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Algebraic reflexivity of the isometry group of some spaces of Lipschitz functions ☆

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ABSTRACT

We show that the isometry groups of $\mathrm{Lip}(X,d)$ and $\mathrm{lip}(X,d^{\alpha})$ with $\alpha \in (0,1)$, for a compact metric space (X,d), are algebraically reflexive. We also prove that the sets of isometric reflections and generalized bi-circular projections on such spaces are algebraically reflexive. In order to achieve this, we characterize generalized bi-circular projections on these spaces.

1. Introduction

This note is a contribution on a topic of current interest, namely Banach spaces whose groups of surjective linear isometries are completely determined by the local behavior of their elements. Given a Banach space X, we denote by $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. For any nonempty subset $\mathcal{S}$ of $\mathcal{B}(X)$, let

$$\mathrm{ref}_{\mathrm{al}}(\mathcal{S}) = \{T \in \mathcal{B}(X)\colon\ T(x) \in \mathcal{S}(x),\ \forall x \in X\},$$

where $\mathcal{S}(x) = \{L(x)\colon\ L \in \mathcal{S}\}$. The set $\mathcal{S}$ is said to be *algebraically reflexive* if $\mathrm{ref}_{\mathrm{al}}(\mathcal{S}) = \mathcal{S}$. If $\mathcal{G}(X)$ denotes the group of all surjective linear isometries of X, we will say that X is *iso-reflexive* if $\mathcal{G}(X)$ is algebraically reflexive. Notice that $T \in \mathrm{ref}_{\mathrm{al}}(\mathcal{G}(X))$ if for every $x \in X$, there exists a $T_x \in \mathcal{G}(X)$ such that $T(x) = T_x(x)$. The elements of $\mathrm{ref}_{\mathrm{al}}(\mathcal{G}(X))$ are called *local surjective isometries*. Hence X is iso-reflexive if and only if every local surjective isometry is a surjective isometry. The iso-reflexivity of some function spaces has been studied by F. Cabello Sánchez [4], F. Cabello Sánchez and L. Molnár [5], K. Jarosz and T.S.S.R.K. Rao [11] and L. Molnár and B. Zalar [15]. For pertinent results in the case of operator spaces, we refer to L. Molnár [14] and T.S.S.R.K. Rao [16].

Let (X,d) be a compact metric space and let $\mathbb{K}$ be the field of real or complex numbers. We denote by $\mathrm{Lip}(X,d)$ the Banach algebra of all functions $f : X \to \mathbb{K}$ such that

$$p_d(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)}\colon\ x, y \in X,\ x \neq y\right\} < \infty,$$

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endowed with the norm

$$\|f\|_d = p_d(f) + \|f\|_\infty,$$

where

$$\|f\|_\infty = \sup\{|f(x)|: x \in X\}.$$

Moreover, $\mathrm{lip}(X,d)$ denotes the closed subalgebra of $\mathrm{Lip}(X,d)$ consisting of all those functions f such that $\lim_{d(x,y)\to 0} |f(x) - f(y)|/d(x,y) = 0$, that is,

$$\forall \varepsilon > 0,\ \exists \delta > 0:\ x, y \in X,\ 0 < d(x,y) < \delta \quad \Rightarrow \quad \frac{|f(x) - f(y)|}{d(x,y)} < \varepsilon.$$

Both $\mathrm{Lip}(X,d)$ and $\mathrm{lip}(X,d)$ are unital semisimple commutative Banach algebras containing the constant functions, but while $\mathrm{Lip}(X,d)$ separates the points of X, $\mathrm{lip}(X,d)$ may contain only constant functions, for example $\mathrm{lip}([0,1], |\cdot|)$. To avoid this, we will consider the algebras $\mathrm{lip}(X,d^\alpha)$ with $\alpha \in (0,1)$, where d^α is the metric on X defined by $d^\alpha(x,y) = d(x,y)^\alpha$ for $x, y \in X$. These algebras were first studied by D. Sherbert [17,18]. Complete information about them can be found in Weaver's book on *Lipschitz algebras* [19].

From now on, given a compact metric space (X,d) and a real parameter $\alpha \in (0,1]$, we will denote by $A_\alpha(X)$ either $\mathrm{Lip}(X,d^\alpha)$ if $\alpha = 1$ or $\mathrm{lip}(X,d^\alpha)$ if $\alpha \in (0,1)$.

