Proc. Indian Acad. Sci. (Math. Sci.) Vol. 119, No. 1, February 2009, pp. 53–62.

Hölder seminorm preserving linear bijections and isometries

A JIMÉNEZ-VARGAS and M A NAVARRO

Departamento de Álgebra y Análisis Matemático, Universidad de Almería, 04120, Almería, Spain
E-mail: ajimenez@ual.es; manav@ual.es

MS received 21 March 2007; revised 26 July 2008

Abstract. Let (X, d) be a compact metric and $0 < \alpha < 1$. The space $\mathrm{Lip}^\alpha(X)$ of Hölder functions of order α is the Banach space of all functions f from X into $\mathbb{K}$ such that $\|f\| = \max\{\|f\|_\infty, L(f)\} < \infty$, where

$$L(f) = \sup\{|f(x) - f(y)|/d^\alpha(x, y): x, y \in X,\ x \neq y\}$$

is the Hölder seminorm of f. The closed subspace of functions f such that

$$\lim_{d(x,y)\to 0} |f(x) - f(y)|/d^\alpha(x, y) = 0$$

is denoted by $\mathrm{lip}^\alpha(X)$. We determine the form of all bijective linear maps from $\mathrm{lip}^\alpha(X)$ onto $\mathrm{lip}^\alpha(Y)$ that preserve the Hölder seminorm.

Keywords. Lipschitz function; isometry; linear preserver problem; Banach–Stone theorem.

1. Introduction

The main linear preserver problems on function algebras concern the characterizations of linear bijections preserving some given norm. The classical Banach–Stone theorem determines all linear bijections preserving the supremum norm of the space $\mathcal{C}(X)$ of all scalar-valued continuous functions on a compact Hausdorff space X. This result has found a large number of extensions, generalizations and variants in many different contexts (see the survey paper [5]).

Let (X, d) be a compact metric space and $0 < \alpha < 1$. Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$. The space $\mathrm{Lip}^\alpha(X)$ of Hölder functions of order α is the Banach space of all functions f from X into $\mathbb{K}$ such that $\|f\| = \max\{\|f\|_\infty, L(f)\} < \infty$, where

$$\|f\|_\infty = \sup\{|f(x)|: x \in X\}$$

is the supremum norm of f, and

$$L(f) = \sup\{|f(x) - f(y)|/d^\alpha(x, y): x, y \in X,\ x \neq y\}$$

is the Hölder seminorm of f. The space $\mathrm{lip}^\alpha(X)$ is the subspace of $\mathrm{Lip}^\alpha(X)$ consisting of all those functions f with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \varepsilon\, d^\alpha(x, y)$. These spaces have been studied extensively (see the survey book [10]).

It is easy to prove that $\mathrm{lip}^\alpha(X)$ is a closed subspace of $\mathrm{Lip}^\alpha(X)$ containing the constant functions. This is also true when $\alpha = 1$, but, in this case, it need not in general contain anything but constant functions. For example, if $X = [0, 1]$ and d is the usual metric, then $f \in \mathrm{lip}^1(X)$ implies that $f'(x) = 0$ for all $x \in [0, 1]$, so that f is constant. To be assured of an abundance of nonconstant functions, we shall concern with the spaces $\mathrm{lip}^\alpha(X)$ for $0 < \alpha < 1$, since $\mathrm{Lip}^1(X) \subset \mathrm{lip}^\alpha(X)$ for each $\alpha \in (0, 1)$ and the family of functions $\{f_x : x \in X\} \subset \mathrm{Lip}^1(X)$, where

$$f_x(t) = d(t, x), \quad \forall t \in X,$$

separates the points of X.

Besides the sup-norm, one of the most natural possibilities to measure a function $f \in \mathrm{Lip}^\alpha(X)$ is to consider the value $L(f)$, which provides a measure of the maximum rate of change of f over X. However the function L defines only a seminorm, not a norm, on $\mathrm{Lip}^\alpha(X)$, since L applied to any constant function on X is zero.

In recent years, another line of research motivated by the Banach–Stone theorem has been to study the linear bijections of $\mathcal{C}(X)$ which leave invariant the diameter seminorm:

$$\mathrm{diam}(f) = \sup\{|f(x) - f(y)| : x, y \in X\}, \quad \forall f \in \mathcal{C}(X).$$

Győry and Molnár [7] obtained the general form of these bijections when X is a first countable compact Hausdorff space. Later, Cabello [2] and González and Uspenskij [6] got similar results for arbitrary compact X. Several papers on diameter preserving linear bijections of function spaces have appeared since then [1, 3, 9].

In the context of Lipschitz functions, it seems more natural to raise this type of linear preserver problems with the seminorm L instead of the diameter seminorm. The aim of this paper is to determine all linear bijections T from $\mathrm{lip}^\alpha(X)$ onto $\mathrm{lip}^\alpha(Y)$ preserving the Hölder seminorm $f \mapsto L(f)$, that is, satisfying that

$$L(T(f)) = L(f), \quad \forall f \in \mathrm{lip}^\alpha(X).$$

For simplicity, we shall say that these mappings are Hölder seminorm preserving.

