



Homomorphisms on real-valued little Lipschitz function spaces

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ARTICLE INFO

Article history:

Received 18 July 2008

Received in revised form 19 December 2008

Accepted 8 January 2009

MSC:

46E40

46E05

46J10

Keywords:

Ring homomorphism

Vector lattice homomorphism

Little Lipschitz function

Hölder function

Banach–Stone theorem

ABSTRACT

We describe the general form of algebra, ring and vector lattice homomorphisms between spaces of real-valued little Lipschitz functions on compact Hölder metric spaces (X, d^α) for $0 < \alpha < 1$.

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1. Introduction

A *ring homomorphism* is a map between rings which preserves both addition and multiplication; an *algebra homomorphism* is a ring homomorphism between algebras which also preserves scalar multiplication; and a *vector lattice homomorphism* is a linear map between vector lattices which preserves the infimum and supremum of any two elements.

Algebra, ring and vector lattice homomorphisms between different types of spaces of Lipschitz functions were first studied by Sherbert [21], Su [23], Luukkainen [15], Weaver [24,25] and Ransford [19] and, more recently, by Arrazola and Bustamante [1] and Garrido and Jaramillo [6,7].

Given a metric space (X, d) , we denote by $\text{Lip}(X, d)$ the real Banach algebra of all bounded Lipschitz functions $f : X \rightarrow \mathbb{R}$ with the norm $\|f\|_d = \|f\|_\infty + L_d(f)$, where

$$L_d(f) = \sup\{|f(x) - f(y)|/d(x, y) : x, y \in X, x \neq y\}.$$

Moreover, we denote by $\text{lip}(X, d)$ the closed subalgebra of $\text{Lip}(X, d)$ formed by all functions f , known as *little Lipschitz functions*, satisfying the property that

$$\forall \varepsilon > 0, \exists \delta > 0: d(x, y) \leq \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon \cdot d(x, y).$$

Besides its vector structure, $\text{Lip}(X, d)$ has a rich lattice structure under the pointwise order; and $\text{lip}(X, d)$ is a sublattice of $\text{Lip}(X, d)$. This order is compatible with the vector structure, and $\text{Lip}(X, d)$ and $\text{lip}(X, d)$ are both vector lattices. Hölder metric spaces (X, d^α) for $0 < \alpha < 1$ are of particular interest in connection with little Lipschitz function spaces.

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These function spaces have been the subject of considerable study. The first works on them are due to de Leeuw [4], Sherbert [22] and Hedberg [9]. They researched into the isometries, the structure of ideals and point derivations, and the Stone–Weierstrass theorem in these spaces. Bade et al. [2] studied the amenability and weak amenability of $\text{lip}(X, d^\alpha)$. The isometries between spaces $\text{lip}(X, d^\alpha)$ were characterized by Mayer-Wolf [16]. The papers [17,18] of Pavlović investigated automatic continuity properties of homomorphisms from algebras $\text{lip}(X, d^\alpha)$. A nice survey on *spaces of Lipschitz and Hölder functions and their applications* is the paper [13] by Kalton, where the structure of these spaces and their preduals on general metric spaces is studied. Precisely, the interest in the spaces $\text{lip}(X, d^\alpha)$ lies especially in the fact that the second dual space of $\text{lip}(X, d^\alpha)$ is isometrically isomorphic to $\text{Lip}(X, d^\alpha)$ (see [2]). We have described also the form of the disjointness preserving operators and the linear bijections preserving the seminorm L_{d^α} between spaces $\text{lip}(X, d^\alpha)$ [10,11]. An interested reader can find a complete information about Lipschitz functions in Weaver's book on *Lipschitz Algebras* [27].

When (X, d_X) and (Y, d_Y) are compact metric spaces and $\alpha, \beta \in (0, 1)$, we gave in [12] the general form of bijective vector lattice homomorphisms from $\text{lip}(X, d_X^\alpha)$ onto $\text{lip}(Y, d_Y^\beta)$. It seems interesting to find representation theorems for such homomorphisms without imposing any additional conditions. In this line, the paper [3] contains a number of Banach–Stone type theorems for not necessarily linear isomorphisms of lattices of real-valued Lipschitz functions. In this note we shall study vector lattice homomorphisms $T : \text{lip}(X, d_X^\alpha) \rightarrow \text{lip}(Y, d_Y^\alpha)$ that are not necessarily bijective. We shall describe the general form of such maps. Roughly speaking, we can divide Y into two subsets. On a unique open subset Y_c , T can be expressed as a weighted composition operator of the form $T(f) = \tau \cdot (f \circ \varphi)$ for a unique nonvanishing positive function $\tau \in \text{lip}(Y_c, d_Y^\alpha)$ and a unique locally Lipschitz map $\varphi : Y_c \rightarrow X$ and, on the closed set $Y \setminus Y_c$, $T(f)$ is the zero map, for every $f \in \text{lip}(X, d_X^\alpha)$.