Our goal in this paper is to show that $A_\alpha(X)$ is iso-reflexive. Our method is to use a known characterization of the isometry group of $A_\alpha(X)$ due to K. Jarosz and V. Pathak [10]. In the complex-valued case, it is possible to give a different proof by applying a characterization of the carrier space of $A_\alpha(X)$ given by D. Sherbert [17,18], together with the famous Gleason–Kahane–Żelazko theorem [8,12].

Furthermore, we will apply the iso-reflexivity of $A_\alpha(X)$ to study the algebraic reflexivity of some subsets of isometries and projections of $A_\alpha(X)$. In order to introduce these sets, we recall that an isometry of a metric space X is a map $\varphi : X \to X$ satisfying that $d(\varphi(x), \varphi(y)) = d(x,y)$ for all $x, y \in X$. If, in addition, $\varphi^2 = Id$ where Id is the identity map of X, then φ is said to be an *involutive isometry* of X. Notice that every involutive isometry of X is surjective. In particular, if X is a Banach space, an involutive linear isometry of X is often called an *isometric reflection* of X.

Let $S_{\mathbb{K}}$ denote the set of elements in $\mathbb{K}$ with modulus 1. Given a Banach space X, a linear map $P : X \to X$ is said to be a *generalized bi-circular projection* if $P^2 = P$ and $P + \lambda(Id - P)$ is an isometry for some $\lambda \in S_{\mathbb{K}}$, $\lambda \neq 1$. Notice that $P + \lambda(Id - P)$ is surjective. The concept of generalized bi-circular projection was introduced by M. Fosner, D. Ilisevic and C. Li in [7]. They characterized these projections in the finite-dimensional case. Since then, a considerable account of work has been done concerning generalized bi-circular projections on various spaces. See, for example, [1–3,6,13].

Using the iso-reflexivity of $A_\alpha(X)$, we will prove that the sets of isometric reflections and generalized bi-circular projections of $A_\alpha(X)$ are also algebraically reflexive. In order to achieve that, we first give a complete description of such operators. An easy application of the aforementioned characterization of the isometries of $A_\alpha(X)$ shows that every isometric reflection of $A_\alpha(X)$ is either a composition operator induced by an involutive isometry of the metric space X or the negative of such a composition operator. On the other hand, we state that every generalized bi-circular projection of $A_\alpha(X)$ is the average of the identity with an isometric reflection. In particular, every generalized bi-circular projection P of $A_\alpha(X)$ is a bi-contractive projection, that is, $\|P\| \leqslant 1$ and $\|Id - P\| \leqslant 1$.

We emphasize that our results are motivated by recent studies of the analogous problems on the space $\mathcal{C}(X)$ of all complex-valued continuous functions on a compact Hausdorff space X. In general, $\mathcal{C}(X)$ is not iso-reflexive (see [5, Theorem 9] for an example), but in the case that X is a first countable compact Hausdorff space, L. Molnár and B. Zalar proved in [15, Theorem 2.2] that $\mathcal{C}(X)$ is iso-reflexive. On the other hand, F. Botelho and J.E. Jamison [2] showed that if X is a connected compact Hausdorff space, then the only projections of $\mathcal{C}(X)$ that can be represented as the average of the identity with an isometric reflection are generalized bi-circular projections. Recently, S. Dutta and T.S.S.R.K. Rao [6] have investigated the algebraic reflexivity of the sets of isometric reflections and generalized bi-circular projections of $\mathcal{C}(X)$.

2. Algebraic reflexivity of the groups of isometries and involutive isometries

Throughout the paper, given a compact metric space (X,d), let $A_\alpha(X)$ stand for either $\mathrm{Lip}(X,d^\alpha)$ if $\alpha = 1$ or $\mathrm{lip}(X,d^\alpha)$ if $\alpha \in (0,1)$, equipped with the norm $\|f\|_{d^\alpha} = p_{d^\alpha}(f) + \|f\|_\infty$. If it is necessary to specify the field, we will write $A_\alpha(X,\mathbb{K})$. In what follows, we will use frequently the easy fact that $\mathrm{Lip}(X,d)$ is contained in $A_\alpha(X)$ for all $\alpha \in (0,1]$. The symbol $\mathbf{1}$ will stand for the function constantly 1 on X.

We also will make use of the following functions. For any $x \in X$ and $\delta > 0$, $h_{x,\delta} : X \to [0,1]$ defined by

$$h_{x,\delta}(z) = \max\left\{0, 1 - \frac{d(z,x)}{\delta}\right\} \quad (z \in X),$$

belongs to $\mathrm{Lip}(X,d)$ with $h_{x,\delta}(x) = 1$ and $h_{x,\delta}(z) = 0$ if $d(z,x) \geqslant \delta$.