We must point out that the same technique of proof has been used by the first author in [8] to study the linear bijections of $\mathrm{Lip}^\alpha(X)$ preserving the Hölder seminorm. Namely, it was proved in Theorem 2.1 of [8] that such a map $T : \mathrm{Lip}^\alpha(X) \to \mathrm{Lip}^\alpha(X)$ is of the form $T(f) = \tau f \circ \varphi + \mu(f)1_X$ for every $f \in \mathrm{Lip}^\alpha(X)$, where $\tau \in \mathbb{K}$ with $|\tau| = 1$, $\varphi : X \to X$ is a surjective isometry and $\mu : \mathrm{Lip}^\alpha(X) \to \mathbb{K}$ is a linear functional with $\mu(1_X) \neq -\tau$. As usual, 1_X denotes the function constantly 1 on X.

2. Statement of the results and proofs

We follow some notation and terminology of [10]. U denotes the set of all unimodular scalars of $\mathbb{K}$. Define $U^+ = \{1\}$ if $\mathbb{K} = \mathbb{R}$ and $U^+ = \{e^{it} : t \in [0, \pi[\}$ if $\mathbb{K} = \mathbb{C}$. For any Banach space E, we write $B(E)$ for its (closed) unit ball. We shall use the letter d to denote the distance in any metric space.

Let $\varphi : Y \to X$ be a map between metric spaces. If for some $k > 0$, we have $d(\varphi(x), \varphi(y)) = k\, d(x, y)$ for all $x, y \in Y$, we say that φ is a k-dilation.

If $\varphi : Y \to X$ is a surjective k-dilation for some $k > 0$ and $\tau \in U$, then the map $f \mapsto (\tau/k^\alpha) f \circ \varphi$ from $\mathrm{lip}^\alpha(X)$ to $\mathrm{lip}^\alpha(Y)$ is Hölder seminorm preserving. Since L is a seminorm whose kernel consists of the constant functions, then the linear map of the form

$$T(f) = (\tau/k^\alpha) f \circ \varphi + \mu(f) 1_Y, \quad \forall f \in \mathrm{lip}^\alpha(X),$$

where μ: $\mathrm{lip}^\alpha(X) \to \mathbb{K}$ is a linear functional and 1_Y is the function constantly 1 on Y, preserves the Hölder seminorm of functions in $\mathrm{lip}^\alpha(X)$. The following theorem says that every Hölder seminorm preserving linear bijection from $\mathrm{lip}^\alpha(X)$ onto $\mathrm{lip}^\alpha(Y)$ has the aforementioned form.

Theorem 2.1. *Let X and Y be compact metric spaces and* $0 < \alpha < 1$. *A linear bijection T: $\mathrm{lip}^\alpha(X) \to \mathrm{lip}^\alpha(Y)$ is Hölder seminorm preserving if and only if there exists a number $\tau \in U$, a bijective k-dilation φ: $Y \to X$ for some $k > 0$ and a linear functional μ: $\mathrm{lip}^\alpha(X) \to \mathbb{K}$ with $\mu(1_X) \neq -\tau/k^\alpha$ such that $T(f) = (\tau/k^\alpha) f \circ \varphi + \mu(f) 1_Y$ for every $f \in \mathrm{lip}^\alpha(X)$.*

The technique employed in this paper to prove Theorem 2.1 is an adaptation to spaces of Lipschitz functions of an ingenious idea of Félix Cabello Sánchez, used in [2] to study the linear bijections of $\mathcal{C}(X)$ preserving the diameter seminorm. He bases the study of these linear bijections on the analysis of the isometry group of the space $\mathcal{C}(X)$ modulo the set of constant functions. We shall proceed in a similar form with the space $\mathrm{lip}^\alpha(X)$.

Let $\mathrm{lip}^\alpha_L(X)$ denote the quotient of the space $\mathrm{lip}^\alpha(X)$ by the kernel of L and let π: $\mathrm{lip}^\alpha(X) \to \mathrm{lip}^\alpha_L(X)$ be the natural quotient map. Clearly, $\mathrm{lip}^\alpha_L(X)$ is a Banach space endowed with the norm $\|[f]\| = L(f)$ for all $f \in \mathrm{lip}^\alpha(X)$, where $[f]$ stands for the class $\pi(f)$.

Suppose that T is a Hölder seminorm preserving linear bijection from $\mathrm{lip}^\alpha(X)$ onto $\mathrm{lip}^\alpha(Y)$. Then there is a (unique) surjective linear isometry T_L from $\mathrm{lip}^\alpha_L(X)$ onto $\mathrm{lip}^\alpha_L(Y)$ such that the diagram

$$\begin{array}{ccc} \mathrm{lip}^\alpha(X) & \xrightarrow{T} & \mathrm{lip}^\alpha(Y) \\ \pi \downarrow & & \downarrow \pi \\ \mathrm{lip}^\alpha_L(X) & \xrightarrow{T_L} & \mathrm{lip}^\alpha_L(Y) \end{array}$$

commutes.