In order to obtain this description of T , we first solve the problem of determining the general form of nonzero vector lattice homomorphisms from $\text{lip}(X, d_X^\alpha)$ into \mathbb{R} (this means that Y consists of one point only). Namely, we show that each such homomorphism is a positive scalar multiple of an evaluation functional at some point of X . This fact requires to take first a look at the properties of automatic norm-continuity of vector lattice homomorphisms with respect to the Lipschitz norm.

The norm-continuity of such maps will permit us to define the *vector lattice carrier space* $\Delta^l(\text{lip}(X, d))$ of $\text{lip}(X, d)$ as the set of all unital vector lattice homomorphisms from $\text{lip}(X, d)$ into \mathbb{R} , endowed with the metric inherited from the dual Banach space $\text{lip}(X, d)^*$. Let us recall that $T : \text{lip}(X, d) \rightarrow \mathbb{R}$ is *unital* if $T(1_X) = 1$ where 1_X denotes the function which is constantly 1 on X . Then, assumed $\alpha \in (0, 1)$, we show that $\Delta^l(\text{lip}(X, d^\alpha))$ is Lipschitz homeomorphic to (X, d^α) .

It will be proved also that the following classes of functions from $\text{lip}(X, d_X^\alpha)$ into \mathbb{R} coincide: the class of unital vector lattice homomorphisms; the class of ring homomorphisms; and the class of algebra homomorphisms. As a consequence, the same is true for algebra homomorphisms and ring homomorphisms between spaces $\text{lip}(X, d^\alpha)$. Moreover, the general form of such maps $T : \text{lip}(X, d_X^\alpha) \rightarrow \text{lip}(Y, d_Y^\beta)$ is given: on a unique closed and open set $Y_1 \subset Y$, $T(f) = f \circ \varphi$ for a unique Lipschitz map $\varphi : Y_1 \rightarrow X$ and, on $Y \setminus Y_1$, $T(f) = 0$ for every $f \in \text{lip}(X, d_X^\alpha)$.

Our study includes only the case in which vector lattice homomorphisms are defined between spaces of little Lipschitz functions which satisfy a Lipschitz condition of the same order. This condition is not restrictive by the fact that if $0 < \beta < \alpha < 1$, then $f \in \text{lip}(X, d_X^\beta)$ if and only if $f \in \text{lip}(X, d_X^\alpha)$ where $(Z, d_Z) = (X, d_X^{\beta/\alpha})$.

2. Preliminaries

We begin by recalling some concepts on vector lattices. Let $T : E \rightarrow F$ be a linear map between vector lattices. It is said that T is *positive* if $T(x) \geq 0$ whenever $x \geq 0$, and T is *order-preserving* if $T(x) \leq T(y)$ provided $x \leq y$. Notice that T is a vector lattice homomorphism if and only if $T(|x|) = |T(x)|$ for all $x \in E$, where $|x| = x \vee (-x)$; hence T is positive and order-preserving.

In order to prove our results, we first need to confirm the automatic norm-continuity of homomorphisms between vector lattices of little Lipschitz functions.

Lemma 1. *Let (X, d) be a metric space. Every vector lattice homomorphism T from $\text{lip}(X, d)$ into \mathbb{R} is norm-continuous.*

Proof. We argue by contradiction. Assume that T is not norm-continuous. Then there exists a sequence $\{f_n\}$ in $\text{lip}(X, d)$ such that $\|f_n\|_d \leq 1/2^n$ and $T(f_n) \geq n$ for all $n \in \mathbb{N}$. Since $L_d(f_n \vee 0) \leq L_d(f_n)$, $\|f_n \vee 0\|_\infty \leq \|f_n\|_\infty$ and $T(f_n) \leq T(f_n \vee 0)$, replacing $f_n \vee 0$ by f_n if necessary, we can in fact suppose that $f_n \geq 0$ for all $n \in \mathbb{N}$. Since $\text{lip}(X, d)$ is norm-complete, the function $f = \sum_{n=1}^{+\infty} f_n$ belongs to $\text{lip}(X, d)$. Clearly, $f_n \leq f$ for all $n \in \mathbb{N}$, and so $n \leq T(f_n) \leq T(f)$ for all $n \in \mathbb{N}$, which is impossible. \square

For each $y \in Y$, let $\delta_y : \text{lip}(Y, d_Y) \rightarrow \mathbb{R}$ be the *point evaluation functional* defined by $\delta_y(f) = f(y)$ for all $f \in \text{lip}(Y, d_Y)$. If $T : \text{lip}(X, d_X) \rightarrow \text{lip}(Y, d_Y)$ is a vector lattice homomorphism, it is easy to see that for each $y \in Y$, the functional $\delta_y \circ T : \text{lip}(X, d_X) \rightarrow \mathbb{R}$ is also a vector lattice homomorphism, hence it is norm-continuous by Lemma 1. Then an application of the closed graph theorem yields the next result.

