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ORDER ISOMORPHISMS OF LITTLE LIPSCHITZ ALGEBRAS

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Abstract. For compact metric spaces (X, d_X) and (Y, d_Y) and scalars $\alpha, \beta \in (0,1)$, we prove that every order isomorphism T between little Lipschitz algebras $lip(X, d_X^\alpha)$ and $lip(Y, d_Y^\beta)$ is a weighted composition operator of the form $T(f) = a \cdot (f \circ h)$ for all $f \in lip(X, d_X^\alpha)$, where a is a nonvanishing positive function in $lip(Y, d_Y^\beta)$ and h is a Lipschitz homeomorphism from (Y, d_Y^β) onto (X, d_X^α).

1. Introduction

Let (X, d) be a metric space, let $\mathbb{K}$ be the set of complex or real numbers and let α be a real number in $(0, 1]$. A function $f : X \to \mathbb{K}$ is called Lipschitz-α if it satisfies the Lipschitz condition with respect to the metric d^α defined by $d^\alpha(x, y) = (d(x, y))^\alpha$, that is, if there exists a constant $k \in \mathbb{R}^+$ such that

$$|f(x) - f(y)| \leq k \cdot d^\alpha(x, y), \quad \forall x, y \in X.$$

Following [7], we denote by $Lip(X, d^\alpha)$ the Banach space of all bounded Lipschitz-α functions $f : X \to \mathbb{K}$ with the norm $\|f\|_\alpha = p_\alpha(f) + \|f\|_\infty$, where

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

and

$$p_\alpha(f) = \sup\{|f(x) - f(y)| / d^\alpha(x, y) : x, y \in X,\ x \neq y\}.$$

Moreover, we denote by $lip(X, d^\alpha)$ the closed subspace of $Lip(X, d^\alpha)$ consisting of all those functions f in $Lip(X, d^\alpha)$ with the property that for each $\varepsilon > 0$, there

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exists $\delta > 0$ such that $0 < d(x,y) < \delta$ implies $|f(x) - f(y)| / d^\alpha(x,y) < \varepsilon$. In the case $\alpha = 1$, we write $Lip(X,d)$ and $lip(X,d)$.

The spaces $Lip(X,d^\alpha)$ and $lip(X,d^\alpha)$ are both unital self-adjoint commutative Banach algebras with respect to pointwise multiplication, but they also are ordered vector spaces with respect to pointwise order defined by $f \geq 0$ if and only if $f(x) \in \mathbb{R}$ and $f(x) \geq 0$ for all $x \in X$. We say that a function f in $Lip(X,d^\alpha)$ is positive if $f \geq 0$.

Following [5, Definition 1.1], a commutative Banach algebra A is called Lipschitz if there exists a metric space (X,d) such that A is either $Lip(X,d^\alpha)$ (known as big Lipschitz algebra) or $lip(X,d^\alpha)$ (called little Lipschitz algebra). The algebraic structure of Lispchitz algebras has been intensively studied, but so much like its order structure (see, for example, [11] and its references).

Weaver has studied the order structure of Lipschitz function spaces in a series of papers [8, 9, 10]. In [10], Weaver focuses his attention on the algebraic and order structures of $Lip_0(X,d)$, the space of all Lipschitz complex-valued functions on X vanishing at some fixed point, when X is a complete metric space with finite diameter. In [8, 9], $Lip(X,d)$-spaces have been abstractly characterized as vector lattices when X is a complete metric space with diameter at most 2. This condition on the diameter is not restrictive in view of [11, Proposition 1.7.1].

Let $A(X)$ and $B(Y)$ be ordered vector spaces (with respect to pointwise order) of real or complex-valued functions on the sets X and Y, respectively. A linear map $T : A(X) \to B(Y)$ is said to be order-preserving if $f \leq g$ implies $T(f) \leq T(g)$ for all $f, g \in A(X)$. If T is bijective and both T and T^{-1} are order-preserving, then we say that T is an order isomorphism. In the case that the spaces $A(X)$ and $B(Y)$ are both vector lattices of real functions, order isomorphisms are known also in the literature as vector lattice isomorphisms. Recall that T is unital if $T(1_X) = 1_Y$, where 1_X and 1_Y denote the functions constantly 1 on X and Y, respectively.

