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# INTO LINEAR ISOMETRIES BETWEEN SPACES OF LIPSCHITZ FUNCTIONS

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ABSTRACT. In this paper we state a Lipschitz version of a known Holsztyński's theorem on linear isometries of $\mathcal{C}(X)$-spaces. Let $\mathrm{Lip}(X)$ be the Banach space of all scalar-valued Lipschitz functions $f$ on a compact metric space $X$ endowed with the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$, where $L(f)$ is the Lipschitz constant of $f$. We prove that any linear isometry $T$ from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ satisfying that $L(T1_X) < 1$ is essentially a weighted composition operator

$$Tf(y) = \tau(y)f(\varphi(y)) \quad (f \in \mathrm{Lip}(X),\ y \in Y_0),$$

where $Y_0$ is a closed subset of $Y$, $\varphi$ is a Lipschitz map from $Y_0$ onto $X$ with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ and $\tau$ is a function in $\mathrm{Lip}(Y)$ with $\|\tau\| = 1$ and $|\tau(y)| = 1$ for all $y \in Y_0$. We improve this representation in the case of onto linear isometries and we classify codimension 1 linear isometries in two types.

## 1. INTRODUCTION

A map $f : X \to Y$ between metric spaces is said to be a Lipschitz map if there exists a constant $k$ such that $d(f(x), f(z)) \leq kd(x, z)$ for all $x, z \in X$. We shall use the letter $d$ to denote the distance in any metric space.

Let $X$ be a compact metric space and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. The space $\mathrm{Lip}(X)$ is the Banach space of all Lipschitz functions $f$ from $X$ into $\mathbb{K}$, with the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$, where $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$ is the

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supremum norm of $f$ and $L(f) = \sup\{|f(x) - f(y)|/d(x,y) : x, y \in X,\ x \neq y\}$ is the Lipschitz constant of $f$.

The study of linear isometries of spaces $\mathrm{Lip}(X)$ goes back to the sixties when Roy [14] described the surjective linear isometries $T$ of $\mathrm{Lip}(X)$ in the case that $X$ is connected and its diameter $\mathrm{diam}(X)$ is at most 1. Namely, he proved that such an isometry $T$ has the canonical form:

$$Tf(y) = \tau f(\varphi(y)) \quad (f \in \mathrm{Lip}(X),\ y \in Y),$$

where $\varphi$ is an isometry from $Y$ onto $X$ and $\tau$ is a scalar of $S_{\mathbb{K}}$, the set of all unimodular elements of $\mathbb{K}$. Novinger [13] extended Roy's result to the case of linear isometries of $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ when $X$ and $Y$ are connected with diameter at most 1. Vasavada's result in [16] generalizes the aforementioned results since it states that if $X$ and $Y$ are $\beta$-connected for some $\beta < 1$ with diameter at most 2, then any linear isometry from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ arises from an isometry from $Y$ onto $X$ as in the aforementioned canonical form. Let us recall that a metric space $X$ is $\beta$-connected if it cannot be decomposed into two nonempty subsets $A$ and $B$ such that $d(a,b) \geq \beta$ for every $a \in A$ and $b \in B$.

As it will have could to notice, the surjective isometries of $\mathrm{Lip}(X)$ have a valuable literature. However little has been published about the into isometries of $\mathrm{Lip}(X)$, that is not necessarily surjective. This fact is also meaningful if we compare it to the formidable literature existing about into isometries in the context of the Banach spaces $\mathcal{C}(X)$ of scalar-valued continuous functions on a compact Hausdorff space $X$ with the supremum norm.

The classical Banach–Stone theorem states that if $T$ is a linear isometry from $\mathcal{C}(X)$ onto $\mathcal{C}(Y)$, then there exists a homeomorphism from $Y$ onto $X$ and a continuous function $\tau$ from $Y$ into $S_{\mathbb{K}}$ such that

$$Tf(y) = \tau(y) f(\varphi(y)) \quad (f \in \mathcal{C}(X),\ y \in Y).$$

An important generalization of this theorem was given by Holsztyński in [9] by considering into isometries. He proved that if $T$ is a linear isometry from $\mathcal{C}(X)$ into $\mathcal{C}(Y)$, then there exists a closed subset $Y_0$ of $Y$, a continuous map $\varphi$ from $Y_0$ onto $X$ and a function $\tau \in \mathcal{C}(Y)$ with $\|\tau\|_\infty = 1$ and $|\tau(y)| = 1$ for all $y \in Y_0$ such that

$$Tf(y) = \tau(y) f(\varphi(y)) \quad (f \in \mathcal{C}(X),\ y \in Y_0).$$

This result has been extended in many directions. We can cite, for example, the generalizations obtained by Cambern [4] for spaces of vector-valued continuous functions, by Moreno and Rodríguez [12] with a bilinear version, by Jeang and Wong for spaces of scalar-valued continuous functions vanishing at infinity [10], by Araujo and Font for certain subspaces of scalar-valued continuous functions [2] and by many other authors.

The object of this paper is to show that Holsztyński's theorem has a natural formulation in the context of the spaces $\mathrm{Lip}(X)$. We focus our attention on linear isometries $T$ from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ for which $T1_X$ is a contraction, where $1_X$ denotes the function constantly equal 1 on $X$. We recall that a Lipschitz function $f$ is a contraction if $L(f) < 1$.

Our main theorem (Theorem 2.4) states that any linear isometry $T$ from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ for which $T1_X$ is a contraction, is essentially a weighted composition operator

$$Tf(y) = \tau(y)f(\varphi(y)) \quad (f \in \mathrm{Lip}(X),\ y \in Y_0),$$

where $Y_0$ is a closed subset of $Y$, $\varphi$ is a Lipschitz map from $Y_0$ onto $X$ with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ and $\tau$ is a function of $\mathrm{Lip}(Y)$ with $\|\tau\| = 1$ and $|\tau(y)| = 1$ for all $y \in Y_0$. Namely, the weight function $\tau$ is $T1_X$. Moreover, we show (Corollary 2.5) that $Y_0$ is the largest subset of $Y$ on which we can define a Lipschitz map $\varphi$ with values in $X$ satisfying the equality above.

We use extreme point techniques to prove our theorem as they do in [11], [13], [14] and [16]. This method of proof is still used to obtain similar results [1] and seems to date from the known proof of the Banach–Stone theorem given by Dunford and Schwartz in [5].

Our theorem is not true when $L(T1_X) = 1$ (see [17, p. 61] for a counterexample). We must point out also that, in the aforementioned papers, the connectedness condition imposed on the metric spaces is used to prove that $T1_X$ is a constant function, in whose case $L(T1_X) = 0$. On the other hand, we want to emphasize that Vasavada's reduction to metric spaces of diameter at most 2 is not restrictive because if $(X, d)$ is a compact metric space and $X'$ is the set $X$ remetrized with the metric $d'(x, y) = \min\{d(x, y), 2\}$, then $\mathrm{diam}(X') \leq 2$ and $\mathrm{Lip}(X')$ is isometrically isomorphic to $\mathrm{Lip}(X)$ (see [17, Proposition 1.7.1]).

Our theorem also provides some new information concerning the onto case. We show (Theorem 3.1) that any linear isometry $T$ from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ such that $T1_X$ is a nonvanishing contraction, is a weighted composition operator

$$Tf(y) = \tau(y)f(\varphi(y)) \quad (f \in \mathrm{Lip}(X),\ y \in Y),$$

where $\varphi$ is a Lipschitz homeomorphism from $Y$ onto $X$ and $\tau$ is a Lipschitz function from $Y$ into $S_{\mathbb{K}}$. Our approach is different from one which Novinger [13], Roy [14] , Vasavada [16] and Weaver [17, Theorem 2.6.7] present since they impose conditions of connectedness. Let us recall that a map between metric spaces $\varphi : X \to Y$ is a Lipschitz homeomorphism if $\varphi$ is a bijection such that $\varphi$ and $\varphi^{-1}$ are both Lipschitz, and a function $f : X \to \mathbb{K}$ is said to be nonvanishing if $f(x) \neq 0$ for all $x \in X$.