Lemma 2. *Let (X, d_X) and (Y, d_Y) be metric spaces. Every vector lattice homomorphism $T : \text{lip}(X, d_X) \rightarrow \text{lip}(Y, d_Y)$ is norm-continuous.*

Proof. Let $\{f_n\}$ be a sequence in $\text{lip}(X, d_X)$ such that $\{f_n\} \rightarrow f \in \text{lip}(X, d_X)$ and $\{T(f_n)\} \rightarrow g \in \text{lip}(Y, d_Y)$. For any $y \in Y$, we see that $\{T(f_n)(y)\} \rightarrow T(f)(y)$ since $\delta_y \circ T$ is continuous, but also $\{T(f_n)(y)\} \rightarrow g(y)$ since convergence in the Lipschitz norm implies pointwise convergence. It follows that $T(f) = g$ and thus it is seen that T has closed graph. Since $\text{lip}(X, d_X)$ and $\text{lip}(Y, d_Y)$ are Banach spaces, we conclude that T is norm-continuous. \square

Our next objective is to characterize nonzero vector lattice homomorphisms from $\text{lip}(X, d^\alpha)$ into \mathbb{R} for $\alpha \in (0, 1)$. This condition on α is imposed to assure that $\text{lip}(X, d^\alpha)$ separates the points of X . If $\alpha \in (0, 1)$, it is easy to see that $\text{Lip}(X, d) \subset \text{lip}(X, d^\alpha)$ and that the family of functions $\{\rho_x: x \in X\}$ given by $\rho_x(z) = d(z, x)$ for all $z \in X$, is contained in $\text{Lip}(X, d)$ and separates the points of X . However, this does not happen in general. For example, equip the real interval $[0, 1]$ with the usual metric $|\cdot|$ and notice that $\text{lip}([0, 1], |\cdot|)$ consists only of constant functions.

In the proof of our main results, we shall use the following lemma which has interest in itself. It is analogous to others stated by Efremovich [5] and Lacruz and Llavona [14] for uniformly continuous functions, and by Sherbert [21] and Garrido and Jaramillo [6,7] for Lipschitz functions.

Lemma 3. Let (X, d_X) and (Y, d_Y) be bounded metric spaces, let $\alpha \in (0, 1)$ and let $\varphi: Y \rightarrow X$. Then φ is Lipschitz if and only if $f \circ \varphi \in \text{lip}(Y, d_Y^\alpha)$ for every $f \in \text{lip}(X, d_X^\alpha)$.

Proof. Assume that φ is Lipschitz. Let $f \in \text{lip}(X, d_X^\alpha)$ and $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that $|f(x) - f(x')| \leq (\varepsilon/(1 + L(\varphi)^\alpha) \cdot d_X(x, x')^\alpha$ whenever $d_X(x, x') \leq \delta_1$ where $L(\varphi)$ denotes the Lipschitz constant of φ . Choose $0 < \delta_2 < \delta_1/(1 + L(\varphi))$. If $d_Y(y, y') \leq \delta_2$, then $d_X(\varphi(y), \varphi(y')) \leq \delta_1$ and thus

$$|f(\varphi(y)) - f(\varphi(y'))| \leq (\varepsilon/(1 + L(\varphi)^\alpha) \cdot d_X(\varphi(y), \varphi(y'))^\alpha \leq \varepsilon \cdot d_Y(y, y')^\alpha.$$

Hence $f \circ \varphi \in \text{lip}(Y, d_Y^\alpha)$.

Conversely, φ induces the vector lattice homomorphism T from $\text{lip}(X, d_X^\alpha)$ into $\text{lip}(Y, d_Y^\alpha)$ defined by $T(f) = f \circ \varphi$. For $y, z \in Y$ with $\varphi(y) \neq \varphi(z)$, define $f_{yz}: X \rightarrow \mathbb{R}$ by

$$f_{yz}(x) = \max\{2d_X(\varphi(y), x)^\alpha - d_X(\varphi(y), \varphi(z))^\alpha, 0\} \quad (x \in X).$$