The isometry group of $A_\alpha(X)$ was described by K. Jarosz and V. Pathak in [10]. They showed that every surjective linear isometry $T : A_\alpha(X) \to A_\alpha(X)$ is of the form

$$T(f)(x) = \tau f\big(\varphi(x)\big), \quad \forall f \in A_\alpha(X),\ \forall x \in X,$$

where $\tau \in S_{\mathbb{K}}$ and $\varphi : X \to X$ is a surjective isometry [10, Example 8].

Our aim in this section is to show that $A_\alpha(X)$ is iso-reflexive. In order to prove this, we first state some preliminary results. The next lemma is surely known but we give a proof for the sake of completeness.

Lemma 2.1. *If X is a compact metric space and ψ is an isometry from X into X, then ψ is onto.*

Proof. Suppose that $x \in X \setminus \psi(X)$. Clearly, $\psi(X)$ is compact. Then $r = (1/2)d(x, \psi(X)) > 0$. Notice that $d(\psi^k(x), \psi^n(x)) \geqslant 2r$ for all $k, n \in \mathbb{N}$, $k \neq n$.

For $z \in X$ and $\delta > 0$, let $B(z, \delta) = \{y \in X\colon\ d(y, z) < \delta\}$. Obviously, the family $\{B(z, r)\colon\ z \in \psi(X)\}$ is an open cover of $\psi(X)$. Then there exist $z_1, \ldots, z_m \in \psi(X)$ such that $\psi(X) \subseteq \bigcup_{j=1}^m B(z_j, r)$. In particular, $\psi(x) \in B(z_k, r)$ for some $k \in \{1, \ldots, m\}$. Hence the set

$$J = \big\{j \in \{1, \ldots, m\}\colon\ B(z_j, r) \cap \big\{\psi^n(x)\colon\ n \in \mathbb{N}\big\} \neq \emptyset\big\}$$

is nonempty. Moreover, given $j \in J$, it is clear that $B(z_j, r) \cap \{\psi^n(x)\colon\ n \in \mathbb{N}\}$ is a singleton. Then we can define $\gamma : J \to \{\psi^n(x)\colon\ n \in \mathbb{N}\}$ by

$$\big\{\gamma(j)\big\} = B(z_j, r) \cap \big\{\psi^n(x)\colon\ n \in \mathbb{N}\big\} \quad (j \in J).$$

Evidently, γ is onto. Since $\{\psi^n(x)\colon\ n \in \mathbb{N}\}$ is infinite, then so is J, a contradiction. This proves the lemma. □

We now give a description of local surjective isometries of $A_\alpha(X)$.

Lemma 2.2. *Let X be a compact metric space and let $T \in \mathrm{ref}_{\mathrm{al}}(\mathcal{G}(A_\alpha(X)))$. Then there exist a scalar $\tau \in S_{\mathbb{K}}$ and a mapping $\psi : X \to X$ such that $T(f)(\psi(x)) = \tau f(x)$ for every $f \in A_\alpha(X)$ and all $x \in X$.*

Proof. For each $f \in A_\alpha(X)$, there exists $T_f \in \mathcal{G}(A_\alpha(X))$ such that $T(f) = T_f(f)$. According to [10, Example 8], there exist a scalar $\tau_f \in S_{\mathbb{K}}$ and a surjective isometry $\varphi_f : X \to X$ such that

$$T_f(g)(z) = \tau_f g\big(\varphi_f(z)\big), \quad \forall g \in A_\alpha(X),\ \forall z \in X.$$

In particular, we have

$$T(f)(z) = \tau_f f\big(\varphi_f(z)\big), \quad \forall z \in X.$$

Taking $f = \mathbf{1}$ in the equality above, we see that $T(\mathbf{1})$ is the function constantly equal to $\tau_{\mathbf{1}}$ on X. Take $\tau = \tau_{\mathbf{1}}$.

In order to define the function $\psi : X \to X$, for each $x \in X$ we consider the sets:

$$F_x = \big\{f \in A_\alpha(X)\colon\ \|f\|_\infty = 1 = \big|f(x)\big|\big\},$$
$$Q_x = \big\{y \in X\colon\ \big|T(f)(y)\big| = 1,\ \forall f \in F_x\big\}.$$

Notice that $F_x \neq \emptyset$ and $Q_x = \bigcap_{f \in F_x} |T(f)|^{-1}(\{1\})$.