The main tool to prove Theorem 2.1 is the following characterization of the isometries from $\mathrm{lip}^\alpha_L(X)$ onto $\mathrm{lip}^\alpha_L(Y)$.

Theorem 2.2. *Let X and Y be compact metric spaces and* $0 < \alpha < 1$. *A linear map T: $\mathrm{lip}^\alpha_L(X) \to \mathrm{lip}^\alpha_L(Y)$ is a surjective isometry if and only if there is a number $\tau \in U$ and a bijective k-dilation φ: $Y \to X$ for some $k > 0$ such that $T[f] = [(\tau/k^\alpha) f \circ \varphi]$ for all $f \in \mathrm{lip}^\alpha(X)$.*

We first deduce Theorem 2.1 from Theorem 2.2.

Proof of Theorem 2.1. It is straightforward to check that every linear map T of the form $T(f) = (\tau/k^\alpha) f \circ \varphi + \mu(f) 1_Y$ for every $f \in \mathrm{lip}^\alpha(X)$ with τ, k, φ, μ being as in the statement of Theorem 2.1, is a Hölder seminorm preserving linear bijection from $\mathrm{lip}^\alpha(X)$ onto $\mathrm{lip}^\alpha(Y)$.

Suppose now that T: $\mathrm{lip}^\alpha(X) \to \mathrm{lip}^\alpha(Y)$ is a linear bijection that preserves the Hölder seminorm of functions in $\mathrm{lip}^\alpha(X)$. Then there exists a surjective linear isometry

T_L: $\mathrm{lip}_L^\alpha(X) \to \mathrm{lip}_L^\alpha(Y)$ such that $T_L[f] = [T(f)]$ for every $f \in \mathrm{lip}^\alpha(X)$. By Theorem 2.2 there is a number $\tau \in U$ and a surjective k-dilation φ: $Y \to X$ for some $k > 0$ such that $T_L[f] = [(\tau/k^\alpha) f \circ \varphi]$ for all $f \in \mathrm{lip}^\alpha(X)$. Then $T(f) - (\tau/k^\alpha) f \circ \varphi$ is a constant function on Y for all $f \in \mathrm{lip}^\alpha(X)$ and hence there is a linear functional μ: $\mathrm{lip}^\alpha(X) \to \mathbb{K}$ such that $T(f) = (\tau/k^\alpha) f \circ \varphi + \mu(f) 1_Y$ for every $f \in \mathrm{lip}^\alpha(X)$. ■

Remark 2.1. Observe that T is not necessarily continuous. In fact, T is continuous if and only if μ is.

From Theorems 2.1 and 2.2, we derive the following Banach–Stone type theorem.

COROLLARY 2.3

Let X and Y be compact metric spaces and $0 < \alpha < 1$. The following statements are equivalent:

(1) *There is a surjective k-dilation from Y onto X for some $k > 0$.*
(2) $\mathrm{lip}_L^\alpha(X)$ *and* $\mathrm{lip}_L^\alpha(Y)$ *are isometric.*
(3) *There is a (not necessarily continuous) linear bijection from* $\mathrm{lip}^\alpha(X)$ *onto* $\mathrm{lip}^\alpha(Y)$ *that preserves the Hölder seminorm.*

In order to prove Theorem 2.2, we need a characterization of the extreme points of the unit ball of $\mathrm{lip}_L^\alpha(X)^*$ (the dual space of $\mathrm{lip}_L^\alpha(X)$).

For any metric space X, $\tilde{X}$ denotes the set $\{(x, y) \in X^2 \colon x \neq y\}$ and $\mathcal{C}_0(\tilde{X})$ the space of all continuous functions from $\tilde{X}$ into $\mathbb{K}$ vanishing at infinity, with the usual supremum norm.

We embed $\mathrm{lip}_L^\alpha(X)$ isometrically as a subspace of $\mathcal{C}_0(\tilde{X})$. This allows us to relate extreme points of $B(\mathrm{lip}_L^\alpha(X)^*)$ to extreme points of $B(\mathcal{C}_0(\tilde{X})^*)$.

Given a compact metric space X, the map Φ: $\mathrm{lip}_L^\alpha(X) \to \mathcal{C}_0(\tilde{X})$ defined by

$$\Phi[f](x, y) = (f(x) - f(y))/d^\alpha(x, y), \quad \forall f \in \mathrm{lip}^\alpha(X), \ \forall (x, y) \in \tilde{X},$$

is an isometric linear embedding.