In [17, Theorem 2.6.7] Weaver obtained a noncompact version of Vasavada's result. He defined $\mathrm{Lip}(X)$ as the space of all bounded Lipschitz scalar-valued functions $f$ on a metric space $X$ with the norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$, and showed that if $X$ and $Y$ are complete 1-connected with diameter at most 2, then every linear isometry $T$ from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ is of the form

$$Tf(y) = \tau f(\varphi(y)) \quad (f \in \mathrm{Lip}(X),\ y \in Y),$$

where $\varphi$ is an isometry from $Y$ onto $X$ and $\tau$ is a unimodular constant.

Theorem 3.1 can be improved with the aid of this Weaver's result. Namely, we show (Theorem 3.3) that if $X$ and $Y$ are compact metric spaces with diameter at most 2 and $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is a surjective linear isometry such that $T1_X$ is a nonvanishing contraction, then there exists a surjective isometry $\varphi : Y \to X$ and a function $\tau : Y \to S_{\mathbb{K}}$ with $\tau(x) = \tau(y)$ whenever $d(x, y) < 2$ such that

$$Tf(y) = \tau(y)f(\varphi(y)), \quad \forall f \in \mathrm{Lip}(X),\ \forall y \in Y.$$

In the final section we classify codimension 1 linear isometries between $\mathrm{Lip}(X)$-spaces in two types. If $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is such an isometry with $L(T1_X) < 1$, Theorem 4.1 asserts the existence of a closed subset $Y_0$ of $Y$ such that either $Y_0 = Y\backslash\{p\}$, where $p$ is an isolated point of $Y$, or $Y_0 = Y$; a surjective Lipschitz map $\varphi : Y_0 \to X$ and a unimodular Lipschitz function $\tau : Y_0 \to \mathbb{K}$ such that $Tf(y) = \tau(y)f(\varphi(y))$ for all $y \in Y_0$. If $Y\backslash Y_0$ is just a single point or $Y_0 = Y$, we label these isometries as of type I and of type II, respectively. These two types are not disjoint. In fact, we give a method for constructing codimension 1 linear isometries which are simultaneously of types I and II (Proposition 4.2). We also give examples of type I codimension 1 linear isometries which are not of type II (Proposition 4.3), and vice versa (Example 4.4). The remainder of section is devoted to study the properties of $\varphi$ (Proposition 4.6).

In the last years, several authors have investigated about codimension 1 linear isometries on the space $\mathcal{C}(X)$ ([3], [6], [8], among others many). However the key article is due to Gutek, Hart, Jamison and Rajagopalan [7]. These authors studied shift operators on $\mathcal{C}(X)$ and classified these operators using the aforementioned Holsztyński's theorem [9]. In this paper we have followed a similar way to study codimension 1 linear isometries between $\mathrm{Lip}(X)$-spaces applying now our Lipschitz version of the cited theorem.

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## 2. Into linear isometries

We begin by recalling some results which describe partially the set of extreme points of the closed unit ball of the dual space of $\mathrm{Lip}(X)$.

For a Banach space $E$, we denote by $B_E$ the closed unit ball of $E$, by $S_E$ the unit sphere of $E$, by $\mathrm{Ext}(B_E)$ the set of extreme points of $B_E$ and by $E^*$ the dual space of $E$.

Given a compact metric space $X$, let $\widetilde{X} = \{(x,y) \in X^2 : x \neq y\}$ and let the compact Hausdorff space $W$ be the disjoint union of $X$ with $\beta\widetilde{X}$, where $\beta\widetilde{X}$ is the Stone-Čech compactification of $\widetilde{X}$. Consider the mapping $\Phi : \mathrm{Lip}(X) \to \mathcal{C}(W)$ defined for each $f \in \mathrm{Lip}(X)$ by

$$\Phi f(w) = \begin{cases} f(w) & \text{if } w \in X, \\ \\ (\beta f^*)(w) & \text{if } w \in \beta\widetilde{X}, \end{cases}$$

where

$$f^*(x,y) = \frac{f(x) - f(y)}{d(x,y)}, \quad \forall (x,y) \in \widetilde{X},$$

and $\beta f^*$ is its norm-preserving extension to $\beta\widetilde{X}$. It is easily seen that $\Phi$ is a linear isometry from $\mathrm{Lip}(X)$ into $\mathcal{C}(W)$.

For each $w \in W$, define the functionals $\delta_w \in \mathcal{C}(W)^*$ and $\widetilde{\delta}_w \in \mathrm{Lip}(X)^*$ by $\delta_w(f) = f(w)$ and $\widetilde{\delta}_w(f) = \Phi f(w)$, respectively. Clearly, $\left|\widetilde{\delta}_w(f)\right| \leq \|f\|$ for all $f \in \mathrm{Lip}(X)$ and therefore $\widetilde{\delta}_w \in B_{\mathrm{Lip}(X)^*}$. It is well known (see [5, p. 441]) that the extreme points of $B_{\mathrm{Lip}(X)^*}$ are essentially of this form:

**Lemma 2.1.** *Every extreme point of $B_{\mathrm{Lip}(X)^*}$ must be either of the form $\tau\widetilde{\delta}_x$ with $\tau \in S_{\mathbb{K}}$ and $x \in X$ or of the form $\tau\widetilde{\delta}_w$ with $\tau \in S_{\mathbb{K}}$ and $w \in \beta\widetilde{X}$.*

We also shall need the following fact which was proved by Roy [14, Lemma 1.2] using a result of de Leeuw [11, Lemma 3.2]:

**Lemma 2.2.** *For each $x \in X$, $\widetilde{\delta}_x$ is an extreme point of $B_{\mathrm{Lip}(X)^*}$.*

An application of the Hahn–Banach and Krein–Milman theorems yields the following fact surely known.

**Lemma 2.3.** *Let $X$ be a normed space and $M$ a vector subspace of $X$. For each $g \in \mathrm{Ext}(B_{M^*})$ there exists $f \in \mathrm{Ext}(B_{X^*})$ such that $f|_M = g$.*

Finally, we present two families of Lipschitz functions which will be used frequently throughout this paper.

Let $X$ be a compact metric space. For each $x \in X$, the real function $f_x$, defined on $X$ by $f_x(z) = d(z, x)$, belongs to $\mathrm{Lip}(X)$ with $L(f_x) \leq 1$ and $\|f_x\|_\infty \leq \mathrm{diam}(X)$. Also, for each $x \in X$ and $\delta > 0$, the function $h_x^\delta : X \to [0, 1]$ given by

$$h_x^\delta(z) = \max\left\{0, 1 - \frac{d(z, x)}{\delta}\right\}$$

is also in $\mathrm{Lip}(X)$ with $L(h_x^\delta) \leq 1/\delta$ and $\left\|h_x^\delta\right\|_\infty \leq 1$.

After this preparation, we formulate our main result which is a version for isometries of $\mathrm{Lip}(X)$-spaces of a known Holsztyński's theorem on isometries of $\mathcal{C}(X)$-spaces [9].

**Theorem 2.4.** *Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a linear isometry and suppose $T1_X$ is a contraction. Then there exists a closed subset $Y_0$ of $Y$, a surjective Lipschitz map $\varphi : Y_0 \to X$ with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ and a function $\tau \in \mathrm{Lip}(Y)$ with $\|\tau\| = 1$ and $|\tau(y)| = 1$ for all $y \in Y_0$ such that*

$$Tf(y) = \tau(y)f(\varphi(y)), \quad \forall f \in \mathrm{Lip}(X),\ \forall y \in Y_0.$$

Proof. Let $\tau = T1_X$. Evidently, $\tau \in \mathrm{Lip}(Y)$ and $\|\tau\| = \|1_X\| = 1$. Let $Z = T(\mathrm{Lip}(X))$ and define

$$Y_0 = \left\{y \in Y : \overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*}),\ |\tau(y)| = 1\right\}.$$

We first prove that $Y_0$ is nonempty. Since $T$ is a linear isometry from $\mathrm{Lip}(X)$ onto $Z$, the adjoint map $T^* : \mathrm{Lip}(Y)^* \to \mathrm{Lip}(X)^*$ is also a linear isometry from $Z^*$ onto $\mathrm{Lip}(X)^*$ and therefore $T^*$ induces a bijection from $\mathrm{Ext}(B_{Z^*})$ onto $\mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$.