We first prove that Q_x is nonempty. It is enough to show that the family $\{|T(f)|^{-1}(\{1\})\colon\ f \in F_x\}$ has the finite intersection property, since each $|T(f)|^{-1}(\{1\})$ is closed in the compact X. Pick $f_1, \ldots, f_n \in F_x$ and put $f = (1/n)\sum_{j=1}^n (f_j/f_j(x))$. Clearly, $f \in F_x$ and since $T \in \mathrm{ref}_{\mathrm{al}}(\mathcal{G}(A_\alpha(X)))$, there are a number $\tau_f \in S_{\mathbb{K}}$ and a surjective isometry $\varphi_f : X \to X$ such that

$$T(f)(z) = \tau_f f\big(\varphi_f(z)\big), \quad \forall z \in X.$$

Let $y \in X$ be such that $\varphi_f(y) = x$. Then $|T(f)(y)| = |f(\varphi_f(y))| = |f(x)| = 1$. Similarly, we can find for each $j \in \{1, \ldots, n\}$, a scalar $\tau_{f_j} \in S_{\mathbb{K}}$ and a surjective isometry $\varphi_{f_j} : X \to X$ for which

$$T(f_j)(z) = \tau_{f_j} f_j\big(\varphi_{f_j}(z)\big), \quad \forall z \in X.$$

Hence $|T(f_j)(y)| = |f_j(\varphi_{f_j}(y))| \leqslant 1$. Suppose that $|T(f_k)(y)| < 1$ for some $k \in \{1, \ldots, n\}$. Then

$$1 = \big|T(f)(y)\big| \leqslant \frac{1}{n}\sum_{j=1}^n \frac{|T(f_j)(y)|}{|f_j(x)|} = \frac{1}{n}\sum_{j=1}^n \big|T(f_j)(y)\big| < 1,$$

a contradiction. Therefore, $|T(f_j)(y)| = 1$ for every $j \in \{1, \ldots, n\}$, and thus $\bigcap_{j=1}^n |T(f_j)|^{-1}(\{1\}) \neq \emptyset$ as desired.

Next we show that Q_x is a singleton. Let us suppose that there exist $y, z \in Q_x$. Consider the function $h_{x,1}$. Clearly, $h_{x,1} \in F_x$ and therefore $|T(h_{x,1})(y)| = |T(h_{x,1})(z)| = 1$. Again, we can write

$$T(h_{x,1})(w) = \tau_{h_{x,1}} h_{x,1}\big(\varphi_{h_{x,1}}(w)\big), \quad \forall w \in X,$$

where $\tau_{h_{x,1}} \in S_{\mathbb{K}}$ and $\varphi_{h_{x,1}}$ is a surjective isometry of X. Then we have

$$1 = \big|T(h_{x,1})(y)\big| = h_{x,1}\big(\varphi_{h_{x,1}}(y)\big),$$
$$1 = \big|T(h_{x,1})(z)\big| = h_{x,1}\big(\varphi_{h_{x,1}}(z)\big),$$

which gives $\varphi_{h_{x,1}}(y) = \varphi_{h_{x,1}}(z) = x$. Since $\varphi_{h_{x,1}}$ is injective, we conclude that $y = z$.

By above-proved, we can consider the application $\psi : X \to X$ defined by

$$\big\{\psi(x)\big\} = Q_x, \quad \forall x \in X.$$

We now claim that if $f \in A_\alpha(X)$, $x \in X$ and $f(x) = 0$, then $T(f)(\psi(x)) = 0$. In order to prove this, assume $T(f)(\psi(x)) \neq 0$. Then $f \neq 0$, and we can consider the function

$$g(z) = 1 - \frac{|f(z)|}{\|f\|_\infty} \quad (z \in X),$$

and the number

$$\eta = \frac{T(f)(\psi(x))}{|T(f)(\psi(x))| T(g)(\psi(x))}.$$

Since $g \in F_x$, it follows that $|T(g)(\psi(x))| = 1$, and thus $\eta \in S_{\mathbb{K}}$. An easy verification shows that the function $(1/\|f\|_\infty) f + \eta g$ belongs to F_x. Using the definition of ψ, it follows that

$$\begin{aligned} 1 &= \left| \frac{1}{\|f\|_\infty} T(f)\big(\psi(x)\big) + \eta T(g)\big(\psi(x)\big) \right| \\ &= \left(\frac{1}{\|f\|_\infty} + \frac{1}{|T(f)(\psi(x))|} \right) \big|T(f)\big(\psi(x)\big)\big| \\ &= \frac{|T(f)(\psi(x))|}{\|f\|_\infty} + 1 > 1, \end{aligned}$$

a contradiction. This proves our claim.