For $(x, y) \in \tilde{X}$, let $\delta_{(x,y)}$ be the evaluation functional on $\mathcal{C}_0(\tilde{X})$ defined by

$$\delta_{(x,y)}(f) = f(x, y), \quad \forall f \in \mathcal{C}_0(\tilde{X}),$$

and let $\tilde{\delta}_{(x,y)}$ be the functional on $\mathrm{lip}_L^\alpha(X)$ given by

$$\tilde{\delta}_{(x,y)}[f] = \delta_{(x,y)}(\Phi[f]), \quad \forall f \in \mathrm{lip}^\alpha(X).$$

Clearly, $\tilde{\delta}_{(x,y)}$ is linear and $|\tilde{\delta}_{(x,y)}[f]| \leq \|[f]\|$ for all $f \in \mathrm{lip}^\alpha(X)$. Therefore $\tilde{\delta}_{(x,y)}$ is in $\mathrm{lip}_L^\alpha(X)^*$ and $\|\tilde{\delta}_{(x,y)}\| \leq 1$.

The next result provides us with a complete description of the extreme points of $B(\mathrm{lip}_L^\alpha(X)^*)$. We shall need the following standard inequalities which are given without proof.

Lemma 2.4. *Let (X, d) be a metric space. If $x, y, z \in X$ with $x \neq y$ and $0 < \beta \leq 1$, then $|d^\beta(z, x) - d^\beta(z, y)| \leq d^\beta(x, y)$, equality ocurring only when $z = x$ or $z = y$.*

Lemma 2.5. *Let X be a compact metric space,* $0 < \alpha < 1$ *and* $\nu \in lip_L^\alpha(X)^*$. *Then* ν *is an extreme point of* $B(lip_L^\alpha(X)^*)$ *if and only if* $\nu = \sigma\tilde{\delta}_{(x,y)}$ *for some* $\sigma \in U^+$ *and* $(x, y) \in \tilde{X}$.

Proof. Suppose that ν is an extreme point of $B(\mathrm{lip}_L^\alpha(X)^*)$. Since Φ is an isometric linear embedding from $\mathrm{lip}_L^\alpha(X)$ into $\mathcal{C}_0(\tilde{X})$, the set

$$K = \{\mu \in B(\mathcal{C}_0(\tilde{X})^*) \colon \Phi^*(\mu) = \nu\},$$

Φ^* being the adjoint map of Φ, is nonempty by the Hahn–Banach theorem. Notice that K is a bounded weak* closed convex subset of $B(\mathcal{C}_0(\tilde{X})^*)$ and therefore it is a weak* compact convex set. By the Krein–Milman theorem, there exists an extreme point μ of K. It is easy to check that μ is also an extreme point of $B(\mathcal{C}_0(\tilde{X})^*)$.

The Riesz representation theorem identifies the dual space $\mathcal{C}_0(\tilde{X})^*$ with the space $M(\tilde{X})$ of all regular Borel measures on $\tilde{X}$ with values in the ground field. The duality is given by

$$\mu(f) = \int_{\tilde{X}} f \, \mathrm{d}\mu, \quad \forall f \in \mathcal{C}_0(\tilde{X}).$$

Moreover, the norm of a measure μ acting as a linear functional on $\mathcal{C}_0(\tilde{X})$ equals its total variation:

$$\sup\{|\mu(f)| \colon f \in \mathcal{C}_0(\tilde{X}),\ \|f\|_\infty \le 1\} = \|\mu\|_1 := |\mu|(\tilde{X})$$

(see, for example, Appendix C of [4]). The extreme points of $B(M(\tilde{X}))$ are the functionals $\sigma\delta_{(x,y)}$, where $\sigma \in U$ and $\delta_{(x,y)}$ is the unit mass at the point $(x, y) \in \tilde{X}$. Therefore we can find some $\sigma \in U$ and $(x, y) \in \tilde{X}$ such that $\nu = \Phi^*(\sigma\delta_{(x,y)})$, and so $\nu = \sigma\tilde{\delta}_{(x,y)}$.

Evidently, either $\sigma \in U^+$ or $-\sigma \in U^+$. In the first case, there is nothing to prove. In the second one, for any $f \in \mathrm{lip}^\alpha(X)$, we get $\tilde{\delta}_{(x,y)}[f] = -\tilde{\delta}_{(y,x)}[f]$ since $\Phi[f](x, y) = -\Phi[f](y, x)$ for every $(x, y) \in \tilde{X}$ and therefore $\nu = -\sigma\tilde{\delta}_{(y,x)}$.

To prove the sufficiency, let $(x, y) \in \tilde{X}$ and suppose that $\tilde{\delta}_{(x,y)} = t\varphi_1 + (1 - t)\varphi_2$ for some $\varphi_1, \varphi_2 \in B(\mathrm{lip}_L^\alpha(X)^*)$ and $t \in (0, 1)$. We must show that $\varphi_1 = \varphi_2 = \tilde{\delta}_{(x,y)}$. Since Φ isometrically embeds $\mathrm{lip}_L^\alpha(X)$ in $\mathcal{C}_0(\tilde{X})$, by the Hahn–Banach theorem the linear functionals φ_1 and φ_2 can be extended to $\mathcal{C}_0(\tilde{X})$ preserving the norm, and thus by the Riesz representation theorem there are measures $\mu_1, \mu_2 \in B(M(\tilde{X}))$ such that $\varphi_1 = \Phi^*(\mu_1)$ and $\varphi_2 = \Phi^*(\mu_2)$.