It is plain that $f_{yz} \in \text{Lip}(X, d_X^\alpha)$ and $\|f_{yz}\|_{d_X^\alpha} \leq 2(1 + \text{diam}(X)^\alpha)$, where $\text{diam}(X)$ denotes the diameter of X . To see that $f_{yz} \in \text{lip}(X, d_X^\alpha)$, it is sufficient to show that $f_{yz} \in \text{Lip}(X, d_X)$. In order to obtain this, define $\rho_y: X \rightarrow \mathbb{R}$ by $\rho_y(x) = d(\varphi(y), x)$ and $h_{yz}: [0, \text{diam}(X)] \rightarrow \mathbb{R}$ by

$$h_{yz}(t) = \max\{2t^\alpha - d_X(\varphi(y), \varphi(z))^\alpha, 0\}.$$

It is easy to check that h_{yz} is differentiable at $[0, \text{diam}(X)] \setminus \{t_0\}$ with bounded derivative by $2\alpha t_0^{\alpha-1}$, where $t_0 = (1/2)^{1/\alpha} d(\varphi(y), \varphi(z))$. Let $t, s \in [0, \text{diam}(X)]$ with $t \neq s$. If $t, s \in [0, t_0]$, we have $h_{yz}(t) = h_{yz}(s) = 0$. If $t, s \in (t_0, \text{diam}(X)]$,

$$|h_{yz}(t) - h_{yz}(s)| = |h'_{yz}(s_0)| |t - s| \leq 2\alpha t_0^{\alpha-1} |t - s|$$

where s_0 is between t and s by using the mean value theorem. If $t \leq t_0 < s$ (or $s \leq t_0 < t$), we check that

$$|h_{yz}(t) - h_{yz}(s)| = 2(s^\alpha - t_0^\alpha) \leq 2\alpha t_0^{\alpha-1}(s - t_0) \leq 2\alpha t_0^{\alpha-1} |t - s|.$$

Hence $h_{yz} \in \text{Lip}([0, \text{diam}(X)], |\cdot|)$. Moreover, $\rho_y \in \text{Lip}(X, d_X)$. Since $f_{yz} = h_{yz} \circ \rho_y$, we obtain the desired conclusion.

Since $\{f_{yz}: y, z \in Y, \varphi(y) \neq \varphi(z)\}$ is bounded in $\text{lip}(X, d_X^\alpha)$ and the linear map T is norm-continuous by Lemma 2, $\{T(f_{yz}): y, z \in Y, \varphi(y) \neq \varphi(z)\}$ is bounded in $\text{lip}(Y, d_Y^\alpha)$. Hence there is a constant $c > 0$ such that

$$|T(f_{yz})(y) - T(f_{yz})(z)| \leq cd_Y^\alpha(y, z), \quad \forall y, z \in Y.$$

Given $y, z \in Y$, a simple calculation yields

$$T(f_{yz})(y) = f_{yz}(\varphi(y)) = 0, \quad T(f_{yz})(z) = f_{yz}(\varphi(z)) = d_X(\varphi(y), \varphi(z))^\alpha,$$

and thus $d_X(\varphi(y), \varphi(z)) \leq c^{1/\alpha} d_Y(y, z)$. \square

The “if” part of Lemma 3 does not hold for $\alpha = 1$. For example, define $\varphi(x) = \sqrt{x}$ for all $x \in [0, 1]$. It is immediate that $f \circ \varphi \in \text{lip}([0, 1], |\cdot|)$ for each $f \in \text{lip}([0, 1], |\cdot|)$ since this space has only constant functions, and however φ is not Lipschitz.

3. Main results

We first give a complete description of vector lattice homomorphisms from $\text{lip}(X, d^\alpha)$ to \mathbb{R} .

Theorem 4. Let (X, d) be a compact metric space, let $\alpha \in (0, 1)$ and let $T: \text{lip}(X, d^\alpha) \rightarrow \mathbb{R}$ be a nonzero map. Then T is a vector lattice homomorphism if and only if $T = cd_X$ for a unique real number $c > 0$ and a unique point $x \in X$.

Proof. The “if” part is straightforward. To prove the “only if” part, suppose T is a vector lattice homomorphism. We first show that there is a point $a \in X$ such that $T(f) = 0$ for all $f \in \text{lip}(X, d^\alpha)$ which vanishes on a neighborhood of a . If one assumes that such a point a does not exist, then for each $x \in X$ there are an $f_x \in \text{lip}(X, d^\alpha)$ and a neighborhood U_x of x such that $f_x(U_x) = \{0\}$ and $T(f_x) \neq 0$. By the compactness of X , it follows that $X = \bigcup_{i=1}^n U_{x_i}$ for some $x_1, \dots, x_n \in X$. Let $f = \bigwedge_{i=1}^n |f_{x_i}|$. Since $f \in \text{lip}(X, d^\alpha)$ and T is a vector lattice homomorphism, we have $T(f) = \bigwedge_{i=1}^n T(|f_{x_i}|) = \bigwedge_{i=1}^n |T(f_{x_i})| > 0$; but also it is clear that $f = 0$, hence $T(f) = 0$ and so we arrive at a contradiction.

