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LINEAR ISOMETRIES BETWEEN SPACES OF VECTOR-VALUED LIPSCHITZ FUNCTIONS

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Abstract. In this paper we state a Lipschitz version of a theorem due to Cambern concerning into linear isometries between spaces of vector-valued continuous functions and deduce a Lipschitz version of a celebrated theorem due to Jerison concerning onto linear isometries between such spaces.

1. Introduction

Given a metric space (X, d) and a Banach space E, we denote by $\mathrm{Lip}(X, E)$ the Banach space of all bounded Lipschitz functions $f : X \to E$ with the norm $\|f\| = \max\{L(f), \|f\|_\infty\}$, where

$$L(f) = \sup\{\|f(x) - f(y)\| / d(x, y) : x, y \in X,\ x \neq y\}.$$

If E is the field of real or complex numbers, we shall write simply $\mathrm{Lip}(X)$.

The study of surjective linear isometries between spaces $\mathrm{Lip}(X)$ was initiated by Roy [9] and Vasavada [10]. In [9, Theorem 1.7], Roy proved that if (X, d) is a compact connected metric space with diameter at most 1, then a map T is a surjective linear isometry from $\mathrm{Lip}(X)$ onto itself if and only if there exist a surjective isometry $\varphi : X \to X$ and a scalar τ of modulus 1 such that

$$T(f)(y) = \tau f(\varphi(y)), \quad \forall y \in Y,\ \forall f \in \mathrm{Lip}(X).$$

In [8, Theorem 2], Novinger improved slightly Roy's result by considering linear isometries from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$. Vasavada [10] proved it for linear isometries from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ when the metric spaces X, Y are compact with diameter at most 2 and β-connected for some $\beta < 1$. Weaver [11] developed a technique to remove the compactness assumption on X and Y and showed that the above-mentioned characterization holds if X, Y are complete and 1-connected with diameter at most 2 [11, Theorem D]. The reduction to metric spaces of diameter at most 2 is not restrictive since if (X, d) is a metric space and X' is the set X remetrized with the metric $d'(x, y) = \min\{d(x, y), 2\}$, then the diameter of X' is at most 2 and $\mathrm{Lip}(X')$ is isometrically isomorphic to $\mathrm{Lip}(X)$ [12, Proposition 1.7.1]. We must also mention the complete research carried out on surjective linear isometries between spaces of Hölder functions [2, 3, 6, 7]. We refer the reader to Weaver's

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book *Lipschitz Algebras* [12] for unexplained terminology and more information on the subject. This is essentially the history of the onto scalar-valued case. Recently, into linear isometries (that is, not necessarily surjective) and codimension 1 linear isometries between spaces $\mathrm{Lip}(X)$ have been studied in [5].

In this note we shall go a step further and give a complete description of linear isometries between spaces of vector-valued Lipschitz functions. To our knowledge, little or nothing is known on the matter in the vector-valued case. Our approach to the problem is not based on extreme points as in all aforementioned papers. We have used here a different method which is influenced by that utilized by Cambern [1] to characterize into linear isometries between spaces $C(X, E)$ of continuous functions from a compact Hausdorff space X into a Banach space E with the supremum norm. In [4], Jerison extended to the vector case the classical Banach–Stone theorem about onto linear isometries between spaces $C(X)$, and Jerison's theorem was generalized by Cambern [1] by considering into linear isometries.

The aim of this paper is to show that Cambern's and Jerison's theorems have a natural formulation in the context of Lipschitz functions.

2. A Lipschitz version of Cambern's theorem

We begin by introducing some notation. Given a Banach space E, S_E will denote its unit sphere and B_E its closed unit ball. Let us recall that a Banach space E is said to be *strictly convex* if every element of S_E is an extreme point of B_E. For Banach spaces E and F, $L(E, F)$ will stand for the Banach space of all bounded linear operators from E into F with the canonical norm of operators. In the case $E = F$, we shall write $L(E)$ instead of $L(E, F)$. Given a metric space (X, d), we shall denote by 1_X the function constantly 1 on X and by $\mathrm{diam}(X)$ the diameter of X. If $\varphi : X \to Y$ is a Lipschitz map between metric spaces, $L(\varphi)$ will be its Lipschitz constant.

