\}$ such that $k|_{\varphi(\overline{W})} = 1$.

Fix $w, z \in \overline{W}$ with $w \neq z$. Choose a real number $\gamma \in (\alpha, 1)$ and define $h_{w,z} : X \to \mathbb{R}$ by

$$h_{w,z}(u) = [d_X(\varphi(z), u)^\gamma - d_X(\varphi(w), u)^\gamma]/2d_X(\varphi(z), \varphi(w))^{\gamma-\alpha}.$$

It is not hard to check that $h_{w,z} \in \mathrm{lip}(X^\alpha)$ and $\|h_{w,z}\|_\alpha = 1 + (1/2)d_X(\varphi(z), \varphi(w))^\alpha$. Hence $\|h_{w,z}\|_\alpha \leq 1 + (1/2)[\mathrm{diam}(\varphi(\overline{W}))]^\alpha$, where $\mathrm{diam}(\varphi(\overline{W}))$ denotes the diameter of $\varphi(\overline{W})$.

Now define $f_{w,z} : X \to E$ by

$$f_{w,z}(u) = h_{w,z}(u)k(u)e.$$

Clearly, $f_{w,z} \in \mathrm{lip}_{00}(X^\alpha, E)$ and, since $\|\cdot\|_\alpha$ is an algebra norm, it follows that

$$\begin{aligned}\|f_{w,z}\|_\alpha &\leq \|h_{w,z}\|_\alpha \|k\|_\alpha \\ &\leq \left(1 + (1/2)[\mathrm{diam}(\varphi(\overline{W}))]^\alpha\right)\left(1 + \max\left\{1/d_X(\varphi(\overline{W}), X \setminus \varphi(G)), 1\right\}\right).\end{aligned}$$

Hence $\left\{f_{w,z} : w, z \in \overline{W},\ w \neq z\right\}$ is bounded in $\mathrm{lip}_0(X^\alpha, E)$. Since T is assumed to be continuous, it follows that $\left\{Tf_{w,z} : w, z \in \overline{W},\ w \neq z\right\}$ is bounded in $\mathrm{lip}_0(Y^\alpha, F)$. Hence there exists a constant $q > 0$ such that

$$\|Tf_{w,z}\|_\alpha \leq q, \quad \forall\, w, z \in \overline{W},\ w \neq z. \tag{4.2}$$

Given $w, z \in \overline{W}$ with $w \neq z$, from (4.2) it is deduced that

$$\|Tf_{w,z}(w) - Tf_{w,z}(z)\|_F \leq q\ d_Y(w, z)^\alpha. \tag{4.3}$$

An easy calculation yields

$$Tf_{w,z}(w) = h(w)(f_{w,z}(\varphi(w))) = (1/2)d_X(\varphi(z), \varphi(w))^\alpha h(w)(e), \tag{4.4}$$

$$Tf_{w,z}(z) = h(z)(f_{w,z}(\varphi(z))) = -(1/2)d_X(\varphi(z), \varphi(w))^\alpha h(z)(e). \tag{4.5}$$

Substituting (4.4) and (4.5) in (4.3), we infer that

$$(1/2)d_X(\varphi(w), \varphi(z))^\alpha \|h(w)(e) + h(z)(e)\|_F \leq q\ d_Y(w, z)^\alpha. \tag{4.6}$$

Now, notice that $\inf_{z \in \overline{W}} \|h(z)(e)\|_F = \|h(z_0)(e)\|_F > 0$ for some $z_0 \in \overline{W}$ since $\overline{W}$ is compact and $z \mapsto h(z)(e)$ from $\overline{W}$ to F is continuous. Taking into account (4.1), we get

$$\begin{aligned}&\|h(w)(e) + h(z)(e)\|_F \\ &\quad\geq 2\|h(y)(e)\|_F - \|h(w)(e) - h(y)(e)\|_F - \|h(z)(e) - h(y)(e)\|_F \\ &\quad\geq \|h(y)(e)\|_F \geq \|h(z_0)(e)\|_F.\end{aligned} \tag{4.7}$$

Combining (4.6) with (4.7) and writing $2/\|h(z_0)(e)\|_F = p$, we deduce that

$$d_X(\varphi(w), \varphi(z)) \leq (pq)^{1/\alpha}\, d_Y(w, z).$$

Hence φ is Lipschitz on the neighborhood $\overline{W}$ of y and so φ is locally Lipschitz. We can apply similar reasoning to $\phi = \varphi^{-1}$ and see that φ^{-1} is also locally Lipschitz. □

Next we formulate two automatic continuity results. The first one is deduced immediately from Theorems 3.1 and 4.1, and the proof of the second one requires more arguments.