Finally, given $f \in A_\alpha(X)$ and $x \in X$, it is clear that $h = f - f(x) \in A_\alpha(X)$ with $h(x) = 0$. Then, by above-proved, $T(h)(\psi(x)) = 0$, that is, $T(f)(\psi(x)) = T(\mathbf{1})(\psi(x)) f(x) = \tau f(x)$. □

We are now ready to prove the main result of this section.

Theorem 2.3. *Let X be a compact metric space. Then $A_\alpha(X)$ is iso-reflexive.*

Proof. Let $T \in \mathrm{ref}_{\mathrm{al}}(\mathcal{G}(A_\alpha(X)))$. Then, for each $f \in A_\alpha(X)$, there exists $T_f \in \mathcal{G}(A_\alpha(X))$ such that $T(f) = T_f(f)$. Hence $\|T(f)\|_{d^\alpha} = \|T_f(f)\|_{d^\alpha} = \|f\|_{d^\alpha}$, and thus T is an isometry. It remains to prove that T is surjective.

By Lemma 2.2, there exist a scalar $\tau \in S_{\mathbb{K}}$ and a mapping $\psi : X \to X$ such that

$$T(f)\big(\psi(x)\big) = \tau f(x), \quad \forall f \in A_\alpha(X),\ \forall x \in X. \tag{1}$$

Now we claim that ψ is an isometry. To prove this, pick $x, y \in X$. If $x = y$, then $d(\psi(x), \psi(y)) = d(x, y) = 0$. Assume $x \neq y$ and consider $k : X \to [0, 1]$ defined by $k(z) = d(z, x)/(d(z, x) + d(z, y))$ for all $z \in X$. Obviously, $k \in \mathrm{Lip}(X, d)$ with $k^{-1}(\{0\}) = \{x\}$ and $k^{-1}(\{1\}) = \{y\}$. By assumption we can write

$$T(k)(z) = \tau_k k\big(\varphi_k(z)\big), \quad \forall z \in X, \tag{2}$$

for some $\tau_k \in S_{\mathbb{K}}$ and some surjective isometry φ_k of X. Applying now (1) and (2) gives

$$\tau_k k\big(\varphi_k\big(\psi(x)\big)\big) = T(k)\big(\psi(x)\big) = \tau k(x) = 0,$$
$$\tau_k k\big(\varphi_k\big(\psi(y)\big)\big) = T(k)\big(\psi(y)\big) = \tau k(y) = \tau.$$

Then $\varphi_k(\psi(x)) = x$ and $\varphi_k(\psi(y)) = y$. Since φ_k is an isometry, we deduce that $d(\psi(x), \psi(y)) = d(x, y)$ and this proves our claim.

Finally, we show that T is surjective. By Lemma 2.1, the isometry $\psi : X \to X$ is onto. Then, given $g \in A_\alpha(X)$, take $f = \overline{\tau} \cdot (g \circ \psi)$. Clearly, $f \in A_\alpha(X)$ and $T(f) = g$. □

In the complex-valued case, Theorem 2.3 can be proved in a different form. Besides the aforementioned description of the isometry group of $A_\alpha(X)$ given by K. Jarosz and V. Pathak [10], this new approach is based on a known characterization of the carrier space of $A_\alpha(X)$ due to D. Sherbert [17,18] together with the Gleason–Kahane–Żelazko theorem [8,12].

The algebraic structure of $A_\alpha(X)$ was studied by D. Sherbert [17,18]. He proved that every nonzero multiplicative linear functional $T : A_\alpha(X) \to \mathbb{C}$ is an evaluation map at a point, that is,

$$T(f) = f(c), \quad \forall f \in A_\alpha(X),$$

where c is a unique point in X [18, p. 246].

The known Gleason–Kahane–Żelazko theorem [8,12] (see also [9]) asserts that if $\mathcal{A}$ is a complex Banach algebra with a unit e and F is a linear functional on $\mathcal{A}$ such that $F(f) \neq 0$ for all f in the set of invertible elements of $\mathcal{A}$, then $F/F(e)$ is multiplicative.