Let $x \in X$. By Lemma 2.2, $\widetilde{\delta}_x \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$. Therefore $T^*\mu = \widetilde{\delta}_x$ for some $\mu \in \mathrm{Ext}(B_{Z^*})$. By Lemma 2.3, $\mu$ is the restriction to $Z$ of an extreme point of $B_{\mathrm{Lip}(Y)^*}$. Hence $\mu = \alpha \left.\widetilde{\delta}_w\right|_Z$ for some $\alpha \in S_{\mathbb{K}}$ and $w \in Y \cup \beta\widetilde{Y}$ by Lemma 2.1 and so $\alpha T^* \left.\widetilde{\delta}_w\right|_Z (1_X) = \widetilde{\delta}_x(1_X)$. We now see that $w \in Y$. If there were $w \in \beta\widetilde{Y}$, we should have $\left|\alpha T^* \left.\widetilde{\delta}_w\right|_Z (1_X)\right| < 1$ since

$$\left|\alpha T^* \left.\widetilde{\delta}_w\right|_Z (1_X)\right| = \left|\widetilde{\delta}_w(T1_X)\right| = |\beta(T1_X)^*(w)| < 1.$$

It suffices to observe that

$$|(T1_X)^*(y, z)| = \frac{|T1_X(y) - T1_X(z)|}{d(y, z)} \leq L(T1_X) < 1$$

for all $(y,z) \in \widetilde{Y}$ and that $\widetilde{Y}$ is dense in $\beta\widetilde{Y}$. But, on the other hand, $\widetilde{\delta}_x(1_X) = 1$. This contradiction gives us $w = y$ for some $y \in Y$. It follows that

$$1 = \widetilde{\delta}_x(1_X) = \alpha\widetilde{\delta}_y(T(1_X)) = \alpha T(1_X)(y) = \alpha\tau(y).$$

From this it is deduced that $|\tau(y)| = 1$ and $\alpha = \overline{\tau(y)}$. Hence $\overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z = \widetilde{\delta}_x \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$ and so $y \in Y_0$.

We next show that for each $y \in Y_0$, there exists a unique point $x \in X$ such that $\overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z = \widetilde{\delta}_x$. Let $y \in Y_0$. Since $\overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$ by definition of $Y_0$, Lemma 2.1 yields $\overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z = \alpha\widetilde{\delta}_w$ for suitable $\alpha \in S_{\mathbb{K}}$ and $w \in X \cup \beta\widetilde{X}$. We show that $w \in X$. If $w \in \beta\widetilde{X}$, it is clear that $\alpha\widetilde{\delta}_w(1_X) = 0$, but

$$\overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z (1_X) = \overline{\tau(y)}\widetilde{\delta}_y(T1_X) = \overline{\tau(y)}T1_X(y) = |\tau(y)|^2 = 1.$$

This contradiction proves that $w = x$ for some $x \in X$. Then

$$\alpha = \alpha\widetilde{\delta}_x(1_X) = \overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z (1_X) = 1$$

and so $\overline{\tau(y)}T^*(\left.\widetilde{\delta}_y\right|_Z) = \widetilde{\delta}_x$. This proves the existence of $x$.

To show its uniqueness, assume there exists $x' \in X$ such that $\overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z = \widetilde{\delta}_{x'}$. Consider the function $f_x \in \mathrm{Lip}(X)$. If there were $x' \neq x$, we would have

$$\overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z (f_x) = \widetilde{\delta}_{x'}(f_x) = f_x(x') = d(x', x) \neq 0,$$

but also

$$\overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z (f_x) = \widetilde{\delta}_x(f_x) = f_x(x) = 0.$$

This contradiction gives us $x' = x$.

Let $\varphi : Y_0 \to X$ be the map defined by $\varphi(y) = x$ whenever $\overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z = \widetilde{\delta}_x$. Clearly, $T^* \left.\widetilde{\delta}_y\right|_Z = \tau(y)\widetilde{\delta}_{\varphi(y)}$ for each $y \in Y_0$ and therefore

$$Tf(y) = \tau(y)f(\varphi(y)), \quad \forall f \in \mathrm{Lip}(X),\ \forall y \in Y_0.$$

The map $\varphi : Y_0 \to X$ is surjective. This is proved as the existence of points in $Y_0$. We now check that $\varphi : Y_0 \to X$ is Lipschitz. Notice that for each $x \in X$, $f_x \in \mathrm{Lip}(X)$ and $\|f_x\| \leq k$ where $k = \max\{1, \mathrm{diam}(X)\}$. Hence $\left\|f_{\varphi(y)}\right\| \leq k$ for all $y \in Y_0$. Since $T$ is a linear isometry, it follows that $\left\|Tf_{\varphi(y)}\right\| \leq k$

for all $y \in Y_0$. Then $L(Tf_{\varphi(y)}) \leq k$ for all $y \in Y_0$. Let $y, z \in Y_0$. We have $\left|Tf_{\varphi(y)}(y) - Tf_{\varphi(y)}(z)\right| \leq kd(y,z)$. An easy calculation yields

$$\begin{aligned} Tf_{\varphi(y)}(y) &= \tau(y)f_{\varphi(y)}(\varphi(y)) = \tau(y)d(\varphi(y),\varphi(y)) = 0, \\ Tf_{\varphi(y)}(z) &= \tau(z)f_{\varphi(y)}(\varphi(z)) = \tau(z)d(\varphi(y),\varphi(z)), \end{aligned}$$

and thus $d(\varphi(y),\varphi(z)) \leq kd(y,z)$.

Finally, we see that $Y_0$ is closed in $Y$. To prove this, given $x \in X$ and $y \in Y$, we notice that $y \in Y_0$ and $\varphi(y) = x$ if and only if $|Tf(y)| = |f(x)|$ for all $f \in \mathrm{Lip}(X)$. Indeed, if $y \in Y_0$ and $\varphi(y) = x$, then $T^* \left.\widetilde{\delta}_y\right|_Z = \tau(y)\widetilde{\delta}_x$; hence, for all $f \in \mathrm{Lip}(X)$, we have

$$Tf(y) = T^* \left.\widetilde{\delta}_y\right|_Z (f) = \tau(y)\widetilde{\delta}_x(f) = \tau(y)f(x),$$

and so $|Tf(y)| = |f(x)|$. Conversely, if $|Tf(y)| = |f(x)|$ for all $f \in \mathrm{Lip}(X)$, then $|\tau(y)| = 1$ and $\ker T^* \left.\widetilde{\delta}_y\right|_Z = \ker \widetilde{\delta}_x$. This last implies that $T^* \left.\widetilde{\delta}_y\right|_Z = \alpha\widetilde{\delta}_x$ for some nonzero scalar $\alpha$. In particular, we deduce that $\tau(y) = \alpha$ and thus $\overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z = \widetilde{\delta}_x$. This says us that $y \in Y_0$ and $\varphi(y) = x$.

To show that $Y_0$ is closed in $Y$, let $\{y_n\}$ be a sequence in $Y_0$ converging to a point $y \in Y$. For each natural $n$, let $x_n = \varphi(y_n)$. By the compactness of $X$, $\{x_n\}$ has a subsequence $\left\{x_{\sigma(n)}\right\}$ which converges to a point $x \in X$. Let $f \in \mathrm{Lip}(X)$. We have $\left|Tf(y_{\sigma(n)})\right| = \left|f(x_{\sigma(n)})\right|$ for all $n \in \mathbb{N}$. It follows that $|T(f)(y)| = |f(x)|$. Since $f$ was arbitrary, the remark above gives us $y \in Y_0$ and so $Y_0$ is closed in $Y$. $\square$

The next result shows that the triple $\{Y_0, \tau, \varphi\}$ associated to the isometry $T$ in Theorem 2.4 possesses a universal property.