Corollary 4.2 *Let $T : \mathrm{lip}_0(X^\alpha, E) \to \mathrm{lip}_0(Y^\alpha, F)$ be a biseparating linear map and suppose that E or F is finite-dimensional. Then T is continuous.*

Theorem 4.3 *Let $T : \mathrm{lip}_0(X^\alpha, E) \to \mathrm{lip}_0(Y^\alpha, F)$ be a linear biseparating map and suppose that X or Y has no isolated points. Then T is continuous.*

Proof By Theorem 3.1, X and Y are homeomorphic. Hence Y has no isolated points. Let $h : Y \to L^{-1}(E, F)$ be as in Theorem 3.1. We will prove that $h(y)$ is continuous for every $y \in Y$, and then T will be continuous by Theorem 4.1. Let $y \in Y$ be given and assume that $h(y)$ is not continuous. Since y is not isolated, there exists a sequence $\{y_n\}$ of distinct points in $Y \setminus \{y\}$ converging to y. Put $x_n = \varphi(y_n)$ ($n \in \mathbb{N}$) and let $\{U_n\}$ be a sequence of pairwise disjoint neighborhoods of x_n for each $n \in \mathbb{N}$. In view of Lemma 2.2 ii), we can find two sequences $\{V_n\}$ and $\{W_n\}$ of relatively compact neighborhoods of x_n with $\overline{V}_n \subseteq W_n \subseteq \overline{W}_n \subseteq U_n$ for each $n \in \mathbb{N}$, and a sequence $\{k_n\}$ in $\mathrm{lip}_{00}(X^\alpha)$ with $\|k_n\|_\alpha \leq r_n := \max\{1/d_X(\overline{V}_n, X \setminus W_n), 1\} + 1$ such that $k_n(x_n) = 1$ and $\mathrm{coz}(k_n) = W_n$ for every $n \in \mathbb{N}$.

Since $h(y)$ is not continuous, we can take a sequence $\{e_n\}$ in E such that $0 < \|e_n\|_E \leq 1/n^2$ and $\|h(y)(e_n)\|_F > r_n$ for all $n \in \mathbb{N}$. By Lemma 2.2 ii), there are a neighborhood V of $\varphi(y)$ and a function $k \in \mathrm{lip}_{00}(X^\alpha)$ such that $k(x) = 1$ for all $x \in V$. Since $\{x_n\} \to \varphi(y)$, we can suppose, passing to a subsequence if necessary, that $x_n \in V$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, define $g_n : X \to E$ by

$$g_n(z) = k(z)e_n, \quad \forall\, z \in X.$$

Since $g_n \in \mathrm{lip}_{00}(X^\alpha, E)$ and $g_n(\varphi(y)) = e_n$, we have $\|Tg_n(y)\|_F = \|h(y)(g_n(\varphi(y)))\|_F = \|h(y)(e_n)\|_F > r_n$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, the sequence $\{Tg_n(y_m)\}_{m\in\mathbb{N}}$ converges to $Tg_n(y)$. Therefore, for each $n \in \mathbb{N}$, there exists a natural $k(n) > n$ such that $\|Tg_n(y_{k(n)})\|_F > r_n$. Define $\sigma : \mathbb{N} \to \mathbb{N}$ by $\sigma(1) = 1$ and $\sigma(n+1) = k(\sigma(n))$ for all $n \in \mathbb{N}$. Clearly, σ is strictly increasing and $\|Tg_{\sigma(n)}(y_{\sigma(n+1)})\|_F > r_{\sigma(n)}$ for all $n \in \mathbb{N}$. Since $Tg_{\sigma(n)}(y_{\sigma(n+1)}) = h(y_{\sigma(n+1)})(g_{\sigma(n)}(x_{\sigma(n+1)})) = h(y_{\sigma(n+1)})(e_{\sigma(n)})$, it follows that $\|h(y_{\sigma(n+1)})(e_{\sigma(n)})\|_F > r_{\sigma(n)}$ for all $n \in \mathbb{N}$.

Now, for every $n \in \mathbb{N}$, define $f_{\sigma(n)} : X \to E$ by

$$f_{\sigma(1)}(z) = 0, \quad f_{\sigma(n+1)}(z) = (1/r_{\sigma(n)})k_{\sigma(n)}(z)e_{\sigma(n)}, \quad \forall\, z \in X.$$

Clearly, $f_{\sigma(n)} \in \mathrm{lip}_{00}(X^\alpha, E)$, $f_{\sigma(n+1)}(x_{\sigma(n+1)}) = (1/r_{\sigma(n)})e_{\sigma(n)}$ and $f_{\sigma(n)}(\varphi(y)) = 0$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned}\|Tf_{\sigma(n+1)}(y_{\sigma(n+1)})\|_F &= \|h(y_{\sigma(n+1)})(f_{\sigma(n+1)}(x_{\sigma(n+1)}))\|_F \\ &= (1/r_{\sigma(n)})\,\|h(y_{\sigma(n+1)})(e_{\sigma(n)})\|_F > 1\end{aligned}$$

for all $n \in \mathbb{N}$. Moreover, since $\left\|f_{\sigma(n)}\right\|_\alpha \leq 1/n^2$ for all $n \in \mathbb{N}$, the function $f := \sum_{n=1}^{\infty} f_{\sigma(n)}$ belongs to $\mathrm{lip}_0(X^\alpha, E)$. Clearly, $f(\varphi(y)) = 0$ and therefore $Tf(y) = 0$.