The preceding descriptions of the carrier space and the isometry group of $A_\alpha(X)$ are valid for real and complex-valued functions. However, the Gleason–Kahane–Żelazko theorem is not available in the real case.

Other proof of complex-valued case. Let X be a compact metric space. We want to show that $A_\alpha(X, \mathbb{C})$ is iso-reflexive. Let $T \in \mathrm{ref}_{\mathrm{al}}(\mathcal{G}(A_\alpha(X, \mathbb{C})))$. As in the proof of Theorem 2.3, T is an isometry and we need only to see that T is surjective. According to [10, Example 8], there exist a scalar $\tau_f \in S_{\mathbb{C}}$ and a surjective isometry $\varphi_f : X \to X$ such that

$$T(f)(z) = \tau_f f\big(\varphi_f(z)\big), \quad \forall z \in X. \tag{3}$$

In particular, $T(\mathbf{1}) = \tau_{\mathbf{1}}$.

Let $x \in X$ be fixed and define the nonzero linear functional $T_x : A_\alpha(X, \mathbb{C}) \to \mathbb{C}$ by

$$T_x(f) = T(f)(x), \quad \forall f \in A_\alpha(X, \mathbb{C}).$$

Take $f \in A_\alpha(X, \mathbb{C})$ and suppose that f is nowhere vanishing. In view of (3), it is clear that $T_x(f) \neq 0$. Then $T_x/T_x(\mathbf{1}) = \overline{\tau_{\mathbf{1}}}\, T_x$ is multiplicative by the Gleason–Kahane–Żelazko theorem. Since every nonzero multiplicative linear functional on $A_\alpha(X, \mathbb{C})$ is an evaluation map at a point, there exists a unique point $c_x \in X$ such that $\overline{\tau_{\mathbf{1}}}\, T_x(f) = f(c_x)$ for all $f \in A_\alpha(X, \mathbb{C})$. Since this is true for each $x \in X$, we have thus a map $\varphi : X \to X$ defined by $\varphi(x) = c_x$ such that

$$T(f)(x) = \tau_{\mathbf{1}} f\big(\varphi(x)\big), \quad \forall f \in A_\alpha(X, \mathbb{C}),\ \forall x \in X. \tag{4}$$

We next show that φ is injective. Let $x, y \in X$ and suppose that $\varphi(x) = \varphi(y)$. Define $h : X \to \mathbb{R}_0^+$ by $h(z) = d(z, \varphi(x))$ for all $z \in X$. Clearly, $h \in \mathrm{Lip}(X, d)$ and $h^{-1}(\{0\}) = \{\varphi(x)\}$. Since $T \in \mathrm{ref}_{\mathrm{al}}(\mathcal{G}(A_\alpha(X, \mathbb{C})))$, we have

$$T(h)(z) = \tau_h h\big(\varphi_h(z)\big), \quad \forall z \in X, \tag{5}$$

where $\tau_h \in S_{\mathbb{C}}$ and φ_h is a surjective isometry of X. By using the equalities (4) and (5), we obtain:

$$\tau_h h\big(\varphi_h(x)\big) = T(h)(x) = \tau_{\mathbf{1}} h\big(\varphi(x)\big) = 0,$$
$$\tau_h h\big(\varphi_h(y)\big) = T(h)(y) = \tau_{\mathbf{1}} h\big(\varphi(y)\big) = 0.$$

This implies that $\varphi_h(x) = \varphi_h(y) = \varphi(x)$ and, since φ_h is injective, we conclude that $x = y$.

Now we claim that φ is an isometry. To prove this, pick $x, y \in X$. If $\varphi(x) = \varphi(y)$, then $x = y$ by the injectivity of φ and thus $d(\varphi(x), \varphi(y)) = d(x, y) = 0$. If $\varphi(x) \neq \varphi(y)$, we deduce that $d(\varphi(x), \varphi(y)) = d(x, y)$ as in the proof of Theorem 2.3.

To see that φ is surjective, suppose that there is an $x \in X \setminus \varphi(X)$. Notice that $\varphi(X)$ is closed since φ is continuous and X is compact. Therefore $\delta = d(x, \varphi(X)) > 0$. Take $h_{x,\delta} \in \mathrm{Lip}(X, d)$ and clearly $h_{x,\delta}(\varphi(z)) = 0$ for all $z \in X$. From (4) it follows that $T(h_{x,\delta})(z) = 0$ for all $z \in X$, but $h_{x,\delta}(x) = 1$, which contradicts that the linear map T is injective. This proves the surjectivity of φ.