**Corollary 2.5.** *Let $T$ be a linear isometry from* $\mathrm{Lip}(X)$ *into* $\mathrm{Lip}(Y)$ *for which $T1_X$ is a contraction. Let $Y_0$, $\tau$ and $\varphi$ be as in Theorem 2.4. If $Y_0'$ is a subspace (not necessarily closed) of $Y$, and $\tau' : Y_0' \to S_{\mathbb{K}}$ and $\varphi' : Y_0' \to X$ are Lipschitz maps such that*

$$Tf(y) = \tau'(y)f(\varphi'(y)), \quad \forall f \in \mathrm{Lip}(X),\ \forall y \in Y_0',$$

*then $Y_0' \subset Y_0$, $\tau' = \tau|_{Y_0'}$ and $\varphi' = \varphi|_{Y_0'}$.*

Proof. Let $y \in Y_0'$. Taking $f = 1_X$ above in the expression of $Tf$, we have $\tau'(y) = T1_X(y) = \tau(y)$, and so $\tau' = \tau|_{Y_0'}$. Now the expression of $Tf$ reads as $\overline{\tau(y)}T^* \left.\widetilde{\delta}_y\right|_Z = \widetilde{\delta}_{\varphi'(y)}$ where $Z = T(\mathrm{Lip}(X))$. Since $\widetilde{\delta}_{\varphi'(y)} \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$ by

Lemma 2.2, it follows that $\overline{\tau(y)}T^*\ \widetilde{\delta}_y\Big|_Z \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$. Moreover,

$$1 = \widetilde{\delta}_{\varphi'(y)}(1_X) = \overline{\tau(y)}\widetilde{\delta}_y(T1_X) = \overline{\tau(y)}\tau(y) = |\tau(y)|^2 .$$

Hence $y \in Y_0$ and thus $Y_0' \subset Y_0$.

Let $f \in \mathrm{Lip}(X)$. Since $Tf(z) = \tau(z)f(\varphi(z))$ for all $z \in Y_0$ by Theorem 2.4, $Y_0' \subset Y_0$ and $\tau' = \tau|_{Y_0'}$, we have $Tf(y) = \tau'(y)f(\varphi(y))$. Moreover, $Tf(y) = \tau'(y)f(\varphi'(y))$ by hypothesis. Therefore $f(\varphi'(y)) = f(\varphi(y))$ for all $f \in \mathrm{Lip}(X)$. This implies $\varphi'(y) = \varphi(y)$, because otherwise we could take the function $f_{\varphi(y)}$ and $f_{\varphi(y)}(\varphi'(y)) = d(\varphi'(y), \varphi(y)) \neq 0 = f_{\varphi(y)}(\varphi(y))$. Thus $\varphi' = \varphi|_{Y_0'}$. □

## 3. Onto linear isometries

In this section we shall apply Theorem 2.4 to study the onto case.

**Theorem 3.1.** *Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a surjective linear isometry. Suppose $T1_X$ is a nonvanishing contraction. Then there exists a Lipschitz homeomorphism $\varphi : Y \to X$ with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ and $L(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)\}$ and a Lipschitz function $\tau : Y \to S_{\mathbb{K}}$ such that*

$$Tf(y) = \tau(y)f(\varphi(y)), \quad \forall f \in \mathrm{Lip}(X),\ \forall y \in Y.$$

Proof. According to Theorem 2.4, there exists a closed subset $Y_0$ of $Y$, a Lipschitz map $\varphi$ from $Y_0$ onto $X$ with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ and a function $\tau \in \mathrm{Lip}(Y)$ with $\|\tau\| = 1$ and $|\tau(y)| = 1$ for all $y \in Y_0$ such that

$$Tf(y) = \tau(y)f(\varphi(y)), \quad \forall f \in \mathrm{Lip}(X),\ \forall y \in Y_0.$$

Since now $T$ is surjective, the set $Y_0$ comes given by

$$\left\{ y \in Y : \overline{\tau(y)}T^*\widetilde{\delta}_y \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*}) \right\}.$$

We next prove that $Y_0 = Y$. Let $y \in Y$. Since $\widetilde{\delta}_y \in \mathrm{Ext}(B_{\mathrm{Lip}(Y)^*})$ by Lemma 2.2, it follows that $T^*\widetilde{\delta}_y \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$ because $T^*$ is a linear isometry from $\mathrm{Lip}(Y)^*$ onto $\mathrm{Lip}(X)^*$. Then $T^*\widetilde{\delta}_y = \alpha\widetilde{\delta}_w$ for some $\alpha \in S_{\mathbb{K}}$ and $w \in X \cup \beta\widetilde{X}$ by Lemma 2.1. We see that $w \in X$. Indeed, if $w \in \beta\widetilde{X}$, then $\alpha\widetilde{\delta}_w(1_X) = 0$, but

$$T^*\widetilde{\delta}_y(1_X) = \widetilde{\delta}_y(T1_X) = T1_X(y) \neq 0$$

because $T1_X$ is nonvanishing. Hence $w = x \in X$. Then

$$\tau(y) = T1_X(y) = T^*\widetilde{\delta}_y(1_X) = \alpha\widetilde{\delta}_x(1_X) = \alpha.$$

Hence $|\tau(y)| = 1$ and so $\overline{\tau(y)}\alpha = 1$. It follows that

$$\overline{\tau(y)}T^*(\widetilde{\delta}_y) = \overline{\tau(y)}\alpha\widetilde{\delta}_x = \widetilde{\delta}_x \in \mathrm{Ext}(B_{\mathrm{Lip}(X)^*})$$

and so $y \in Y_0$.

To prove the injectivity of $\varphi$, let $y, y' \in Y$ be for which $\varphi(y) = \varphi(y')$ and let us suppose $y \neq y'$. Since $Tf(z) = \tau(z)f(\varphi(z))$ for all $f \in \mathrm{Lip}(X)$ and $z \in Y$, it is clear that $|Tf(y)| = |Tf(y')|$ for all $f \in \mathrm{Lip}(X)$ and, since $T$ is surjective, it follows that $|h(y)| = |h(y')|$ for all $h \in \mathrm{Lip}(Y)$. However this can not be because $|f_y(y)| = 0 \neq d(y, y') = |f_y(y')|$.

On the other hand, $T^{-1}$ is a linear isometry from $\mathrm{Lip}(Y)$ onto $\mathrm{Lip}(X)$. It comes given indeed by

$$T^{-1}g(x) = \overline{\tau(\varphi^{-1}(x))}g(\varphi^{-1}(x)), \quad \forall g \in \mathrm{Lip}(Y),\ \forall x \in X.$$

Using this we can deduce that $\varphi^{-1}$ is Lipschitz with $L(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)\}$. The proof is similar to that given in Theorem 2.4 to prove that $\varphi$ is Lipschitz. $\square$

Under the conditions of Theorem 3.1, if $T$ is also unital, that is $T(1_X) = 1_Y$, then $T$ is an algebra isomorphism. Namely, we have the following:

**Corollary 3.2.** *Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a surjective linear isometry. Suppose $T$ is unital. Then there exists a Lipschitz homeomorphism $\varphi : Y \to X$ with $L(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ and $L(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)\}$ such that*

$$Tf(y) = f(\varphi(y)), \quad \forall f \in \mathrm{Lip}(X),\ \forall y \in Y.$$

Next we see that Theorem 3.1 can be improved. If $\tau : Y \to S_{\mathbb{K}}$ is a function such that $\tau(y) = \tau(y')$ whenever $d(y, y') < 2$, and $\varphi : Y \to X$ is a surjective isometry, it is easily seen that $Tf = \tau(f \circ \varphi)$ ($f \in \mathrm{Lip}(X)$) is a linear isometry from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$. Conversely, we have the following improvement of Theorem 3.1.

**Theorem 3.3.** *Let $X$ and $Y$ be compact metric spaces with diameter at most 2 and let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a surjective linear isometry such that $T1_X$ is a nonvanishing contraction. Then there exist a Lipschitz function $\tau : Y \to S_{\mathbb{K}}$ with $\tau(y) = \tau(y')$ whenever $d(y, y') < 2$ and a surjective isometry $\varphi : Y \to X$ such that $T$ is of the form*

$$Tf(y) = \tau(y)f(\varphi(y)), \quad \forall f \in \mathrm{Lip}(X),\ \forall y \in Y.$$

Proof. By Theorem 3.1, there exists a Lipschitz function $\tau : Y \to S_{\mathbb{K}}$ and a Lipschitz homeomorphism $\varphi : Y \to X$ such that

$$Tf(y) = \tau(y)f(\varphi(y)), \quad \forall f \in \mathrm{Lip}(X),\ \forall y \in Y.$$

Let us define an equivalence relation on $X$ by setting $x \sim z$ ($x, z \in X$) if and only if there is a finite sequence of points $x_1, ..., x_k$ in $X$ such that $x_1 = x$, $x_k = z$ and $d(x_i, x_{i+1}) < 1$ for $1 \leq i < k$. The equivalence class of this relation are called the 1-connected components of $X$. Since $X$ is compact, there exist $z_1, ..., z_m \in X$ such that $X = \cup_{i=1}^m B(z_i, 1)$ where $B(z_i, 1) = \{x \in X : d(x, z_i) < 1\}$. For each $i \in \{1, ..., m\}$, the ball $B(z_i, 1)$ is contained in the 1-connected component of $X$ which contains to $z_i$ and therefore the number of 1-connected components of $X$ is $n \leq m$. Let $X_1, ..., X_n$ be the 1-connected components of $X$. Notice that $X_1, ..., X_n$ are pairwise disjoint closed sets.