For any $f \in \mathrm{Lip}(X)$ and $e \in E$, define $f \otimes e : X \to E$ by $(f \otimes e)(x) = f(x)e$. It is easy to check that $f \otimes e \in \mathrm{Lip}(X, E)$ with $\|f \otimes e\|_\infty = \|f\|_\infty \|e\|$ and $L(f \otimes e) = L(f) \|e\|$, and thus $\|f \otimes e\| = \|f\| \|e\|$.

Theorem 2.1. *Let X and Y be compact metric spaces and let E be a strictly convex Banach space. Let T be a linear isometry from $\mathrm{Lip}(X, E)$ into $\mathrm{Lip}(Y, E)$ such that $T(1_X \otimes e) = 1_Y \otimes e$ for some $e \in S_E$. Then there exists a Lipschitz map φ from a closed subset Y_0 of Y onto X with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$, and a Lipschitz map $y \mapsto T_y$ from Y into $L(E)$ with $\|T_y\| = 1$ for all $y \in Y$, such that*

$$T(f)(y) = T_y(f(\varphi(y))), \quad \forall y \in Y_0, \ \forall f \in \mathrm{Lip}(X, E).$$

Proof. For each $x \in X$, define

$$F(x) = \{f \in \mathrm{Lip}(X, E) : f(x) = \|f\|_\infty e\}.$$

Clearly, $1_X \otimes e \in F(x)$. For each $\delta > 0$, the map $h_{x,\delta} \otimes e : X \to E$, defined by

$$h_{x,\delta}(z) = \max\{0, 1 - d(z, x)/\delta\} \quad (z \in X),$$

belongs to $F(x)$. Indeed, an easy verification shows that $h_{x,\delta} \in \mathrm{Lip}(X)$ with $\|h_{x,\delta}\|_\infty = 1 = h_{x,\delta}(x)$. Hence $h_{x,\delta} \otimes e \in \mathrm{Lip}(X, E)$ with $\|h_{x,\delta} \otimes e\|_\infty = 1$ and $(h_{x,\delta} \otimes e)(x) = e$. Then $(h_{x,\delta} \otimes e)(x) = \|h_{x,\delta} \otimes e\|_\infty e$ and thus $h_{x,\delta} \otimes e \in F(x)$.

We shall prove the theorem in a series of steps.

Step 1. Let $x \in X$. For each $f \in F(x)$, the set

$$P(f) = \{y \in Y : T(f)(y) = f(x)\}$$

is nonempty and closed.

Let $f \in F(x)$. If $f = 0$, then $P(f) = Y$ and there is nothing to prove. Suppose $f \neq 0$ and consider $g = \|f\|_\infty f + \|f\|^2 (1_X \otimes e)$. Clearly, $g \in \mathrm{Lip}(X, E)$ with $L(g) = \|f\|_\infty L(f)$ and $g(x) = \left(\|f\|_\infty^2 + \|f\|^2\right) e$. The latter equality implies $g \neq 0$. Since

$$L(g) \leq \|f\|_\infty \|f\| \leq \|f\|_\infty^2 + \|f\|^2 = \|g(x)\| \leq \|g\|_\infty ,$$

it follows that $\|g\| = \|g\|_\infty$. Moreover, $\|g\|_\infty = \|g(x)\| = \|f\|_\infty^2 + \|f\|^2$ since

$$\|g\|_\infty = \left\| \|f\|_\infty f + \|f\|^2 (1_X \otimes e) \right\|_\infty \leq \|f\|_\infty^2 + \|f\|^2 = \|g(x)\| .$$