We now choose a real number β with $\alpha < \beta < 1$ and let $f \colon X \to \mathbb{R}$ be the function defined by

$$f(z) = \frac{d^\beta(z, y) - d^\beta(z, x)}{2d^{\beta-\alpha}(x, y)}, \quad \forall z \in X.$$

Clearly, $f \in \mathrm{Lip}^\alpha(X)$ with $L(f) = 1$, since

$$\begin{aligned}\frac{|f(z) - f(w)|}{d^\alpha(z, w)} &= \frac{|d^\beta(z, y) - d^\beta(z, x) + d^\beta(w, x) - d^\beta(w, y)|}{2d^{\beta-\alpha}(x, y)d^\alpha(z, w)} \\ &\le \frac{2\min\{d^\beta(z, w), d^\beta(x, y)\}}{2d^{\beta-\alpha}(x, y)d^\alpha(z, w)} \\ &= \min\left\{\frac{d^\alpha(x, y)}{d^\alpha(z, w)}, \frac{d^{\beta-\alpha}(z, w)}{d^{\beta-\alpha}(x, y)}\right\} \le 1\end{aligned}$$

for all $(z, w) \in \tilde{X}$ and

$$\frac{f(x) - f(y)}{d^\alpha(x, y)} = \frac{d^\beta(x, y) - (-d^\beta(y, x))}{2d^{\beta-\alpha}(x, y)d^\alpha(x, y)} = 1.$$

As well $f \in \mathrm{lip}^\alpha(X)$ since, given $\varepsilon > 0$, define $\delta = d(x, y)\varepsilon^{1/(\beta-\alpha)}$ and then

$$d(z, w) < \delta \quad \Rightarrow \quad |f(z) - f(w)| \leq \frac{d^{\beta-\alpha}(z, w)}{d^{\beta-\alpha}(x, y)} d^\alpha(z, w) < \varepsilon d^\alpha(z, w).$$

Since $\tilde{\delta}_{(x,y)}[f] = 1$, we have

$$1 = (t\varphi_1 + (1-t)\varphi_2)[f] = t\int_{\tilde{X}} \Phi[f]\mathrm{d}\mu_1 + (1-t)\int_{\tilde{X}} \Phi[f]\mathrm{d}\mu_2.$$

As $\|\Phi[f]\|_\infty = 1$ and $\mu_1, \mu_2 \in B(M(\tilde{X}))$, it follows that

$$\left|\int_{\tilde{X}} \Phi[f]\mathrm{d}\mu_1\right| \leq 1 \quad \text{and} \quad \left|\int_{\tilde{X}} \Phi[f]\mathrm{d}\mu_2\right| \leq 1.$$

We deduce that

$$\int_{\tilde{X}} \Phi[f]\mathrm{d}\mu_1 = \int_{\tilde{X}} \Phi[f]\mathrm{d}\mu_2 = 1.$$

We next shall prove that given $(z, w) \in \tilde{X}$, it holds that $|\Phi[f](z, w)| = 1$ if and only if $(z, w) = (x, y)$ or $(z, w) = (y, x)$. The "if" part follows immediately.

To prove the "only if" part, firstly suppose $d(z, w) \leq d(x, y)$. By using Lemma 2.4, we have

$$\begin{aligned} |\Phi[f](z, w)| &\leq \frac{|d^\beta(z, y) - d^\beta(w, y)| + |d^\beta(w, x) - d^\beta(z, x)|}{2d^{\beta-\alpha}(x, y)d^\alpha(z, w)} \\ &\leq \frac{2d^\beta(z, w)}{2d^{\beta-\alpha}(x, y)d^\alpha(z, w)} = \frac{d^{\beta-\alpha}(z, w)}{d^{\beta-\alpha}(x, y)} \leq 1. \end{aligned} \tag{1}$$

For equality in $|\Phi[f](z, w)| \leq 1$, we must have equality at (1). By Lemma 2.4,

$$|d^\beta(z, y) - d^\beta(w, y)| = d^\beta(z, w)$$

if and only if $z = y$ or $w = y$. Similarly, we have

$$|d^\beta(w, x) - d^\beta(z, x)| = d^\beta(z, w)$$

if and only if $z = x$ or $w = x$. Consequently, since $x \neq y$, then $z = y$, $w = x$ or $z = x$, $w = y$. Hence $|\Phi[f](z, w)| < 1$ if $(z, w) \neq (y, x)$ and $(z, w) \neq (x, y)$.