Fix $1 \leq i \leq n$ and identify $\mathrm{Lip}(X_i)$ with the functions in $\mathrm{Lip}(X)$ which are supported on $X_i$. Let $Y_i = \varphi^{-1}(X_i)$. Clearly, $T$ takes $\mathrm{Lip}(X_i)$ isometrically onto $\mathrm{Lip}(Y_i)$ and then, by [17, Theorem 2.6.7], there exists a constant $\tau_i \in S_{\mathbb{K}}$ and an isometry $\varphi_i$ from $Y_i$ onto $X_i$ such that

$$Tf(y) = \tau_i f(\varphi_i(y)), \quad \forall f \in \mathrm{Lip}(X_i),\ \forall y \in Y_i.$$

A simple verification shows that $\tau|_{Y_i} = \tau_i 1_Y$ and $\varphi|_{Y_i} = \varphi_i$.

Fix $i, j \in \{1, ..., n\}$ with $i \neq j$. Suppose $d(x_0, x_0') < 2$ for some $x_0 \in X_i$ and $x_0' \in X_j$. Let $a = \inf\{d(x, x') : x \in X_i,\ x' \in X_j\}$ and consider $f : X \to \mathbb{R}$ defined by $f(x) = -a/2$ if $x \in X_i$, $f(x) = a/2$ if $x \in X_j$ and $f(x) = 0$ if $x \notin X_i \cup X_j$. We claim that $f \in \mathrm{Lip}(X)$ with $L(f) = 1$. Let $x, x' \in X$. If $x \in X_i$ and $x' \in X_j$, we have

$$\frac{|f(x) - f(x')|}{d(x, x')} = \frac{a}{d(x, x')} \leq 1.$$

If $x \in X_i \cup X_j$ and $x' \notin X_i \cup X_j$, we obtain

$$\frac{|f(x) - f(x')|}{d(x, x')} = \frac{\frac{a}{2}}{d(x, x')} \leq \frac{a}{2} < 1.$$

It follows that $L(f) \leq 1$. Using the definition of $a$ it is easy to see that $L(f) = 1$. Since $\|f\|_\infty = a/2 < 1$, we have $\|f\| = 1$ and thus $\|Tf\| = 1$. Moreover, since $Tf(y) = -\tau_i a/2$ if $y \in Y_i$, $Tf(y) = \tau_j a/2$ if $y \in Y_j$ and $Tf(y) = 0$ elsewhere, we have $\|Tf\|_\infty = a/2 < 1$ and thus $L(Tf) = 1$. Let

$$b = \inf\{d(y, y') : y \in Y_i,\ y' \in Y_j\}.$$

Since $Y_i$ and $Y_j$ are pairwise disjoint closed sets, we have $b > 0$. Next we check that $L(Tf) \leq a/b$. Let $y, y' \in Y$. If $y \in Y_i$ and $y' \in Y_j$, we have

$$\frac{|Tf(y) - Tf(y')|}{d(y, y')} = \frac{\frac{a}{2}|\tau_i + \tau_j|}{d(y, y')} \leq \frac{a}{b}.$$

If $y \in Y_i \cup Y_j$ and $y' \notin Y_i \cup Y_j$, we get

$$\frac{|Tf(y) - Tf(y')|}{d(y, y')} = \frac{\frac{a}{2}}{d(y, y')} \leq \frac{a}{2} \leq \frac{a}{b}.$$

Therefore $L(Tf) \leq a/b$ and so $1 \leq a/b$.

We now prove that $\tau_i = \tau_j$. Suppose $\tau_i \neq \tau_j$ and let $g : X \to \mathbb{K}$ be the function given by $g(x) = -\overline{\tau_i}a/2$ if $x \in X_i$, $g(x) = \overline{\tau_j}a/2$ if $x \in X_j$ and $g(x) = 0$ elsewhere. To see that $g \in \mathrm{Lip}(X)$, let $x, x' \in X$. If $x \in X_i$ and $x' \in X_j$, we have

$$\frac{|g(x) - g(x')|}{d(x,x')} = \frac{\frac{a}{2}|\tau_i + \tau_j|}{d(x,x')} \leq \frac{|\tau_i + \tau_j|}{2} < 1$$

since $\tau_i, \tau_j \in S_{\mathbb{K}}$ and $\tau_i \neq \tau_j$. If now $x \in X_i \cup X_j$ and $x' \notin X_i \cup X_j$, we obtain

$$\frac{|g(x) - g(x')|}{d(x,x')} = \frac{\frac{a}{2}}{d(x,x')} \leq \frac{a}{2} < 1.$$

Hence $g \in \mathrm{Lip}(X)$ with $L(g) \leq \max\{|\tau_i + \tau_j|/2, a/2\} < 1$. Since $\|g\|_\infty = a/2$, it follows that $\|g\| < 1$ and so $L(Tg) \leq \|Tg\| = \|g\| < 1$. On the other hand, since $Tg(y) = -a/2$ if $y \in Y_i$ and $Tg(y') = a/2$ if $y' \in Y_j$, we deduce

$$\frac{|Tg(y) - Tg(y')|}{d(y,y')} = \frac{a}{d(y,y')} \leq L(Tg),$$

which implies $a/b \leq L(Tg)$. Then $a/b < 1$, which contradicts that $1 \leq a/b$. This proves that $\tau_i = \tau_j$.

It is an easily checked fact, which is contained in [16], that $\left\|\widetilde{\delta}_y - \widetilde{\delta}_{y'}\right\| = d(y,y')$ whenever $d(y,y') \leq 2$. Using this fact we now prove that $\varphi$ is an isometry. Let $y, y' \in Y$ be such that $d(\varphi(y), \varphi(y')) < 2$. Clearly, $y \in Y_i$ and $y' \in Y_j$ for some $i, j \in \{1, ..., n\}$. By what has been proved above, we have $\tau(y) = \tau_i = \tau_j = \tau(y')$ and it follows that

$$d(\varphi(y), \varphi(y')) = \left\|\widetilde{\delta}_{\varphi(y)} - \widetilde{\delta}_{\varphi(y')}\right\| = \left\|\tau(y)\widetilde{\delta}_{\varphi(y)} - \tau(y')\widetilde{\delta}_{\varphi(y')}\right\|$$
$$= \left\|\widetilde{\delta}_y \circ T - \widetilde{\delta}_{y'} \circ T\right\| = \left\|T^*(\widetilde{\delta}_y - \widetilde{\delta}_{y'})\right\| = \left\|\widetilde{\delta}_y - \widetilde{\delta}_{y'}\right\| = d(y,y').$$

If now $y, y' \in Y$ with $d(\varphi(y), \varphi(y')) = 2$, then $d(y,y') = 2$. In contrary case, it would be $d(\varphi^{-1}(\varphi(y)), \varphi^{-1}(\varphi(y'))) < 2$ and, by applying to $T^{-1}$ what has already been proved, we would have $d(\varphi^{-1}(\varphi(y)), \varphi^{-1}(\varphi(y'))) = d(\varphi(y), \varphi(y'))$, that is, $2 = d(\varphi(y), \varphi(y')) = d(y,y') < 2$, a contradiction. $\square$

## 4. Codimension 1 linear isometries

Applying Theorem 2.4, we can describe codimension 1 linear isometries between $\mathrm{Lip}(X)$-spaces as follows.