We now see that $T(f) = 0$ for all $f \in \text{lip}(X, d^\alpha)$ such that $f(a) = 0$. If f is such a function, according to [22, Lemma 4.1] there exists a sequence $\{f_n\}$ in $\text{lip}(X, d^\alpha)$ satisfying: (i) $f_n = f$ in a neighborhood of a ; and (ii) $\{\|f_n\|_{d^\alpha}\} \rightarrow 0$. Then $T(f_n) = T(f)$ for all $n \in \mathbb{N}$ by proved above, and $\{T(f_n)\} \rightarrow 0$ by the continuity of T . Hence $T(f) = 0$.

We claim that T is a strictly positive scalar multiple of δ_a . For each $f \in \text{lip}(X, d^\alpha)$, let $k_f : X \rightarrow \mathbb{R}$ be defined by $k_f = f - f(a)1_X$. Obviously, $k_f \in \text{lip}(X, d^\alpha)$ and $k_f(a) = 0$. Hence $T(k_f) = 0$, that is, $T(f) = f(a)T(1_X)$. Let $c = T(1_X)$ and thus $T = c\delta_a$. Since T is a nonzero positive functional, it follows that $c > 0$ and this proves our claim.

To prove the uniqueness of c and a , suppose that there exist $c' > 0$ and $a' \in X$ such that $T = c'\delta_{a'}$. A simple calculation gives $c = c' = T(1_X)$. Hence $\delta_a = \delta_{a'}$, which implies $a = a'$ since $\text{lip}(X, d^\alpha)$ for $\alpha \in (0, 1)$ separates the points of X . \square

Next we characterize unital vector lattice homomorphisms from $\text{lip}(X, d^\alpha)$ to \mathbb{R} .

Theorem 5. Let (X, d) be a compact metric space, let $\alpha \in (0, 1)$ and let $T : \text{lip}(X, d^\alpha) \rightarrow \mathbb{R}$ be a nonzero map. Then the following conditions are equivalent:

- (1) T is a unital vector lattice homomorphism.
- (2) There exists a unique $x \in X$ such that $T(f) = f(x)$ for each $f \in \text{lip}(X, d^\alpha)$.
- (3) T is a ring homomorphism.
- (4) T is an algebra homomorphism.

Proof. (1) \Rightarrow (2): Assume that T is a unital vector lattice homomorphism. According to Theorem 4, $T = c\delta_x$ for a unique scalar $c > 0$ and a unique $x \in X$. Since T is unital, $c = 1$; hence $T = \delta_x$ and thus $T(f) = f(x)$ for all $f \in \text{lip}(X, d^\alpha)$.

(2) \Rightarrow (3): The proof is straightforward.

(3) \Rightarrow (4): We follow [15, Lemma 2.2]. Suppose that T is a ring homomorphism. Let $f \in \text{lip}(X, d^\alpha)$ and $\lambda \in \mathbb{R}$. If $T(f) \neq 0$, define $\xi : \mathbb{R} \rightarrow \mathbb{R}$ by $\xi(t) = T(tf)/T(f)$. An easy calculation shows that ξ is a nonzero ring homomorphism. Hence ξ is the identity [8, 0.22]. Thus $T(\lambda f) = \lambda T(f)$. On the other hand, if $T(f) = 0$, then $T(\lambda f) = 0$ since, otherwise, we would have $\lambda \neq 0$ and $T(f) = T(\lambda^{-1}(\lambda f)) = \lambda^{-1}T(\lambda f) \neq 0$ by proved above. Hence $T(\lambda f) = 0 = \lambda T(f)$.

(4) \Rightarrow (1): Assume that T is an algebra homomorphism. As $T(1_X) = T(1_X)^2$, we have $T(1_X) = 1$ since otherwise $T(f) = T(f)T(1_X) = 0$ for all $f \in \text{lip}(X, d^\alpha)$. We claim that T is positive. Indeed, suppose there exists $f \in \text{lip}(X, d^\alpha)$ with $f \geq 0$ and $T(f) < 0$. Define $g = f - T(f)1_X$. Clearly, $g \in \text{lip}(X, d^\alpha)$ and $g \geq -T(f)1_X > 0$. It is well known (see [27, Proposition 3.1.4]) that if (X, d) is a compact metric space, then $\text{lip}(X, d)$ is inverse-closed, that is, if $f \in \text{lip}(X, d)$ and $f(x) \neq 0$ for all $x \in X$, then $1_X/f \in \text{lip}(X, d)$. Hence $1_X/g \in \text{lip}(X, d^\alpha)$. Then $T(1_X) = T(g)T(1_X/g) = 0$, a contradiction, and this proves the claim. For every $f \in \text{lip}(X, d^\alpha)$, we have $T(|f|)^2 = T(|f|^2) = T(f^2) = T(f)^2$, and since $T(|f|) \geq 0$, we obtain $T(|f|) = |T(f)|$. Hence T is a vector lattice homomorphism. \square

As we have said in the Introduction, $\Delta^l(\text{lip}(X, d))$ denotes the set of all unital vector lattice homomorphisms from $\text{lip}(X, d)$ into \mathbb{R} , endowed with the metric inherited from the dual Banach space $\text{lip}(X, d)^*$. Let us recall that a map between metric spaces $\varphi : X \rightarrow Y$ is a Lipschitz homeomorphism if it is bijective and both φ and φ^{-1} are Lipschitz.