Secondly, if $d(x, y) \leq d(z, w)$, then

$$\begin{aligned} |\Phi[f](z, w)| &\leq \frac{|d^\beta(z, y) - d^\beta(z, x)| + |d^\beta(w, x) - d^\beta(w, y)|}{2d^{\beta-\alpha}(x, y)d^\alpha(z, w)} \\ &\leq \frac{2d^\beta(x, y)}{2d^{\beta-\alpha}(x, y)d^\alpha(z, w)} = \frac{d^\alpha(x, y)}{d^\alpha(z, w)} \leq 1, \end{aligned} \tag{2}$$

and applying similar arguments to (2) yields the same conclusion.

So we have proved that $|\Phi[f](z, w)| < 1$ for all $(z, w) \in \tilde{X}$ except at $(z, w) = (x, y)$ or $(z, w) = (y, x)$. This implies that μ_1 and μ_2 must be supported on these two points. Thus φ_1 and φ_2 are linear combinations of $\tilde{\delta}_{(x,y)}$ and $\tilde{\delta}_{(y,x)} = -\tilde{\delta}_{(x,y)}$, that is, $\varphi_1 = a\tilde{\delta}_{(x,y)}$ and $\varphi_2 = b\tilde{\delta}_{(x,y)}$ for some $a, b \in \mathbb{K}$. Since $\varphi_1, \varphi_2 \in B(\mathrm{lip}_L^\alpha(X)^*)$ and $\|\tilde{\delta}_{(x,y)}\| = 1$, it follows easily that $\varphi_1 = \varphi_2 = \tilde{\delta}_{(x,y)}$. ∎

The extreme points of $B(\mathrm{lip}_L^\alpha(X)^*)$ are all distinct as we see next.

Lemma 2.6. *Let X be a compact metric space and $0 < \alpha < 1$. Let σ_1, σ_2 be in U^+ and $(x, y), (z, w)$ in $\tilde{X}$. Suppose that $\sigma_1\tilde{\delta}_{(x,y)} = \sigma_2\tilde{\delta}_{(z,w)}$. Then $\{x, y\} = \{z, w\}$ and this implies that $(x, y) = (z, w)$ and $\sigma_1 = \sigma_2$.*

Proof. To obtain a contradiction, suppose $\{x, y\} \neq \{z, w\}$. Then there exists at least a point in $\{x, y, z, w\}$ which is distinct of the other three. There is no loss of generality in assuming that such a point is x. Let $\varepsilon = d(x, \{y, z, w\}) > 0$ and let $g: X \to \mathbb{R}$ be the function defined by

$$g(t) = \max\{0, \varepsilon - d(t, x)\}, \quad \forall t \in X.$$

It is easily seen that $g \in \mathrm{lip}^\alpha(X)$ and an easy computation shows that $\sigma_1\tilde{\delta}_{(x,y)}[g] = \sigma_1\varepsilon$ and $\sigma_2\tilde{\delta}_{(z,w)}[g] = 0$. This contradicts that $\sigma_1\varepsilon \neq 0$ and so $\{x, y\} = \{z, w\}$.

This implies that either $x = z$, $y = w$ or $x = w$, $y = z$. In the former case we have $\sigma_1\tilde{\delta}_{(x,y)} = \sigma_2\tilde{\delta}_{(x,y)}$ and since $\|\tilde{\delta}_{(x,y)}\| = 1$, $\sigma_1 = \sigma_2$. In the latter case, $\sigma_1\tilde{\delta}_{(x,y)} = -\sigma_2\tilde{\delta}_{(x,y)}$ and therefore $\sigma_1 = -\sigma_2$, which is impossible. ∎

Now we are ready to prove Theorem 2.2. The basic technique of the proof comes essentially from [2].

Proof of Theorem 2.2. It is easy to check that the linear map T of the form $T[f] = [(\tau/k^\alpha)f \circ \varphi]$ for all $f \in \mathrm{lip}^\alpha(X)$ with τ, k, φ under the assumptions of Theorem 2.2, is an isometry from $\mathrm{lip}_L^\alpha(X)$ onto $\mathrm{lip}_L^\alpha(Y)$.

Suppose now that T is a linear isometry from $\mathrm{lip}_L^\alpha(X)$ onto $\mathrm{lip}_L^\alpha(Y)$. Then the adjoint map T^* is also a linear isometry from $\mathrm{lip}_L^\alpha(Y)^*$ onto $\mathrm{lip}_L^\alpha(X)^*$, and therefore T^* maps extreme points to extreme points. By Lemma 2.5 the extreme points of $B(\mathrm{lip}_L^\alpha(Y)^*)$ are exactly the functionals of the form $\sigma\tilde{\delta}_{(x,y)}$ for $\sigma \in U^+$ and $(x, y) \in \tilde{Y}$, and similarly for the extreme points of $B(\mathrm{lip}_L^\alpha(X)^*)$.