**Theorem 4.1.** *Let $T$ be a codimension 1 linear isometry from* $\mathrm{Lip}(X)$ *into* $\mathrm{Lip}(Y)$. *Suppose $T1_X$ is a contraction. Then there exists a closed subset $Y_0$ of $Y$*

*where either* $Y_0 = Y\backslash\{p\}$ *being* $p$ *an isolated point of* $Y$ *or* $Y_0 = Y$, *a surjective Lipschitz map* $\varphi : Y_0 \to X$ *and a Lipschitz function* $\tau : Y_0 \to S_{\mathbb{K}}$ *such that*

$$Tf(y) = \tau(y)f(\varphi(y)), \quad \forall f \in \mathrm{Lip}(X),\ \forall y \in Y_0.$$

PROOF. By Theorem 2.4 there exists a nonempty closed subset $Y_0$ of $Y$, a Lipschitz map $\varphi$ from $Y_0$ onto $X$ and a Lipchitz function $\tau$ from $Y_0$ into $S_{\mathbb{K}}$ such that

$$Tf(y) = \tau(y)f(\varphi(y)), \quad \forall f \in \mathrm{Lip}(X),\ \forall y \in Y_0.$$

Suppose $Y\backslash Y_0$ has two distinct points $y_1, y_2$. For $i \in \{1,2\}$, let

$$\delta_i = d(Y_0 \cup \{y_j : j \neq i\}, y_i).$$

Clearly, $\delta_i > 0$ and $h_{y_i}^{\delta_i} \in \mathrm{Lip}(Y)$ satisfies that $h_{y_i}^{\delta_i}(y_i) = 1$ and $h_{y_i}^{\delta_i}(y) = 0$ for all $y \in Y_0 \cup \{y_j : j \neq i\}$.

We see that $h_{y_1}^{\delta_1}$ and $h_{y_2}^{\delta_2}$ are linearly independent. Suppose $\alpha h_{y_1}^{\delta_1} + \beta h_{y_2}^{\delta_2} = 0$ for some scalars $\alpha, \beta$. Since $h_{y_i}^{\delta_i}(y_j) = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker's delta, it follows that $\alpha = \beta = 0$.

We now prove that no nonzero linear combination of $h_{y_1}^{\delta_1}$ and $h_{y_2}^{\delta_2}$ belongs to the range of $T$. Let $\alpha, \beta \in \mathbb{K}$ and suppose there exists a $f \in \mathrm{Lip}(X)$ such that $Tf = \alpha h_{y_1}^{\delta_1} + \beta h_{y_2}^{\delta_2}$. Then, for all $y \in Y_0$, we have $Tf(y) = 0$, but also $Tf(y) = \tau(y)f(\varphi(y))$. In consequence, $\tau(y)f(\varphi(y)) = 0$ for all $y \in Y_0$. Then $f$ is the zero function because $\tau$ is unimodular and $\varphi$ is surjective, and thus $\alpha h_{y_1}^{\delta_1} + \beta h_{y_2}^{\delta_2} = 0$ by the linearity of $T$.

From the above it is deduced that $h_{y_1}^{\delta_1}$ is not in the range of $T$. Since the range of $T$ has codimension 1, we have $h_{y_2}^{\delta_2} = \alpha h_{y_1}^{\delta_1} + T(g)$ for some $\alpha \in \mathbb{K}$ and $g \in \mathrm{Lip}(X)$. Then $h_{y_2}^{\delta_2} - \alpha h_{y_1}^{\delta_1} = 0$, a contradiction. Therefore $Y\backslash Y_0$ has at most a point. Then either $Y_0 = Y$ or $Y_0 = Y\backslash\{p\}$ for some point $p \in Y$. This point $p$ must be isolated since $Y\backslash\{p\}$ is closed. □

Theorem 4.1 allows us to classify codimension 1 linear isometries between $\mathrm{Lip}(X)$-spaces in two types:

**Definition 1.** Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a codimension 1 linear isometry such that $T1_X$ is a contraction. We say:

(1) $T$ is of type I when there exists an isolated point $p$ of $Y$, a surjective Lipschitz map $\varphi : Y\backslash\{p\} \to X$ and a Lipschitz function $\tau : Y\backslash\{p\} \to S_{\mathbb{K}}$ such that

$$Tf(y) = \tau(y)f(\varphi(y)), \quad \forall y \in Y\backslash\{p\}.$$

(2) $T$ is of type II if there is a surjective Lipschitz map $\varphi : Y \to X$ and a Lipschitz function $\tau : Y \to S_{\mathbb{K}}$ such that

$$Tf(y) = \tau(y)f(\varphi(y)), \qquad \forall y \in Y.$$

These two types are not necessarily disjoint as the next result shows.

**Proposition 4.2.** *Let $Y$ be a metric compact space with an isolated point $p$. Let $X = Y\backslash\{p\}$ and suppose there exists a point $x_0 \in X$ such that $d(x, x_0) \leq d(x, p)$ for all $x \in X$ (in particular, this happens when $\operatorname{diam}(X) \leq d(p, X)$). Then the map $T : \operatorname{Lip}(X) \to \operatorname{Lip}(Y)$ defined by*

$$Tf(y) = f(y), \qquad \forall y \in X, \quad Tf(p) = f(x_0),$$

*is a codimension 1 linear isometry with $L(T1_X) < 1$, which is simultaneously of types I and II.*

Proof. Let $f \in \operatorname{Lip}(X)$. Obviously, $\|Tf\|_\infty = \|f\|_\infty$ and $L(f) \leq L(Tf)$. Moreover, we check at once that

$$|Tf(x) - Tf(p)| = |f(x) - f(x_0)| \leq L(f)d(x, x_0) \leq L(f)d(x, p), \qquad \forall x \in X.$$

Therefore $L(Tf) \leq L(f)$ and so $\|Tf\| = \|f\|$. Clearly, $T$ is linear. However $T$ is not surjective, since the function $h_p^r \in \operatorname{Lip}(Y)$ with $r = d(p, X) > 0$ is not in $T(\operatorname{Lip}(X))$. Moreover, $T1_X = 1_Y$ and thus $L(T1_X) = 0 < 1$. Finally, $T(\operatorname{Lip}(X))$ is of codimension 1 since every $g \in \operatorname{Lip}(Y)$ may be expressed as

$$g = Tf + (g(p) - g(x_0))h_p^r,$$

where $f$ is the function in $\operatorname{Lip}(X)$ defined by $f(x) = g(x)$ for all $x \in X$.

Hence $T$ is of type I taking as $\varphi$ the identity map on $Y\backslash\{p\}$ and as $\tau$ the function $1_{Y\backslash\{p\}}$. But also $T$ is of type II if we now put $\tau = 1_Y$ and $\varphi$ the function from $Y$ to $Y\backslash\{p\}$ given by $\varphi(y) = y$ if $y \neq p$ and $\varphi(p) = x_0$. □

We next give a method for constructing type I codimension 1 linear isometries which are not of type II.

**Proposition 4.3.** *Let $X$ and $Y$ be metric compact spaces. Let $p$ be a point of $Y$ such that $1 < d(p, Y\backslash\{p\})$, $\varphi : Y\backslash\{p\} \to X$ a surjective isometry and $\tau$ a unimodular constant. Then the map $T : \operatorname{Lip}(X) \to \operatorname{Lip}(Y)$ defined by*

$$Tf(y) = \tau f(\varphi(y)), \qquad \forall y \in Y\backslash\{p\}, \quad Tf(p) = 0,$$

*for all $f \in \operatorname{Lip}(X)$, is a codimension 1 linear isometry of type I with $L(T1_X) < 1$, but it is not of type II.*

Proof. Obviously, $T$ is linear and preserves the supremum norm. Let $f \in \mathrm{Lip}(X)$. For all $x, w \in X$, we have

$$\begin{aligned}|f(x) - f(w)| &= \left|Tf(\varphi^{-1}(x)) - Tf(\varphi^{-1}(w))\right| \\ &\leq L(Tf)d(\varphi^{-1}(x), \varphi^{-1}(w)) = L(Tf)d(x, w).\end{aligned}$$

Hence $L(f) \leq L(Tf)$ and so $\|f\| \leq \|Tf\|$ . On the other hand, it is clear that

$$\begin{aligned}|Tf(y) - Tf(z)| &= |f(\varphi(y)) - f(\varphi(z))| \\ &\leq L(f)d(\varphi(y), \varphi(z)) \leq \|f\| \, d(y, z)\end{aligned}$$

for all $y, z \in Y \backslash \{p\}$, and

$$\begin{aligned}|Tf(y) - Tf(p)| &= |Tf(y)| = |f(\varphi(y))| \leq \|f\|_\infty \\ &\leq \|f\|_\infty \, d(p, Y \backslash \{p\}) \leq \|f\| \, d(p, y)\end{aligned}$$

for all $y \in Y \backslash \{p\}$. This implies that $L(Tf) \leq \|f\|$ and thus $\|Tf\| \leq \|f\|$. Hence $T$ is an isometry. Moreover, we have

$$|T1_X(y) - T1_X(p)| = |T1_X(y)| = 1 < d(p, Y \backslash \{p\}) \leq d(p, y)$$

for all $y \in Y \backslash \{p\}$, which gives $L(T1_X) < 1$.