Theorem 6. Let (X, d) be a compact metric space and let $\alpha \in (0, 1)$. Then the map $x \mapsto \delta_x$ is a Lipschitz homeomorphism from (X, d^α) onto $\Delta^l(\text{lip}(X, d^\alpha))$ satisfying $\|\delta_x - \delta_y\| \leq d(x, y)^\alpha \leq ((1/2) \text{diam}(X)^\alpha + 1) \cdot \|\delta_x - \delta_y\|$ for all $x, y \in X$.

Proof. For each $x \in X$, it is clear that $\delta_x : \text{lip}(X, d^\alpha) \rightarrow \mathbb{R}$ is a unital vector lattice homomorphism. Conversely, Theorem 4 shows that every element of $\Delta^l(\text{lip}(X, d^\alpha))$ is of this form. Therefore, the map $x \mapsto \delta_x$ from X to $\Delta^l(\text{lip}(X, d^\alpha))$ is surjective. Moreover, this map is injective since $\text{lip}(X, d^\alpha)$ separates the points of X .

Next we see that this bijection is a Lipschitz homeomorphism. Fix $x, y \in X$ with $x \neq y$. For any $f \in \text{lip}(X, d^\alpha)$, it is immediate that

$$|\delta_x(f) - \delta_y(f)| = |f(x) - f(y)| \leq \|f\|_{d^\alpha} d(x, y)^\alpha,$$

and thus $\|\delta_x - \delta_y\| \leq d(x, y)^\alpha$. Conversely, we show that $d(x, y)^\alpha \leq k\|\delta_x - \delta_y\|$ where $k = (1/2) \text{diam}(X)^\alpha + 1$. Indeed, choose $\beta \in (\alpha, 1)$ and define

$$f_{xy}(z) = \frac{d(z, y)^\beta - d(z, x)^\beta}{2d(x, y)^{\beta-\alpha}}, \quad \forall z \in X.$$

A simple calculation gives $f_{xy}(x) - f_{xy}(y) = d(x, y)^\alpha$. It is not hard to check that $f_{xy} \in \text{lip}(X, d^\alpha)$ and $\|f_{xy}\|_{d^\alpha} = (1/2)d(x, y)^\alpha + 1$ (see, for example, [12]). It follows that $d(x, y)^\alpha = \delta_x(f_{xy}) - \delta_y(f_{xy}) \leq k\|\delta_x - \delta_y\|$, which was the inequality desired. \square

The general form of vector lattice homomorphisms between spaces $\text{lip}(X, d_X^\alpha)$ is given next. Let us recall that a map between metric spaces $\varphi : X \rightarrow Y$ is *locally Lipschitz* if for every $x \in X$, there exists a neighborhood U_x of x such that φ is Lipschitz on U_x . In [20, Theorem 2.1], Scanlon showed that $\varphi : X \rightarrow Y$ is locally Lipschitz if and only if φ is Lipschitz on each compact subset of X .

Theorem 7. *Let (X, d_X) and (Y, d_Y) be compact metric spaces, let $\alpha \in (0, 1)$ and let T be a nonzero vector lattice homomorphism from $\text{lip}(X, d_X^\alpha)$ into $\text{lip}(Y, d_Y^\alpha)$. Then there exist a unique nonempty open set $Y_c \subset Y$, a unique nonvanishing positive function $\tau \in \text{lip}(Y_c, d_Y^\alpha)$ and a unique locally Lipschitz map $\varphi : Y_c \rightarrow X$ such that, for all $f \in \text{lip}(X, d_X^\alpha)$,*

$$T(f)(y) = \begin{cases} \tau(y)f(\varphi(y)) & \text{if } y \in Y_c, \\ 0 & \text{if } y \in Y \setminus Y_c. \end{cases}$$