Order isomorphisms between $Lip(X,d)$-spaces have been studied by several authors. Weaver [8, Main theorem] and Garrido–Jaramillo [3, Theorem 3.10] have characterized the Lipschitz structure of an arbitrary complete metric space X in terms of the (purely algebraic) unital vector lattice structure of the family $Lip(X,d)$. More precisely, they proved that if (X,d_X) and (Y,d_Y) are complete metric spaces, then $Lip(X,d_X)$ is isomorphic to $Lip(Y,d_Y)$ as unital vector lattices if and only if X is Lipschitz homeomorphic to Y. Furthermore, they stated that every unital vector lattice isomorphism T from $Lip(X,d_X)$ onto $Lip(Y,d_Y)$ is a composition operator of the form $T(f) = f \circ h$ where $h : Y \to X$ is a

Lipschitz homeomorphism, in a natural connection with the classical Banach–Stone theorem.

The aim of this note is to determine the form of all order isomorphisms between little Lipschitz algebras.

Let (X, d_X) and (Y, d_Y) be compact metric spaces and let $\alpha, \beta \in (0, 1)$. Obviously, if $a : Y \to (0, \infty)$ is a function in $lip(Y, d_Y^\beta)$ and h is a Lipschitz homeomorphism from (Y, d_Y^β) onto (X, d_X^α), then the map $T : lip(X, d_X^\alpha) \to lip(Y, d_Y^\beta)$ defined by $T(f) = a \cdot (f \circ h)$ for every $f \in lip(X, d_X^\alpha)$, is an order isomorphism.

Our purpose is to prove that the converse is also true: every order isomorphism T from $lip(X, d_X^\alpha)$ onto $lip(Y, d_Y^\beta)$ is a weighted composition operator of the form above. In particular, T is an algebra isomorphism followed by multiplication with an invertible positive element. Thus, the pointwise order in $lip(X, d_X^\alpha)$ determines the Lipschitz structure of compact metric space X. Analogous assertions hold for order isomorphisms from $Lip(X, d_X^\alpha)$ onto $Lip(Y, d_Y^\beta)$, as a consequence of [3, Theorem 3.10].

We must point out that our approach is different of those of Weaver [8] and Garrido–Jaramillo [3], and depends on the analysis of the support map associated with every order isomorphism. Precisely, the support map of an order isomorphism T from $lip(X, d_X^\alpha)$ onto $lip(Y, d_Y^\beta)$ is the Lipschitz homeomorphism $h : Y \to X$. The concept of support map appears in the study of the multiplicative representation of disjointness preserving operators on vector lattices [1], similar in form to our Banach–Stone type representation. Following this line of research, order isomorphisms between some types of Fourier algebras were studied in [2, 4].

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2. The results

We begin by proving that every order-preserving linear map between little Lipschitz algebras is automatically continuous for the respective Lipschitz norms.

Lemma 2.1. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $\alpha, \beta \in (0, 1]$. Every order-preserving linear map $T : lip(X, d_X^\alpha) \to lip(Y, d_Y^\beta)$ is continuous.*

Proof. In order to prove the continuity of the linear map T, we use the closed graph theorem. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $lip(X, d_X^\alpha)$ such that $\|f_n\|_\alpha$ converges to 0, and $\|T(f_n) - g\|_\beta$ converges to 0 for some $g \in lip(Y, d_Y^\beta)$. We have to show that $g = 0$. Since convergence in the Lipschitz norm implies uniform convergence, then $\|f_n\|_\infty$ and $\|T(f_n)-g\|_\infty$ converge to 0. From the inequalities