We now claim that $T$ has codimension 1. First observe that the function $h_p^1 \in \mathrm{Lip}(Y)$ does not belong to $T(\mathrm{Lip}(X))$, since if $h_p^1 = Tf$ for some $f \in \mathrm{Lip}(X)$), then $1 = h_p^1(p) = Tf(p) = 0$, a contradiction. Then, given $g \in \mathrm{Lip}(Y)$, we take $f = \overline{\tau}(g \circ \varphi^{-1}) \in \mathrm{Lip}(X)$ and it is clear that $g = Tf + g(p)h_p^1$, which proves our claim.

Evidently, $T$ is of type I. However $T$ is not of type II since, in contrary case, we could write $Tf = \tau'(f \circ \varphi')$ for some Lipschitz surjection $\varphi' : Y \to X$ and some Lipschitz function $\tau' : Y \to S_{\mathbb{K}}$, and we would have $T1_X(p) = \tau'(p) \neq 0$, which contradicts the definition of $T$. □

We next provide a example of a type II codimension 1 linear isometry which is not of type I.

**Example 4.4.** *For $X = [0, 2]$ and $Y = [0, 1] \cup [2, 3]$, let $\varphi : Y \to X$ be the map defined by*

$$\varphi(y) = \begin{cases} y & \text{if } y \in [0, 1], \\ y - 1 & \text{if } y \in [2, 3], \end{cases}$$

*and let $\tau : Y \to S_{\mathbb{C}}$ be the function given by*

$$\tau(y) = \begin{cases} 1 & \text{if } y \in [0, 1], \\ \frac{2}{3} + \frac{\sqrt{5}}{3} i & \text{if } y \in [2, 3]. \end{cases}$$

*We claim that* $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ *defined by*

$$Tf(y) = \tau(y)f(\varphi(y)), \quad \forall y \in Y, \ \forall f \in \mathrm{Lip}(X),$$

*is a codimension* 1 *linear isometry.*

*Since* $\varphi$ *and* $\tau$ *are Lipschitz,* $T$ *is well defined. Obviously,* $T$ *is linear and, by the surjectivity of* $\varphi$, $T$ *preserves the supremum norm.*

*We now check that* $L(f) \leq L(Tf)$ *for all* $f \in \mathrm{Lip}(X)$. *Let* $f \in \mathrm{Lip}(X)$ *and* $x_1, x_2 \in X$ *with* $x_1 \neq x_2$. *If* $x_1, x_2 \in [0,1]$ *or* $x_1, x_2 \in [1,2]$, *then*

$$|f(x_1) - f(x_2)| = |Tf(y_1) - Tf(y_2)| \leq L(Tf)\,|y_1 - y_2| = L(Tf)\,|x_1 - x_2|$$

*for suitable points* $y_1, y_2$ *in* $Y$ *satisfying* $\varphi(y_1) = x_1$ *and* $\varphi(y_2) = x_2$. *If* $x_1 \in [0,1]$ *and* $x_2 \in [1,2]$, *suppose that* $x_1 \neq 1 \neq x_2$ *and*

$$\frac{|f(1) - f(x_1)|}{1 - x_1} < \frac{|f(x_2) - f(x_1)|}{x_2 - x_1}, \quad \frac{|f(x_2) - f(1)|}{x_2 - 1} < \frac{|f(x_2) - f(x_1)|}{x_2 - x_1}.$$

*Then we arrive at the following contradiction:*

$$|f(x_2) - f(x_1)| \leq |f(x_2) - f(1)| + |f(1) - f(x_1)|$$

$$< \frac{|f(x_2) - f(x_1)|}{x_2 - x_1}\left[(x_2 - 1) + (1 - x_1)\right] = |f(x_2) - f(x_1)|\,.$$

*Therefore we have*

$$\frac{|f(x_2) - f(x_1)|}{x_2 - x_1} \leq \frac{|f(1) - f(x_1)|}{1 - x_1}$$

*or*

$$\frac{|f(x_2) - f(x_1)|}{x_2 - x_1} \leq \frac{|f(x_2) - f(1)|}{x_2 - 1}.$$

*Applying the above-proved gives* $|f(x_2) - f(x_1)|/(x_2 - x_1) \leq L(Tf)$ *and so* $L(f) \leq L(Tf)$. *As also* $\|Tf\|_\infty = \|f\|_\infty$, *we deduce that* $\|f\| \leq \|Tf\|$.

*On the other hand, let* $y_1, y_2 \in Y$. *A simple calculation yields*

$$\begin{aligned} |Tf(y_1) - Tf(y_2)| &= |\tau(y_1)f(\varphi(y_1)) - \tau(y_2)f(\varphi(y_2))| \\ &\leq |\tau(y_1)f(\varphi(y_1)) - \tau(y_1)f(\varphi(y_2))| + |\tau(y_1)f(\varphi(y_2)) - \tau(y_2)f(\varphi(y_2))| \\ &\leq L(f)\,|\varphi(y_1) - \varphi(y_2)| + \|f\|_\infty\,|\tau(y_1) - \tau(y_2)|\,. \end{aligned}$$

*If* $y_1, y_2 \in [0,1]$ *or* $y_1, y_2 \in [2,3]$, *it follows that*

$$|Tf(y_1) - Tf(y_2)| \leq L(f)|y_1 - y_2| \leq \|f\|\,|y_1 - y_2|\,,$$

*whereas for* $y_1 \in [0,1]$ *and* $y_2 \in [2,3]$*, we obtain that*

$$|Tf(y_1) - Tf(y_2)| \leq \|f\| \left[ |y_1 - y_2 + 1| + \left| \frac{1}{3} - \frac{\sqrt{5}}{3} i \right| \right]$$

$$= \|f\| \left( y_2 - y_1 - 1 + \sqrt{\frac{2}{3}} \right) \leq \|f\| (y_2 - y_1) = \|f\| |y_1 - y_2| .$$

*This proves that* $L(Tf) \leq \|f\|$ *and therefore* $\|Tf\| \leq \|f\|$*. Hence* $T$ *is a linear isometry. Furthermore,* $L(T1_X) = \sqrt{2/3} < 1$.

*Finally, we claim that* $T$ *has codimension* 1*. Clearly, the function*

$$g(y) = 1 \text{ if } y \in [0,1], \quad g(y) = 0 \text{ if } y \in [2,3].$$

*is in* $\mathrm{Lip}(Y)$*, but not in* $T(\mathrm{Lip}(X))$ *since* $|g(1)| = 1 \neq 0 = |g(2)|$*. Given* $h \in \mathrm{Lip}(Y)$*, we can take the scalar* $\alpha = h(1) - h(2) \left( \frac{2}{3} - \frac{\sqrt{5}}{3} i \right)$ *and the function*

$$f(x) = \begin{cases} h(x) - \alpha & \text{if } x \in [0,1], \\ \left( \frac{2}{3} - \frac{\sqrt{5}}{3} i \right) h(x+1) & \text{if } x \in [1,2]. \end{cases}$$

*It is readily seen that* $f \in \mathrm{Lip}(X)$ *with* $L(f) \leq L(h)$*. Taking into account that*

$$Tf(y) + \alpha g(y) = \tau(y) f(\varphi(y)) + \alpha g(y) = f(y) + \alpha = h(y)$$

*for all* $y \in [0,1]$*, and*

$$Tf(y) + \alpha g(y) = \tau(y)\, f(\varphi(y)) + \alpha g(y) = \left( \frac{2}{3} + \frac{\sqrt{5}}{3} i \right) f(y-1)$$

$$= \left( \frac{2}{3} + \frac{\sqrt{5}}{3} i \right) \left( \frac{2}{3} - \frac{\sqrt{5}}{3} i \right) h(y) = h(y)$$

*for all* $y \in [2,3]$*, we have* $h = Tf + \alpha g$*, which proves our claim.*

*Observe that* $T$ *is of type II, but not of type I since* $Y$ *has not isolated points.*

Next we study the properties of the map $\varphi$ and state some conditions under which $\varphi$ is a Lipschitz homeomorphism.