Proof. Define $Y_c = \{y \in Y : \delta_y \circ T \neq 0\}$. Since T is nonzero, then Y_c is nonempty. For each $f \in \text{lip}(X, d_X^\alpha)$, $T(f)$ is a continuous function on Y ; hence $T(f)^{-1}(\{0\})$ is closed in Y and so is also $Y \setminus Y_c = \bigcap_{f \in \text{lip}(X, d_X^\alpha)} T(f)^{-1}(\{0\})$. In others words, Y_c is open in Y . For each $y \in Y_c$, $\delta_y \circ T : \text{lip}(X, d_X^\alpha) \rightarrow \mathbb{R}$ is a nonzero vector lattice homomorphism. Then, according to Theorem 4, there exist a unique real number $\tau(y) > 0$ and a unique point $\varphi(y) \in X$ such that $\delta_y \circ T = \tau(y)\delta_{\varphi(y)}$. This defines the maps $\tau : Y_c \rightarrow (0, \infty)$ and $\varphi : Y_c \rightarrow X$. Given any $f \in \text{lip}(X, d_X^\alpha)$, we have $T(f)(y) = \tau(y)f(\varphi(y))$ for every $y \in Y_c$, and $T(f)(y) = 0$ for every $y \in Y \setminus Y_c$. It follows that τ is the restriction function of $T(1_X)$ on Y_c and thus τ belongs to $\text{lip}(Y_c, d_Y^\alpha)$. Next we see that φ is continuous. Let $y \in Y_c$ and let $\{y_n\}$ be a sequence in Y_c converging to y . Since τ is continuous, $\{\tau(y_n)\} \rightarrow \tau(y)$. For the same reason, $\{T(f)(y_n)\} \rightarrow T(f)(y)$ for each $f \in \text{lip}(X, d_X^\alpha)$. In summary, $\{f(\varphi(y_n))\} \rightarrow f(\varphi(y))$ for all $f \in \text{lip}(X, d_X^\alpha)$. Take $f \in \text{lip}(X, d_X^\alpha)$ given by $f(z) = d(z, \varphi(y))$ for all $z \in X$. It follows that $\{d(\varphi(y_n), \varphi(y))\} \rightarrow 0$, that is, $\{\varphi(y_n)\} \rightarrow \varphi(y)$, and thus φ is continuous. To prove that φ is locally Lipschitz, let $K \subset Y_c$ be nonempty compact. Since φ is continuous, $\varphi(K) \subset X$ is compact. Each $f \in \text{lip}(\varphi(K), d_X^\alpha)$ has an extension $\tilde{f} \in \text{lip}(X, d_X^\alpha)$ according to [26, Theorem 4]. We know that $\text{lip}(K, d_Y^\alpha)$ is inverse-closed. Since $f \circ \varphi|_K = (\tilde{f} \circ \varphi)|_K = \frac{T(\tilde{f})}{\tau}|_K \in \text{lip}(K, d_Y^\alpha)$, we conclude that $\varphi|_K$ is Lipschitz by Lemma 3.

To prove the uniqueness, assume there are a nonempty open set $Y'_c \subset Y$, a nonvanishing positive function $\tau' \in \text{lip}(Y'_c, d_Y^\alpha)$ and a locally Lipschitz map $\varphi' : Y'_c \rightarrow X$ such that $T(f)(y) = \tau'(y)f(\varphi'(y))$ if $y \in Y'_c$ and $T(f)(y) = 0$ if $y \in Y \setminus Y'_c$, for all $f \in \text{lip}(X, d_X^\alpha)$. If $y \in Y \setminus Y'_c$, it is clear that $\delta_y \circ T = 0$; and, conversely, if $\delta_y \circ T = 0$ for some $y \in Y$, then $T(f)(y) = 0$ for each $f \in \text{lip}(X, d_X^\alpha)$, hence $T(1_X)(y) = 0$ which implies that $y \in Y \setminus Y'_c$ since, otherwise, we would have $\tau'(y) = 0$, which is impossible. Hence $Y \setminus Y'_c = \{y \in Y : \delta_y \circ T = 0\}$ and thus $Y'_c = Y_c$. Then $\tau' = T(1_X)|_{Y'_c} = \tau$ and therefore, given any $y \in Y_c$, we have $f(\varphi'(y)) = f(\varphi(y))$ for all $f \in \text{lip}(X, d_X^\alpha)$. Since $\text{lip}(X, d_X^\alpha)$ separates the points of X , we infer that $\varphi'(y) = \varphi(y)$ and so $\varphi' = \varphi$. \square

Finally, we characterize ring homomorphisms between spaces $\text{lip}(X, d_X^\alpha)$. Notice that, as a consequence from Theorem 5, such a homomorphism is an algebra homomorphism.

Theorem 8. *Let (X, d_X) and (Y, d_Y) be compact metric spaces and let $\alpha \in (0, 1)$. A map $T : \text{lip}(X, d_X^\alpha) \rightarrow \text{lip}(Y, d_Y^\alpha)$ is a ring homomorphism if and only if*

$$T(f)(y) = \begin{cases} f(\varphi(y)) & \text{if } y \in Y_1, \\ 0 & \text{if } y \in Y \setminus Y_1, \end{cases}$$

for all $f \in \text{lip}(X, d_X^\alpha)$, where Y_1 is a unique open and closed subset of Y with $d_Y(Y_1, Y \setminus Y_1) > 0$ if $Y_1 \neq \emptyset \neq Y \setminus Y_1$ and $\varphi : Y_1 \rightarrow X$ is a unique Lipschitz map.