Finally, the surjectivity of T follows as in the proof of Theorem 2.3. □

Definition 2.1. Let X be a Banach space. An isometric reflection of X is a linear isometry $T : X \to X$ satisfying that $T^2 = Id$.

From the Banach–Stone type theorem that describes the isometry group of $A_\alpha(X)$, we deduce easily the form of isometric reflections of $A_\alpha(X)$ in the following result.

Corollary 2.4. *Let X be a compact metric space. A map $T : A_\alpha(X) \to A_\alpha(X)$ is an isometric reflection if and only if there exist a constant $\tau \in \{-1, 1\}$ and an involutive isometry φ of X such that*

$$T(f)(x) = \tau f\big(\varphi(x)\big), \quad \forall f \in A_\alpha(X),\ \forall x \in X.$$

Theorem 2.5. *Let X be a compact metric space. Then the set of all isometric reflections of $A_\alpha(X)$ is algebraically reflexive.*

Proof. Denote by $\mathcal{G}^2(A_\alpha(X))$ the set of all isometric reflections of $A_\alpha(X)$. Let $T \in \text{ref}_{\text{al}}(\mathcal{G}^2(A_\alpha(X)))$. According to Corollary 2.4, for every $f \in A_\alpha(X)$, there are a scalar $\tau_f \in \{-1, 1\}$ and an involutive isometry φ_f of X such that

$$T(f)(x) = \tau_f f\big(\varphi_f(x)\big), \quad \forall x \in X. \tag{6}$$

By Theorem 2.3, it follows that T is a surjective isometry. Hence there exist a constant $\tau \in S_{\mathbb{K}}$ and a surjective isometry $\varphi : X \to X$ such that

$$T(g)(x) = \tau g\big(\varphi(x)\big), \quad \forall g \in A_\alpha(X),\ \forall x \in X. \tag{7}$$

From (6) and (7), it is deduced that $\tau = T(\mathbf{1}) = \tau_{\mathbf{1}}$ and thus $\tau \in \{-1, 1\}$. To see that T is involutive, it remains to prove that $\varphi^2(x) = x$ for all $x \in X$. Let $x \in X$. If $\varphi(x) = x$, then $\varphi^2(x) = x$. Otherwise, assume that $\varphi(x) \neq x$ and take $k : X \to [0, 1]$ defined by $k(z) = d(z, \varphi(x))/(d(z, \varphi(x)) + d(z, x))$ for all $z \in X$. Clearly, $k \in \text{Lip}(X, d)$ with $k^{-1}(\{0\}) = \{\varphi(x)\}$ and $k^{-1}(\{1\}) = \{x\}$. By hypothesis we can write

$$T(k)(z) = \tau_k k\big(\varphi_k(z)\big), \quad \forall z \in X, \tag{8}$$

where $\tau_k \in \{-1, 1\}$ and φ_k is an involutive isometry of X. From (7) and (8) we infer that

$$\tau_k k\big(\varphi_k(x)\big) = T(k)(x) = \tau k\big(\varphi(x)\big) = 0,$$
$$\tau k\big(\varphi\big(\varphi_k(x)\big)\big) = T(k)\big(\varphi_k(x)\big) = \tau_k k\big(\varphi_k^2(x)\big) = \tau_k k(x) = \tau_k.$$

The first equality implies that $\varphi_k(x) = \varphi(x)$ and the second one that $\varphi(\varphi_k(x)) = x$. It follows that $\varphi^2(x) = x$ as required. □

3. Algebraic reflexivity of the set of generalized bi-circular projections

Before proving the results we recall the concept of generalized bi-circular projection.

Definition 3.1. Let X be a Banach space. A linear map $P : X \to X$ is a generalized bi-circular projection if $P^2 = P$ and there exists a $\lambda \in S_{\mathbb{K}}$, $\lambda \neq 1$ such that $P + \lambda(Id - P)$ is an isometry.

We next give a complete description of the form of generalized bi-circular projections of $A_\alpha(X)$.