We now claim that there exists a point $y \in Y$ such that $T(g/\|g\|)(y) = e$. Contrary to our claim, assume $e \neq T(g/\|g\|)(y)$ for all $y \in Y$. Let $\varepsilon > 0$ and take $h = g/\|g\| + \varepsilon(1_X \otimes e)$. Clearly, $h \in \mathrm{Lip}(X, E)$ and $T(h) = T(g)/\|g\| + \varepsilon(1_Y \otimes e)$. A simple calculation yields

$$L(T(h)) = L(T(g))/\|g\| \leq \|T(g)\| / \|g\| = 1.$$

Next we show that $\|T(h)\|_\infty < 1 + \varepsilon$. For any $y \in Y$, we have

$$\|T(h)(y)\| = \|T(g/\|g\|)(y) + \varepsilon e\| \leq 1 + \varepsilon$$

since $\|T(g/\|g\|)(y)\| \leq \|T(g)\| / \|g\| = 1$. Indeed,

$$\|T(g/\|g\|)(y) + \varepsilon e\| < 1 + \varepsilon.$$

Otherwise the vector $u = (1/(1+\varepsilon))(T(g/\|g\|)(y) + \varepsilon e)$ would be an extreme point of B_E by the strict convexity of E, and since u is a convex combination of $T(g/\|g\|)(y)$ and e, which are in B_E, we infer that $T(g/\|g\|)(y) = e$, a contradiction. Hence $\|T(h)(y)\| < 1 + \varepsilon$ for all $y \in Y$. Since $\|T(h)\|_\infty = \|T(h)(y)\|$ for some $y \in Y$, we conclude that $\|T(h)\|_\infty < 1 + \varepsilon$. From what we have proved above it is deduced that $\|T(h)\| < 1 + \varepsilon$, but, on the other hand,

$$1 + \varepsilon = \|g(x)/\|g\| + \varepsilon e\| = \|h(x)\| \leq \|h\|_\infty \leq \|h\| = \|T(h)\| ,$$

which is impossible. This proves our claim.

Now, let $y \in Y$ be such that $T(g/\|g\|)(y) = e$. Since $e = g(x)/\|g\|$, $Tg(y) = g(x)$, that is,

$$\|f\|_\infty Tf(y) + \|f\|^2 T(1_X \otimes e)(y) = \left(\|f\|_\infty^2 + \|f\|^2\right) e.$$

Since $T(1_X \otimes e) = 1_Y \otimes e$, we have

$$\|f\|_\infty T(f)(y) + \|f\|^2 e = \left(\|f\|_\infty^2 + \|f\|^2\right) e,$$

and thus $T(f)(y) = \|f\|_\infty e$, which is $T(f)(y) = f(x)$ since $f \in F(x)$. Hence $P(f) \neq \emptyset$. Moreover, $P(f)$ is closed in Y since $P(f) = T(f)^{-1}(\{f(x)\})$ and $T(f)$ is continuous.

Step 2. For each $x \in X$, the set

$$B(x) = \{y \in Y : T(f)(y) = f(x),\ \forall f \in F(x)\}$$

is nonempty and closed.

Let $x \in X$. For each $f \in F(x)$, $P(f)$ is a nonempty closed subset of Y by Step 1. Since $B(x) = \bigcap_{f\in F(x)} P(f)$, $B(x)$ is closed. To prove that $B(x) \neq \emptyset$, since Y is compact and $B(x) = \bigcap_{f\in F(x)} P(f)$, it suffices to check that if $f_1, ..., f_n \in F(x)$, then $\bigcap_{j=1}^n P(f_j) \neq \emptyset$.