Therefore, for every $(x, y) \in \tilde{Y}$, there exist $\sigma(x, y) \in U^+$ and $\gamma_1(x, y), \gamma_2(x, y) \in X$ with $\gamma_1(x, y) \neq \gamma_2(x, y)$ such that

$$T^*(\tilde{\delta}_{(x,y)}) = \sigma(x, y)\tilde{\delta}_{(\gamma_1(x,y),\gamma_2(x,y))}.$$

So we have established a map σ from $\tilde{Y}$ into U^+ and two maps γ_1 and γ_2 from $\tilde{Y}$ into $\tilde{X}$. These maps are well-defined by Lemma 2.6.

Switching x and y, we obtain

$$\begin{aligned} T^*(\tilde{\delta}_{(x,y)}) = -T^*(\tilde{\delta}_{(y,x)}) &= -\sigma(y, x)\tilde{\delta}_{(\gamma_1(y,x),\gamma_2(y,x))} \\ &= \sigma(y, x)\tilde{\delta}_{(\gamma_2(y,x),\gamma_1(y,x))} \end{aligned}$$

and then $\sigma(x, y) = \sigma(y, x)$, $\gamma_1(x, y) = \gamma_2(y, x)$ and $\gamma_2(x, y) = \gamma_1(y, x)$ also by Lemma 2.6. Let $\gamma\colon \tilde{Y} \to X$ be the map γ_1. Clearly, we have

$$T^*(\tilde{\delta}_{(x,y)}) = \sigma(x, y)\tilde{\delta}_{(\gamma(x,y),\gamma(y,x))}.$$

Let Y_2 and X_2 be the collections of all subsets of Y and X having exactly two elements. Obviously, T^* induces a mapping $\Gamma\colon Y_2 \to X_2$ defined by

$$\Gamma(\{x, y\}) = \{\gamma(x, y), \gamma(y, x)\}.$$

We next see that Γ is bijective. To prove that Γ is injective, suppose $\Gamma(\{x, y\}) = \Gamma(\{z, w\})$. Then either $\gamma(x, y) = \gamma(z, w)$ and $\gamma(y, x) = \gamma(w, z)$ or $\gamma(x, y) = \gamma(w, z)$ and $\gamma(y, x) = \gamma(z, w)$. In the first case we have

$$\overline{\sigma(x, y)}T^*(\tilde{\delta}_{(x,y)}) = \tilde{\delta}_{(\gamma(x,y),\gamma(y,x))} = \tilde{\delta}_{(\gamma(z,w),\gamma(w,z))} = \overline{\sigma(z, w)}T^*(\tilde{\delta}_{(z,w)})$$

and as T^* is linear and injective, it follows that $\sigma(z, w)\tilde{\delta}_{(x,y)} = \sigma(x, y)\tilde{\delta}_{(z,w)}$, that implies $\{x, y\} = \{z, w\}$ by Lemma 2.6 as desired. Similarly, we obtain the same conclusion in the other case.

The surjectivity of Γ is proved easily by taking into account that T^* is a bijection between the extreme points of $B(\mathrm{lip}_L^\alpha(Y)^*)$ and $B(\mathrm{lip}_L^\alpha(X)^*)$.

Let us observe that

$$\Gamma(\{x, y\}) = \mathrm{supp}(T^*(\tilde{\delta}_{(x,y)}))$$

and therefore there exists a bijection $\varphi\colon Y \to X$ such that

$$\Gamma(\{x, y\}) = \{\varphi(x), \varphi(y)\}, \quad \forall x, y \in Y$$

(see Lemma 3 of [2]). Clearly,

$$T^*(\tilde{\delta}_{(x,y)}) = \tau(x, y)\tilde{\delta}_{(\varphi(x),\varphi(y))}$$

where $\tau(x, y) \in U$.

We next prove that $\frac{d^\alpha(x,y)}{d^\alpha(\varphi(x),\varphi(y))}$ does not depend on x, y. For it notice that the equality

$$T^*(\tilde{\delta}_{(x,y)}) = \tau(x, y)\tilde{\delta}_{(\gamma(x,y),\gamma(y,x))}$$

can be rewritten in the form

$$T^*\left(\frac{\delta_x - \delta_y}{d^\alpha(x, y)}\right) = \tau(x, y)\frac{\delta_{\gamma(x,y)} - \delta_{\gamma(y,x)}}{d^\alpha(\gamma(x, y), \gamma(y, x))},$$

where for every $x, y \in Y$, δ_x and $\delta_{\gamma(x,y)}$ denote respectively the functionals

$$\delta_x(f) = f(x), \quad \forall f \in \mathrm{lip}^\alpha(Y)$$

and

$$\delta_{\gamma(x,y)}(f) = f(\gamma(x, y)), \quad \forall f \in \mathrm{lip}^\alpha(X).$$