Theorem 3.1. *Let X be a compact metric space. A map $P : A_\alpha(X) \to A_\alpha(X)$ is a generalized bi-circular projection if and only if there exist a number $\tau \in \{-1, 1\}$ and an involutive isometry $\varphi : X \to X$ such that P is of the form*

$$P(f)(x) = \frac{1}{2}\big[f(x) + \tau f\big(\varphi(x)\big)\big] \quad \big(f \in A_\alpha(X),\ x \in X\big).$$

Proof. Only the "only if" part deserves to be proved. If P is a generalized bi-circular projection of $A_\alpha(X)$, then $P + \lambda(Id - P)$ is an isometry of $A_\alpha(X)$ for some $\lambda \in S_{\mathbb{K}}$, $\lambda \neq 1$. Then we can find a constant $\tau \in S_{\mathbb{K}}$ and a surjective isometry $\varphi : X \to X$ such that

$$\big[P + \lambda(Id - P)\big](f)(x) = \tau f\big(\varphi(x)\big) \quad \big(f \in A_\alpha(X),\ x \in X\big).$$

From above it is deduced that

$$P(f)(x) = (1 - \lambda)^{-1}\big[-\lambda f(x) + \tau f\big(\varphi(x)\big)\big] \quad \big(f \in A_\alpha(X),\ x \in X\big). \tag{9}$$

Since P is a projection, we have

$$\lambda f(x) - (\lambda + 1)\tau f\big(\varphi(x)\big) + \tau^2 f\big(\varphi^2(x)\big) = 0, \quad \forall f \in A_\alpha(X),\ \forall x \in X. \tag{10}$$

Let us suppose that there exists a point $x \in X$ for which $x \neq \varphi(x)$ and $x \neq \varphi^2(x)$. Take $h_{x,\delta} \in \text{Lip}(X, d)$ with $\delta = \min\{d(x, \varphi(x)),\ d(x, \varphi^2(x))\}$. Evaluating in the formula (10) this x and $h_{x,\delta}$, we obtain that $\lambda = 0$, a contradiction. Thus $\varphi(x) = x$ or $\varphi^2(x) = x$. In either case, $\varphi^2(x) = x$ for all $x \in X$.

If $\varphi \neq Id$, take some $x_0 \in X$ such that $x_0 \neq \varphi(x_0)$. Consider $h_{x_0,\delta_0} \in \text{Lip}(X)$ with $\delta_0 = d(x_0, \varphi(x_0))$. Substituting in (10), first x_0 and h_{x_0,δ_0}, and after $\mathbf{1}$, we get that $\lambda + \tau^2 = 0$ and $\lambda - (\lambda + 1)\tau + \tau^2 = 0$, respectively. Hence $\lambda = -1$ and $\tau^2 = 1$. Then $P(f)(x) = (1/2)[f(x) + \tau f(\varphi(x))]$ for all $x \in X$ by (9).

If $\varphi = Id$, taking $f = \mathbf{1}$ in (10) we obtain $\lambda - (\lambda + 1)\tau + \tau^2 = 0$. Hence $\tau = \lambda$ or $\tau = 1$. From (9) it follows that $P(f)(x) = 0$ for all $f \in A_\alpha(X)$ and $x \in X$, or $P(f)(x) = f(x)$ for all $f \in A_\alpha(X)$ and $x \in X$. □

The following result is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let X be a compact metric space. Every generalized bi-circular projection of $A_\alpha(X)$ is a bi-contractive projection, that is, $\|P\| \leqslant 1$ and $\|Id - P\| \leqslant 1$.*

We finish the paper with the following application of the preceding results.

Corollary 3.3. *Let X be a compact metric space. Then the set of all generalized bi-circular projections of $A_\alpha(X)$ is algebraically reflexive.*

Proof. Let $\mathrm{GBP}(A_\alpha(X))$ denote the set of all generalized bi-circular projections of $A_\alpha(X)$. Let $P \in \mathrm{ref_{al}}(\mathrm{GBP}(A_\alpha(X)))$. Then, by Theorem 3.1, for each $f \in A_\alpha(X)$ there exist a scalar $\tau_f \in \{-1, 1\}$ and an involutive isometry φ_f of X such that $P(f) = (1/2)[f + \tau_f \cdot (f \circ \varphi_f)]$. Hence, for every $f \in A_\alpha(X)$, we have $(2P - Id)(f) = \tau_f \cdot (f \circ \varphi_f)$ and then $2P - Id \in \mathrm{ref_{al}}(\mathcal{G}^2(A_\alpha(X)))$ by Corollary 2.4. From Theorem 2.5 we infer that $2P - Id \in \mathcal{G}^2(A_\alpha(X))$, and thus $P \in \mathrm{GBP}(A_\alpha(X))$ as desired. □

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