We can suppose, without loss of generality, that $f_j \neq 0$ for all $j \in \{1, ..., n\}$ since $P(f_j) = Y$ if $f_j = 0$. For each $j \in \{1, ..., n\}$ define $g_j = \|f_j\|_\infty f_j + \|f_j\|^2 (1_X \otimes e)$. As in the proof of Step 1, $g_j \in \mathrm{Lip}(X, E)$ with $g_j(x) = \left(\|f_j\|_\infty^2 + \|f_j\|^2\right) e$ and $\|g_j\| = \|f_j\|_\infty^2 + \|f_j\|^2$. Hence $g_j \neq 0$ and we can define $h = (1/n) \sum_{j=1}^n (g_j / \|g_j\|)$. Clearly, $h \in \mathrm{Lip}(X, E)$, $h(x) = e$ and $\|h\|_\infty = 1$. Hence $h(x) = \|h\|_\infty e$ and thus $h \in F(x)$. Then, by Step 1, there exists a point $y \in Y$ such that $T(h)(y) = h(x)$. Since $T(h)(y) = (1/n) \sum_{j=1}^n (T(g_j)(y)/\|g_j\|)$ and $h(x) = e$, it follows that $e = (1/n) \sum_{j=1}^n (T(g_j)(y)/\|g_j\|)$. Since E is strictly convex and $\|T(g_j)(y)\| / \|g_j\| \leq \|T(g_j)\| / \|g_j\| = 1$ for all $j \in \{1, \ldots, n\}$, we infer that $T(g_j)(y) = \|g_j\| e$ for all $j \in \{1, \ldots, n\}$. Reasoning as in Step 1, we obtain $T(f_j)(y) = f_j(x)$ for all $j \in \{1, ..., n\}$ and thus $y \in \bigcap_{j=1}^n P(f_j)$.

Step 3. Let $f \in \mathrm{Lip}(X, E)$, $x \in X$ and $y \in B(x)$. If $f(x) = 0$, then $T(f)(y) = 0$.

If $f = 0$, then there is nothing to prove. Suppose $f \neq 0$ and let $\delta = \|f\|_\infty / \|f\|$. Clearly, $L(f)/\|f\|_\infty \leq 1/\delta$. Consider $h_{x,\delta} \otimes e \in F(x)$. We next prove that $f/\|f\|_\infty + (h_{x,\delta} \otimes e)$ belongs to $F(x)$. Since $f/\|f\|_\infty + (h_{x,\delta} \otimes e) \in \mathrm{Lip}(X, E)$ and $f(x)/\|f\|_\infty + (h_{x,\delta} \otimes e)(x) = e$, it suffices to check that $\|f/\|f\|_\infty + (h_{x,\delta} \otimes e)\|_\infty = 1$. Let $z \in X$. If $d(z, x) \geq \delta$, we have $(h_{x,\delta} \otimes e)(z) = 0$ and so

$$\|f(z)/\|f\|_\infty + (h_{x,\delta} \otimes e)(z)\| = \|f(z)\| / \|f\|_\infty \leq 1.$$

If $d(z, x) < \delta$, then $(h_{x,\delta} \otimes e)(z) = (1 - d(z, x)/\delta) e$, and therefore

$$\|f(z)/\|f\|_\infty + (h_{x,\delta} \otimes e)(z)\| \leq \|f(z)\| / \|f\|_\infty + 1 - d(z, x)/\delta \leq 1,$$

since

$$\|f(z)\| / \|f\|_\infty = \|f(z) - f(x)\| / \|f\|_\infty \leq L(f) d(z, x)/\|f\|_\infty \leq d(z, x)/\delta.$$

Hence $\|f/\|f\|_\infty + (h_{x,\delta} \otimes e)(z)\|_\infty \leq 1$. Since

$$\|f(x)/\|f\|_\infty + (h_{x,\delta} \otimes e)(x)\| = \|e\| = 1,$$

we obtain the desired condition.

By the definition of $B(x)$ it follows that

$$T\left(f/\|f\|_\infty + (h_{x,\delta} \otimes e)\right)(y) = \left(f/\|f\|_\infty + (h_{x,\delta} \otimes e)\right)(x),$$

that is, $T(f)(y)/\|f\|_\infty + T(h_{x,\delta} \otimes e)(y) = e$. Moreover, since $y \in B(x)$ and $h_{x,\delta} \otimes e \in F(x)$, we have $T(h_{x,\delta} \otimes e)(y) = (h_{x,\delta} \otimes e)(x) = e$. Hence $T(f)(y)/\|f\|_\infty + e = e$ and thus $T(f)(y) = 0$.