**Lemma 4.5.** *Let* $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ *be a codimension* 1 *linear isometry such that* $T1_X$ *is a contraction. For any* $f \in \mathrm{Lip}(X)$ *and* $x \in X$*, the function* $|Tf|$ *is constant on* $\varphi^{-1}(\{x\})$.

Proof. By Theorem 4.1, for all $y \in Y_0$ we have $Tf(y) = \tau(y) f(\varphi(y))$ and since $\tau(y) \in S_{\mathbb{K}}$, it follows that $|Tf(y)| = |f(\varphi(y))|$. If now $y \in \varphi^{-1}(\{x\})$, we get $|Tf(y)| = |f(x)|$. □

**Proposition 4.6.** *Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a codimension 1 linear isometry such that $T1_X$ is a contraction. We take $Y_0$, $\varphi$ and $\tau$ as in Theorem 4.1. The following assertions hold:*

i) *For each $x \in X$, $\varphi^{-1}(\{x\})$ has at most two elements.*
ii) *If there exists a point $x_0 \in X$ and two distinct points $a, b \in Y_0$ such that $\varphi(a) = \varphi(b) = x_0$, then $\varphi^{-1}(\{x\})$ is a singleton for each $x \in X \backslash \{x_0\}$.*
iii) *If $T$ is of type I ($Y_0 \neq Y$), then $\varphi$ is injective and hence a homeomorphism.*
iv) *If $T$ is of type I with $Y_0 = Y \backslash \{p\}$ and $T^*(\widetilde{\delta}_p) \in \mathrm{Lip}(X)^*$ is zero, then $\varphi$ is a Lipschitz homeomorphism.*

Proof. i) Suppose that there exist three distinct points $y_1, y_2, y_3 \in Y_0$ such that $\varphi(y_1) = \varphi(y_2) = \varphi(y_3)$. Put $\rho = \min\{d(y_1, y_2), d(y_2, y_3), d(y_1, y_3)\}$ and consider the functions $h^\rho_{y_1}$ and $h^\rho_{y_3}$. Since the codimension of range of $T$ is 1, there exist constants $\alpha$ and $\beta$, not both zero, such that $\alpha h^\rho_{y_1} + \beta h^\rho_{y_3} \in T(\mathrm{Lip}(X))$. By Lemma 4.5 we obtain

$$\left|\alpha h^\rho_{y_1}(y_1) + \beta h^\rho_{y_3}(y_1)\right| = \left|\alpha h^\rho_{y_1}(y_2) + \beta h^\rho_{y_3}(y_2)\right| = \left|\alpha h^\rho_{y_1}(y_3) + \beta h^\rho_{y_3}(y_3)\right|.$$

This gives $|\alpha| = 0 = |\beta|$, a contradiction. Hence $\varphi^{-1}(\{x\})$ contains at most two points for all $x \in X$.

ii) Suppose there are distinct points $x_0, x \in X$ and $a, b, p, q \in Y_0$ such that

$$a \neq b, \quad \varphi(a) = \varphi(b) = x_0, \quad p \neq q, \quad \varphi(p) = \varphi(q) = x.$$

Take the positive numbers

$$\begin{aligned} \epsilon &= \min\{d(a,b), d(p,b), d(q,b)\}, \\ r_1 &= \min\{d(a,b), d(a,p), d(a,q)\}, \\ r_2 &= \min\{d(p,a), d(p,b), d(p,q)\}, \end{aligned}$$

and consider the functions $f_1 = h^\epsilon_b$ and $f_2 = h^{r_1}_a + h^{r_2}_p$ in $\mathrm{Lip}(Y)$. Then

$$f_1(b) = 1, \ \ f_1(a) = f_1(p) = f_1(q) = 0, \ \ f_2(a) = f_2(p) = 1, \ \ f_2(b) = f_2(q) = 0.$$

Again, the codimension 1 of range of $T$ provides two scalars $\alpha, \beta$, not both zero, such that $\alpha f_1 + \beta f_2 \in T(\mathrm{Lip}(X))$. By Lemma 4.5, it follows that

$$|\alpha f_1(p) + \beta f_2(p)| = |\alpha f_1(q) + \beta f_2(q)|$$

and

$$|\alpha f_1(a) + \beta f_2(a)| = |\alpha f_1(b) + \beta f_2(b)|.$$

Therefore $|\alpha| = |\beta| = 0$, a contradiction. Hence ii) is true.

iii) Let us assume $T$ is of type I ($Y_0 \neq Y$). Then $Y \backslash Y_0 = \{p\}$ for some isolated point $p \in Y$. Therefore we may take $r = d(p, Y_0) > 0$ and consider $h^r_p \in \mathrm{Lip}(Y)$.

If $h_p^r \in T(\mathrm{Lip}(X))$, it is easy to show that $h_p^r = 0$, a contradiction. Hence $h_p^r$ does not belong to the range of $T$.

Suppose there exist $y_0, y \in Y_0$ such that $y_0 \neq y$ and $\varphi(y_0) = \varphi(y)$. Let $\epsilon = \min\{d(y_0, y), d(y, p)\}$. Since $h_y^\epsilon$ satisfies $h_y^\epsilon(y) = 1 \neq 0 = h_y^\epsilon(y_0)$, Lemma 4.5 gives $h_y^\epsilon \notin T(\mathrm{Lip}(X))$. As a consequence, there exists $\alpha \in \mathbb{K}$ and $f \in \mathrm{Lip}(X)$ such that $h_p^r = Tf + \alpha h_y^\epsilon$. Then Lemma 4.5 gives $\left|h_p^r(y_0) - \alpha h_y^\epsilon(y_0)\right| = \left|h_p^r(y) - \alpha h_y^\epsilon(y)\right|$, that is $0 = |\alpha|$, but then $h_p^r \in T(\mathrm{Lip}(X))$, a contradiction. Hence $\varphi$ is injective.

iv) Let $y, z \in Y_0$ with $y \neq z$. Putting $\gamma = \min\{d(y,z), d(y,p)\}$, define $g = d(y,z)h_y^\gamma \in \mathrm{Lip}(Y)$. A trivial verification yields $L(g) \leq \max\{1, \mathrm{diam}(Y_0)/d(p, Y_0)\}$ and $\|g\|_\infty = g(y) = d(y,z) \leq \mathrm{diam}(Y_0)$. As a consequence, $\|g\| \leq k$ where $k = \max\{\delta, \mathrm{diam}(Y_0)\}$ and $\delta = \max\{1, \mathrm{diam}(Y_0)/d(p, Y_0)\}$.

We now consider the function $h_p^r \in \mathrm{Lip}(Y)$ with $r = d(p, Y_0)$. Since $h_p^r \notin T(\mathrm{Lip}(X))$ (see iii)) and $T(\mathrm{Lip}(X))$ has codimension 1, there exist $\alpha \in \mathbb{K}$ and $f \in \mathrm{Lip}(X)$ such that $g = Tf + \alpha h_p^r$. Since

$$0 = g(p) = Tf(p) + \alpha = T^* \widetilde{\delta}_p(f) + \alpha = \alpha,$$

it follows that $g = Tf$. Then

$$\begin{aligned} d(y,z) = |g(y)| = \left|g(y) - \tau(y)\overline{\tau(z)}g(z)\right| &= \left|Tf(y) - \tau(y)\overline{\tau(z)}Tf(z)\right| \\ = \left|\tau(y)f(\varphi(y)) - \tau(y)\overline{\tau(z)}\tau(z)f(\varphi(z))\right| &= |f(\varphi(y)) - f(\varphi(z))| \\ \leq L(f)d(\varphi(y), \varphi(z)) \leq \|g\|\, d(\varphi(y), \varphi(z)) &\leq kd(\varphi(y), \varphi(z)). \end{aligned}$$

Hence $\varphi^{-1}$ is Lipschitz and so $\varphi$ is a Lipschitz homeomorphism. □

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