Let x, y, z be distinct three elements of Y. We have

$$\begin{aligned}\tau(x,y)&\frac{d^\alpha(x,y)}{d^\alpha(\varphi(x),\varphi(y))}(\delta_{\varphi(x)}-\delta_{\varphi(y)})\\ &= T^*(\delta_x-\delta_y)=T^*(\delta_x-\delta_z+\delta_z-\delta_y)=T^*(\delta_x-\delta_z)+T^*(\delta_z-\delta_y)\\ &= \tau(x,z)\frac{d^\alpha(x,z)}{d^\alpha(\varphi(x),\varphi(z))}(\delta_{\varphi(x)}-\delta_{\varphi(z)})+\tau(z,y)\frac{d^\alpha(z,y)}{d^\alpha(\varphi(z),\varphi(y))}(\delta_{\varphi(z)}-\delta_{\varphi(y)}).\end{aligned}$$

Since the functionals $\delta_{\varphi(x)}$'s are linearly independent, it follows that

$$\tau(x,z)\frac{d^\alpha(x,z)}{d^\alpha(\varphi(x),\varphi(z))}=\tau(x,y)\frac{d^\alpha(x,y)}{d^\alpha(\varphi(x),\varphi(y))}=\tau(z,y)\frac{d^\alpha(z,y)}{d^\alpha(\varphi(z),\varphi(y))}.$$

Since $\tau(\cdot,\cdot)$ has the unit modulus, we have

$$\frac{d^\alpha(x,z)}{d^\alpha(\varphi(x),\varphi(z))}=\frac{d^\alpha(x,y)}{d^\alpha(\varphi(x),\varphi(y))}=\frac{d^\alpha(z,y)}{d^\alpha(\varphi(z),\varphi(y))}, \tag{3}$$

and so we obtain

$$\tau(x,z)=\tau(x,y)=\tau(z,y).$$

Since x, y and z are arbitrary, the first equality in (3) means that $\frac{d^\alpha(\cdot,\cdot)}{d^\alpha(\varphi(\cdot),\varphi(\cdot))}$ does not depend on the second variable, while the second equality in (3) says us that the same occurs with the first one. Hence there exists a constant $k>0$ such that

$$d(\varphi(x),\varphi(y))=kd(x,y).$$

Therefore φ is a k-dilation from Y onto X. Reasoning as before we see that $\tau(\,\cdot,\cdot)$ does not depend on x, y. Hence $\tau(x,y)=\tau$ for a suitable $\tau\in U$.

Finally, we prove that $T[f]=[(\tau/k^\alpha)f\circ\varphi]$ for every $f\in \mathrm{lip}^\alpha(X)$. Given $f\in\mathrm{lip}^\alpha(X)$ we have

$$\begin{aligned}\tilde\delta_{(x,y)}(T[f])=T^*(\tilde\delta_{(x,y)})[f]=\tau\tilde\delta_{(\varphi(x),\varphi(y))}[f]&=\tau\frac{f(\varphi(x))-f(\varphi(y))}{d^\alpha(\varphi(x),\varphi(y))}\\ &=\frac{((\tau/k^\alpha)f\circ\varphi)(x)-((\tau/k^\alpha)f\circ\varphi)(y)}{d^\alpha(x,y)}\\ &=\tilde\delta_{(x,y)}[(\tau/k^\alpha)f\circ\varphi]\end{aligned}$$

for every $(x,y)\in\tilde Y$. Then the Krein–Milman theorem implies that

$$S(T[f])=S[(\tau/k^\alpha)f\circ\varphi]$$

for every $S\in\mathrm{lip}_L^\alpha(Y)^*$ and so $T[f]=[(\tau/k^\alpha)f\circ\varphi]$. ■

Acknowledgements

The authors are very grateful to the referee for his/her observations that helped to improve the presentation of the paper. They also acknowledge support from Junta de Andalucía, projects P06-FQM-1215 and P06-FQM-1438, and from MEC, project MTM2006-4837.

References

[1] Barnes B A and Roy A K, Diameter preserving maps on various classes of function spaces, *Studia Math.* **153** (2002) 127–145

[2] Cabello Sánchez F, Diameter preserving linear maps and isometries, *Arch. Math. (Basel)* **73** (1999) 373–379

[3] Cabello Sánchez F, Diameter preserving linear maps and isometries II, *Proc. Indian Acad. Sci. (Math. Sci.)* **110** (2000) 205–211

[4] Conway J B, A Course in Functional Analysis (1990) (New York: Springer-Verlag)

[5] Garrido M I and Jaramillo J A, Variations on the Banach–Stone Theorem, *Extracta Math.* **17** (2002) 351–383

[6] González F and Uspenskij V V, On homomorphisms of groups of integer-valued functions, *Extracta Math.* **14** (1999) 19–29

[7] Győry M and Molnár L, Diameter preserving linear bijections of $\mathcal{C}(X)$, *Arch. Math. (Basel)* **71** (1998) 301–310

[8] Jiménez-Vargas A, Linear bijections preserving the Hölder seminorm, *Proc. Amer. Math. Soc.* **135** (2007) 2539–2547

[9] Rao T S S R K and Roy A K, Diameter-preserving linear bijections of function spaces, *J. Austral. Math. Soc.* **A70** (2001) 323–335

[10] Weaver N, Lipschitz Algebras (1999) (London: World Scientific)