Step 4. Let $x, x' \in X$ with $x \neq x'$. Then $B(x) \cap B(x') = \emptyset$.

Suppose $y \in B(x) \cap B(x')$. Let $\delta = d(x, x') > 0$ and consider $h_{x,\delta} \otimes e$. Since $y \in B(x)$ and $h_{x,\delta} \otimes e \in F(x)$, we have $T(h_{x,\delta} \otimes e)(y) = (h_{x,\delta} \otimes e)(x) = e$ by Step 2, but Step 3 also yields $T(h_{x,\delta} \otimes e)(y) = 0$ since $y \in B(x')$ and $(h_{x,\delta} \otimes e)(x') = 0$. So we arrive at a contradiction. Hence $B(x) \cap B(x') = \emptyset$.

Steps 3 and 4 motivate the following:

Definition 1. Let $Y_0 = \bigcup_{x\in X} B(x)$. Define $\varphi : Y_0 \to X$ by $\varphi(y) = x$ if $y \in B(x)$.

Clearly, φ is surjective. Moreover, given $y \in Y_0$, there exists $x \in X$ such that $y \in B(x)$, and hence $\varphi(y) = x$ and $T(f)(y) = f(x)$ for all $f \in F(x)$.

We shall obtain the representation of T in terms of the following functions.

Definition 2. For each $y \in Y$, define $T_y : E \to E$ by $T_y(u) = T(1_X \otimes u)(y)$.

It is easy to show that $T_y \in L(E)$ with $\|T_y\| = 1 = \|T_y(e)\|$ for all $y \in Y$.

Step 5. The map $y \mapsto T_y$ from Y into $L(E)$ is Lipschitz.

Let $y, z \in Y$. Given $u \in E$, we have

$$\begin{aligned}\|(T_y - T_z)(u)\| &\leq L(T(1_X \otimes u))d(y,z)\\ &\leq \|T(1_X \otimes u)\|\, d(y,z) = \|u\|\, d(y,z),\end{aligned}$$

and thus $\|T_y - T_z\| \leq d(y,z)$.

Step 6. $T(f)(y) = T_y(f(\varphi(y)))$ for all $f \in \mathrm{Lip}(X,E)$ and $y \in Y_0$.

Let $f \in \mathrm{Lip}(X,E)$ and $y \in Y_0$. Let $x = \varphi(y) \in X$ and define $h = f - (1_X \otimes f(x))$. Obviously, $h \in \mathrm{Lip}(X,E)$ with $h(x) = 0$. From Step 3, we have $T(h)(y) = 0$ and therefore $T(f)(y) = T(1_X \otimes f(x))(y) = T_y(f(x)) = T_y(f(\varphi(y)))$.

Step 7. Y_0 is closed in Y.

Let $y \in Y$ and let $\{y_n\}$ be a sequence in Y_0 which converges to y. Let $x_n = \varphi(y_n)$ for all $n \in \mathbb{N}$. Since X is compact, there exists a subsequence $\left\{x_{\sigma(n)}\right\}$ converging to a point $x \in X$. Let $f \in F(x)$. Clearly, $\left\{T(f)(y_{\sigma(n)})\right\}$ converges to $T(f)(y)$, but also to $f(x)$ as we see at once. Indeed, for each $n \in \mathbb{N}$, we have

$$T(f)(y_{\sigma(n)}) = T_{y_{\sigma(n)}}(f(x_{\sigma(n)})) = T(1_X \otimes f(x_{\sigma(n)}))(y_{\sigma(n)}),$$

by Step 6, and

$$\begin{aligned}f(x) &= \|f\|_\infty\, e = \|f\|_\infty\, (1_Y \otimes e)(y_{\sigma(n)})\\ &= \|f\|_\infty\, T(1_X \otimes e)(y_{\sigma(n)}) = T(1_X \otimes f(x))(y_{\sigma(n)}),\end{aligned}$$