On the other hand, as the sets $\mathrm{coz}(f_n) = W_n$ are pairwise disjoint, given any $m \in \mathbb{N}$, we deduce that $f_{\sigma(n)}|_{W_{\sigma(m)}} = 0$ for all $n \neq m$; hence $f_{\sigma(n)} \in I_{x_{\sigma(m)}}$ for all $n \neq m$. This implies that $\sum_{m\neq n=1}^{+\infty} f_{\sigma(n)} \in I_{x_{\sigma(m)}}$ and therefore $T(\sum_{m\neq n=1}^{+\infty} f_{\sigma(n)})(y_{\sigma(m)}) = 0$. Hence $\|Tf(y_{\sigma(m)})\|_F = \|Tf_{\sigma(m)}(y_{\sigma(m)})\|_F > 1$ for all $m > 1$. Letting $m \to \infty$, we arrive at a contradiction. □

In a sense, the next result tells us when there exists a discontinuous linear biseparating map between spaces $\mathrm{lip}_0(X^\alpha, E)$ and how we can construct it explicitly.

Corollary 4.4 *If X, Y are Lipschitz homeomorphic locally compact metric spaces, α is in $(0,1)$ and E is an infinite-dimensional Banach space, then there exists a discontinuous linear biseparating map T from $\mathrm{lip}_0(X^\alpha, E)$ onto $\mathrm{lip}_0(Y^\alpha, E)$ if and only if X or Y has isolated points.*

Proof The proof of the "only if" part follows immediately from Theorem 4.3. To prove the "if" part, since X and Y are homeomorphic, we can assume, without loss of generality, that Y has an isolated point y_0. Then $d_Y(y, y_0) > r$ for all $y \neq y_0$, for some $r > 0$. Take a Lipschitz homeomorphism $\varphi : Y \to X$ and a discontinuous linear bijection $S : E \to E$. Define $T : \mathrm{lip}_0(X^\alpha, E) \to \mathrm{lip}_0(Y^\alpha, E)$ by

$$Tf(y) = f(\varphi(y)) \quad (y \neq y_0), \quad Tf(y_0) = S(f(\varphi(y_0))).$$

First we prove that T is well defined, that is, $Tf \in \mathrm{lip}_0(Y^\alpha, E)$ if $f \in \mathrm{lip}_0(X^\alpha, E)$. Fix $f \in \mathrm{lip}_0(X^\alpha, E)$. A simple verification shows that $Tf \in \mathrm{Lip}(Y^\alpha, E)$. Let $\varepsilon > 0$ be given. We can find a $\delta > 0$ such that $\|f(x) - f(w)\|_E \leq (\varepsilon/L(\varphi)^\alpha) \cdot d_X(x, w)^\alpha$ whenever $d_X(x, w) \leq \delta$, where $L(\varphi)$ denotes the Lipschitz constant of φ. Then $d_Y(y, z) \leq \min\{r, \delta/L(\varphi)\}$ implies $y \neq y_0 \neq z$ and $d_X(\varphi(y), \varphi(z)) \leq \delta$ and, in consequence, we have

$$\|Tf(y) - Tf(z)\|_E = \|f(\varphi(y)) - f(\varphi(z))\|_E \leq \varepsilon \cdot d_Y(y, z)^\alpha.$$

This shows that $Tf \in \mathrm{lip}(Y^\alpha, E)$. To prove that Tf vanishes at infinity, note that as so is f, then there exists a compact $K \subseteq X$ such that $\|f(x)\|_E < \varepsilon$ for all $x \in X \setminus K$. Taking the compact $K \cup \{\varphi(y_0)\}$ if necessary, we can suppose that $\varphi(y_0) \in K$. Obviously, $\varphi^{-1}(K)$ is compact, and if $y \in Y \setminus \varphi^{-1}(K)$, we have $\|Tf(y)\|_E = \|f(\varphi(y))\|_E < \varepsilon$.

Clearly, T is linear biseparating. If it were continuous, we would have

$$\|S(f(\varphi(y_0)))\|_E \leq \|T\| \, \|f\|_\alpha, \quad \forall\, f \in \mathrm{lip}_0(X^\alpha, E).$$

Since $\varphi(y_0)$ is an isolated point of X, there is an $s > 0$ such that $d_X(x, \varphi(y_0)) > s$ for all $x \neq \varphi(y_0)$. For each $e \in E$ define f_e on X by $f_e(x) = 0$ if $x \neq \varphi(y_0)$ and $f_e(\varphi(y_0)) = e$. Clearly, $f_e \in \mathrm{lip}_0(X^\alpha, E)$ and $\|f_e\|_\alpha \leq (1/s^\alpha + 1)\, \|e\|_E$. We have then $\|S(e)\|_E \leq (1/s^\alpha + 1)\, \|T\|\, \|e\|_E$ for all $e \in E$, which contradicts the discontinuity of S. Hence T is not continuous. □

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