Proof. Assume first that T is a ring homomorphism. Note that $T(1_X) = T(1_X)^2$. It follows that $Y_1 = T(1_X)^{-1}(\{1\})$ and $Y \setminus Y_1 = T(1_X)^{-1}(\{0\})$ are open and closed subsets of Y . Assume $Y_1 \neq \emptyset \neq Y \setminus Y_1$. Since Y_1 and $Y \setminus Y_1$ are disjoint compact, we have $d_Y(Y_1, Y \setminus Y_1) > 0$. Since $T(1_X)|_{Y \setminus Y_1} = 0$, given any $f \in \text{lip}(X, d_X^\alpha)$, we have $T(f)(y) = 0$ for all $y \in Y \setminus Y_1$. Let $T_1 : \text{lip}(X, d_X^\alpha) \rightarrow \text{lip}(Y_1, d_Y^\alpha)$ be defined by $T_1(f) = T(f)|_{Y_1}$. Clearly, T_1 is a unital ring homomorphism. Then, for each $y \in Y_1$, $\delta_y \circ T_1$ is a unital ring homomorphism from $\text{lip}(X, d_X^\alpha)$ into \mathbb{R} . According to Theorem 5, there exists a unique $x \in X$ such that, for all $f \in \text{lip}(X, d_X^\alpha)$, we have $\delta_y \circ T_1(f) = f(x)$, that is, $T(f)(y) = f(x)$. Define $\varphi : Y_1 \rightarrow X$ by $\varphi(y) = x$ and thus $T(f)(y) = f(\varphi(y))$. Hence $f \circ \varphi = T_1(f) \in \text{lip}(Y_1, d_Y^\alpha)$ for all $f \in \text{lip}(X, d_X^\alpha)$, and therefore φ is Lipschitz by Lemma 3. The uniqueness of Y_1 and φ is proved easily.

To prove the converse statement assume that a map T is defined by

$$T(f)(y) = \begin{cases} f(\varphi(y)) & \text{if } y \in Y_1, \\ 0 & \text{if } y \in Y \setminus Y_1, \end{cases}$$

for all $f \in \text{lip}(X, d_X^\alpha)$ and that the set Y_1 and the map φ satisfy all the requirements of Theorem 8. Then T defines a map from $\text{lip}(X, d_X^\alpha)$ into $\text{lip}(Y, d_Y^\alpha)$. Indeed, let $f \in \text{lip}(X, d_X^\alpha)$ and $\varepsilon > 0$. Then $f \circ \varphi \in \text{lip}(Y_1, d_Y^\alpha)$ by Lemma 3 and, consequently, there exists a $\delta_1 > 0$ such that $|f(\varphi(y)) - f(\varphi(y'))| \leq \varepsilon \cdot d_Y(y, y')^\alpha$ whenever $y, y' \in Y_1$ and $d_Y(y, y') \leq \delta_1$. Denote $a = d_Y(Y_1, Y \setminus Y_1)$ and take $\delta_2 = \min\{\delta_1, (\varepsilon a / \|f\|_\infty)^{1/(1-\alpha)}\}$. Let $y, y' \in Y$ with $d_Y(y, y') \leq \delta_2$. If $y, y' \in Y_1$, we have

$$|T(f)(y) - T(f)(y')| = |f(\varphi(y)) - f(\varphi(y'))| \leq \varepsilon \cdot d_Y(y, y')^\alpha;$$

if $y, y' \in Y \setminus Y_1$, it is immediate that

$$|T(f)(y) - T(f)(y')| = 0 \leq \varepsilon \cdot d_Y(y, y')^\alpha;$$

and for $y \in Y_1$ and $y' \in Y \setminus Y_1$, we conclude that

$$\begin{aligned} |T(f)(y) - T(f)(y')| &= |f(\varphi(y))| \leq \frac{\|f\|_\infty}{d_Y(y, y')} d_Y(y, y')^{1-\alpha} d_Y(y, y')^\alpha \\ &\leq \frac{\|f\|_\infty}{a} (\delta_2)^{1-\alpha} d_Y(y, y')^\alpha \leq \varepsilon \cdot d_Y(y, y')^\alpha. \end{aligned}$$

Hence $T(f) \in \text{lip}(Y, d_Y^\alpha)$. Clearly, T preserves both addition and multiplication. \square

Acknowledgements

The authors thank the referee for comments which have improved this paper. They also acknowledge the support from Junta de Andalucía grants P06-FQM-01215 and P06-FQM-01438, and M.E.C. grant MTM2006-04837.

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