since $f \in F(x)$. We deduce that

$$\begin{aligned}\left\|T(f)(y_{\sigma(n)}) - f(x)\right\| &= \left\|T(1_X \otimes (f(x_{\sigma(n)}) - f(x)))(y_{\sigma(n)})\right\|\\ &\leq \left\|T(1_X \otimes (f(x_{\sigma(n)}) - f(x)))\right\| = \left\|1_X \otimes (f(x_{\sigma(n)}) - f(x))\right\|\\ &= \left\|f(x_{\sigma(n)}) - f(x)\right\|\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\left\{f(x_{\sigma(n)})\right\} \to f(x)$, we conclude that $\left\{T(f)(y_{\sigma(n)})\right\} \to f(x)$. Hence $T(f)(y) = f(x)$ and thus $y \in B(x) \subset Y_0$.

Step 8. The map $\varphi : Y_0 \to X$ is Lipschitz and $L(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$.

Let $y, z \in Y_0$ be such that $\varphi(y) \neq \varphi(z)$ and put $\delta = d(\varphi(y), \varphi(z))/2$. Define $f_{y,z} = \delta(h_{\varphi(y),\delta} - h_{\varphi(z),\delta})$ on X. It is easy to see that $f_{y,z} \in \mathrm{Lip}(X)$ and $\|f_{y,z}\| \leq k := \max\{1, \mathrm{diam}(X)/2\}$. Since T is an isometry, $\|T(f_{y,z} \otimes e)\| \leq k$. This inequality implies $L(T(f_{y,z} \otimes e)) \leq k$. It follows that

$$\|T(f_{y,z} \otimes e)(y) - T(f_{y,z} \otimes e)(z)\| \leq kd(y,z).$$

Using Step 6 we get

$$\begin{aligned}T(f_{y,z} \otimes e)(y) &= T_y((f_{y,z} \otimes e)(\varphi(y))) = T_y(\delta e) = \delta e,\\ T(f_{y,z} \otimes e)(z) &= T_z((f_{y,z} \otimes e)(\varphi(z))) = T_z(-\delta e) = -\delta e.\end{aligned}$$

We conclude that $d(\varphi(y), \varphi(z)) \leq kd(y,z)$. $\square$

The condition in Theorem 2.1, $T(1_X \otimes e) = 1_Y \otimes e$ for some $e \in S_E$, is not too restrictive if we analyse the known results in the scalar case. In this case our condition means $T(1_X) = 1_Y$; notice that the connectedness assumptions on the metric spaces in [9, Lemma 1.5] and [11, Lemma 6] yield a similar condition, namely, that $T(1_X)$ is a constant function.

3. A Lipschitz version of Jerison's theorem

Recall that a map between metric spaces $\varphi : X \to Y$ is said to be a *Lipschitz homeomorphism* if φ is bijective and φ and φ^{-1} are both Lipschitz.

Theorem 3.1. *Let X, Y be compact metric spaces and let E be a strictly convex Banach space. Let T be a linear isometry from $\mathrm{Lip}(X,E)$ onto $\mathrm{Lip}(Y,E)$ such that $T(1_X \otimes e) = 1_Y \otimes e$ for some $e \in S_E$. Then there exists a Lipschitz homeomorphism $\varphi : Y \to X$ with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$ and $L(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)/2\}$, and a Lipschitz map $y \mapsto T_y$ from Y into $L(E)$ where T_y is an isometry from E onto itself for all $y \in Y$ such that*

$$T(f)(y) = T_y(f(\varphi(y))), \quad \forall y \in Y,\ \forall f \in \mathrm{Lip}(X,E).$$

Proof. Let Y_0 and φ be as in Theorem 2.1. Since $T^{-1} : \mathrm{Lip}(Y,E) \to \mathrm{Lip}(X,E)$ is a linear isometry and $T^{-1}(1_Y \otimes e) = 1_X \otimes e$, applying Theorem 2.1 we have

$$T^{-1}(g)(x) = (T^{-1})_x(g(\psi(x))), \quad \forall x \in X_0,\ \forall g \in \mathrm{Lip}(Y,E),$$

where ψ is a Lipschitz map from a closed subset X_0 of X onto Y with $L(\psi) \leq \max\{1, \mathrm{diam}(Y)/2\}$, and $x \mapsto (T^{-1})_x$ is a Lipschitz map from X into $L(E)$. Namely, $X_0 = \bigcup_{y \in Y} B(y)$ where, for each $y \in Y$,

$$B(y) = \left\{x \in X : T^{-1}(g)(x) = g(y),\ \forall g \in F(y)\right\}$$

with

$$F(y) = \{g \in \mathrm{Lip}(Y,E) : g(y) = \|g\|_\infty e\},$$

and $\psi : X_0 \to Y$ is the Lipschitz map defined by $\psi(x) = y$ if $x \in B(y)$. Moreover, using the same arguments as in Step 3, the following can be proved:

Claim 1. Let $g \in \mathrm{Lip}(Y,E)$, $y \in Y$ and $x \in B(y)$. If $g(y) = 0$, then $T^{-1}(g)(x) = 0$.

After this preparation we proceed to prove the theorem. Fix $x \in X$ and let $y \in B(x)$. We first prove that $x \in B(y)$. Suppose that $x \notin B(y)$. Since $B(y) \neq \emptyset$, there exists $x' \in B(y)$ with $x' \neq x$. Take $f \in \mathrm{Lip}(X,E)$ for which $f(x) = 0$ and $f(x') \neq 0$. Since $y \in B(x)$ and $f(x) = 0$, we have $T(f)(y) = 0$ by Step 3. Then $T^{-1}(T(f))(x') = 0$ since $x' \in B(y)$ by Claim 1, and thus $f(x') = 0$, a contradiction. Therefore $x \in B(y) \subset X_0$ and thus $X_0 = X$. Next we see that $Y_0 = Y$. Let $y \in Y$. We can take a point $x \in B(y)$. As above it is proved that $y \in B(x)$ and thus $y \in Y_0$.

To see that φ is a Lipschitz homeomorphism, let $y \in Y$. Then $y \in B(x)$ for some $x \in X$, that is, $\varphi(y) = x$. Moreover, by what we have proved above, $x \in B(y)$ and so $\psi(x) = y$. As a consequence, $\psi(\varphi(y)) = y$. Since φ was surjective, φ is bijective with $\varphi^{-1} = \psi$ and thus φ is a Lipschitz homeomorphism.

To check that T_y is an isometry from E into itself for every $y \in Y$, we first show that T sends nonvanishing functions of $\mathrm{Lip}(X,E)$ into nonvanishing functions of $\mathrm{Lip}(Y,E)$. Assume there exists $f \in \mathrm{Lip}(X,E)$ such that $f(x) \neq 0$ for all $x \in X$, but $T(f)(y) = 0$ for some $y \in Y$. By the surjectivity of ψ, there is a point $x \in X_0$ such that $\psi(x) = y$, that is, $x \in B(y)$. Since $T(f)(y) = 0$, by Claim 1 we have $f(x) = T^{-1}(T(f))(x) = 0$, a contradiction. Hence T maps nonvanishing functions into nonvanishing functions. If, for some $y \in Y$, T_y is not an isometry, then there exists a $u \in S_E$ such that $\|T_y(u)\| = \|T(1_X \otimes u)(y)\| < 1$. Since T is surjective, there is an $f \in \mathrm{Lip}(X, E)$ such that $T(f) = 1_Y \otimes T(1_X \otimes u)(y)$. Thus $\|f\|_\infty \leq \|f\| = \|T(f)\| = \|T(1_X \otimes u)(y)\| < 1$ and $(1_X \otimes u) - f$ never vanishes on X. As $T(1_X \otimes u)(y) = T(f)(y)$, we arrive at a contradiction.

Next we prove that $T_y : E \to E$ is surjective for every $y \in Y$. Fix $y \in Y$ and let $v \in E$. Since T is surjective, there exists $f \in \mathrm{Lip}(X, E)$ such that $T(f) = 1_Y \otimes v$. Let $u = (f \circ \varphi)(y) \in E$. Using Step 6, we have $T_y(u) = T_y(f(\varphi(y))) = T(f)(y) = v$. Hence T_y is surjective. □

Finally, as a direct consequence of Theorem 3.1, we obtain the following:

Corollary 3.2. *Let X, Y be compact metric spaces with diameter at most 2 and let E be a strictly convex Banach space. Then every surjective linear isometry T from $\mathrm{Lip}(X, E)$ into $\mathrm{Lip}(Y, E)$ satisfying that $T(1_X \otimes e) = 1_Y \otimes e$ for some $e \in S_E$, can be expressed as $T(f)(y) = T_y(f(\varphi(y)))$ for all $y \in Y$ and $f \in \mathrm{Lip}(X, E)$, where $\varphi : Y \to X$ is a surjective isometry and $y \mapsto T_y$ is a Lipschitz map from Y into $L(E)$ such that T_y is an isometry from E onto E for all $y \in Y$.*

In the special case that E is a Hilbert space, Theorems 2.1 and 3.1 can be improved as follows. For a Hilbert space E, let us recall that a *unitary operator* is a linear map $\Phi : E \to E$ that is a surjective isometry.

Corollary 3.3. *Let X and Y be compact metric spaces and let E be a Hilbert space. Let T be a linear isometry from $\mathrm{Lip}(X, E)$ into $\mathrm{Lip}(Y, E)$ such that $T(1_X \otimes e)$ is a constant function for some $e \in S_E$. Then there exists a Lipschitz map φ from a closed subset Y_0 of Y onto X with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$ and a Lipschitz map $y \mapsto T_y$ from Y into $L(E)$ with $\|T_y\| = 1$ for all $y \in Y$ such that*

$$T(f)(y) = T_y(f(\varphi(y))), \quad \forall y \in Y_0, \ \forall f \in \mathrm{Lip}(X, E).$$

If, in addition, T is surjective, then $Y_0 = Y$, φ is a Lipschitz homeomorphism with $L(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)/2\}$ and, for each $y \in Y$, T_y is a unitary operator.

Proof. Assume that $T(1_X \otimes e) = 1_Y \otimes u$ for some $u \in E$. Obviously, $\|u\| = 1$. Since E is a Hilbert space, we can construct a unitary operator $\Phi : E \to E$ such that $\Phi(u) = e$. Define $S : \mathrm{Lip}(Y, E) \to \mathrm{Lip}(Y, E)$ by

$$S(g)(y) = \Phi(g(y)), \quad \forall y \in Y, \ \forall g \in \mathrm{Lip}(Y, E).$$

It is easy to prove that S is a surjective linear isometry satisfying that $S(1_Y \otimes u) = 1_Y \otimes e$. Hence $R = S \circ T$ is a linear isometry from $\mathrm{Lip}(X, E)$ into $\mathrm{Lip}(Y, E)$ with $R(1_X \otimes e) = 1_Y \otimes e$. Then Theorem 2.1 guarantees the existence of a Lipschitz map φ from a closed subset Y_0 of Y onto X with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$ and a Lipschitz map $y \mapsto R_y$ from Y into $L(E)$ with $\|R_y\| = 1$ for all $y \in Y$ such that

$$R(f)(y) = R_y(f(\varphi(y))), \quad \forall y \in Y_0, \ \forall f \in \mathrm{Lip}(X, E).$$

For each $y \in Y$, consider $T_y = \Phi^{-1} \circ R_y \in L(E)$. It is easily seen that the map $y \mapsto T_y$ from Y into $L(E)$ is Lipschitz with $\|T_y\| = 1$ for all $y \in Y$. Moreover, for any $y \in Y_0$ and $f \in \mathrm{Lip}(X, E)$, we have

$$T(f)(y) = \Phi^{-1}(R_y(f(\varphi(y)))) = T_y(f(\varphi(y))).$$

If, in addition, T is surjective, the rest of the corollary follows by applying Theorem 3.1 to R. □

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