$$-\|f_n\|_\infty 1_X \leq Re(f_n) \leq \|f_n\|_\infty 1_X,$$

$$-\|f_n\|_\infty 1_X \leq Im(f_n) \leq \|f_n\|_\infty 1_X,$$

we deduce that

$$-\|f_n\|_\infty T(1_X) \leq T(Re(f_n)) \leq \|f_n\|_\infty T(1_X),$$

$$-\|f_n\|_\infty T(1_X) \leq T(Im(f_n)) \leq \|f_n\|_\infty T(1_X),$$

which yield

$$\|T(f_n)\|_\infty \leq 2\|f_n\|_\infty \|T(1_X)\|_\infty .$$

Then $\|T(f_n)\|_\infty$ converges to 0 and thus $g=0$. □

We now need to recall some separation properties of Lipschitz algebras. Given a compact metric space (X,d), Sherbert proved that $Lip(X,d^\alpha)$ for $\alpha\in(0,1]$ is regular [6, Corollary 4.3]. Using the regularity of $Lip(X,d)$ and the easily checked fact that $Lip(X,d)\subset lip(X,d^\alpha)$ when $\alpha\in(0,1)$, Sherbert deduced the regularity of $lip(X,d^\alpha)$ for $\alpha\in(0,1)$ [7, Proposition 2.1]. On the other hand, $lip(X,d)$ is not regular in general since there are spaces $lip(X,d)$ which consist only of constant functions (see [7, p. 245] for an example). Moreover, Sherbert stated without proof in [7, p. 253] that $Lip(X,d)$ is normal. From the two methods suggested by him to prove this fact, the more direct one is perhaps the following: if A_k and B_k are disjoint closed subsets of X, then $d(A_k,B_k)>0$ and the function $h_k : X\to[0,1]$ defined by $h_k(z)=\max\{0,1-d(z,B_k)/d(A_k,B_k)\}$ belongs to $Lip(X,d)$ and satisfies that $h_k(z)=0$ for all $z\in A_k$ and $h_k(z)=1$ for all $z\in B_k$.

Given a set X and a function $f : X\to\mathbb{K}$, let $coz(f)$ denote the set of all points $x\in X$ such that $f(x)\neq 0$, and let $\overline{coz}(f)$ denote the closure of $coz(f)$ in X. The following lemma is a version for $Lip(X,d)$ of the classical result on the existence of a partition of the unity on X subordinate to a covering. We include it for the sake of completeness.

Lemma 2.2. *Let (X,d) be a compact metric space and let $\{U_1,...,U_n\}$ be an open covering of X. Then there exist positive functions $f_1,...,f_n$ in $Lip(X,d)$ such that $\sum_{k=1}^n f_k = 1_X$ and $coz(f_k)\subset U_k$ for each $k\in\{1,...,n\}$.*

PROOF. For each $x\in X$, let $W(x)$ be a compact neighborhood of x such that $W(x)\subset U_k$ for some $k\in\{1,...,n\}$. Since X is compact, there exist $x_1,...,x_m\in X$ such that $X\subset\cup_{j=1}^m W(x_j)$. For each $k\in\{1,...,n\}$, let V_k be the union of all compact sets $W(x_j)$ such that $W(x_j) \subset U_k$. Since each V_k is compact and $V_k \subset U_k$, we have $d(X\backslash U_k, V_k) > 0$ and the function $h_k : X \to [0,1]$ defined by

$$h_k(z) = \max\{0, 1 - d(z, V_k)/d(X\backslash U_k, V_k)\}$$

is in $Lip(X,d)$ with $h_k(z) = 1$ for all $z \in V_k$ and $coz(h_k) \subset U_k$. Let us define the following functions:

$$\begin{array}{lcl} f_1 & = & h_1, \\ f_2 & = & (1_X - h_1)h_2, \\ \vdots & & \vdots \\ f_n & = & (1_X - h_1)(1_X - h_2)\cdots(1_X - h_{n-1})h_n. \end{array}$$

Clearly, $coz(f_k) \subset coz(h_k) \subset U_k$ for $k = 1, ..., n$. Given $x \in X$, we have

$$f_1(x) + f_2(x) = h_1(x) + (1 - h_1(x))h_2(x)$$

$$= 1 - (1 - h_1(x)) + (1 - h_1(x))h_2(x) = 1 - (1 - h_1(x))(1 - h_2(x)).$$

By induction, we prove at once that

$$f_1(x) + f_2(x) + \cdots + f_n(x)$$

$$= 1 - (1 - h_1(x))(1 - h_2(x))\cdots(1 - h_n(x)).$$

Since $x \in X \subset \cup_{j=1}^m W(x_j)$, there exists $k_0 \in \{1, ..., n\}$ such that $x \in V_{k_0}$. Then $h_{k_0}(x) = 1$ and $f_1(x) + f_2(x) + \cdots + f_n(x) = 1$. $\square$

For any functional F in the dual space $lip(X, d^\alpha)^*$ of $lip(X, d^\alpha)$, we define $supp(F)$ to be the set of all points $x \in X$ such that for each neighborhood U of x, there exists a function $f \in lip(X, d^\alpha)$ with $coz(f) \subset U$ such that $F(f) \neq 0$.

In $lip(X, d^\alpha)^*$ we can define the following order which we also denote by $\geq$: if $F \in lip(X, d^\alpha)^*$, let $F \geq 0$ if and only if $F(f) \geq 0$ for all $f \in lip(X, d^\alpha)$ such that $f \geq 0$. In the case that $F \geq 0$, we can improve the definition of $supp(F)$ with the next observation.

Lemma 2.3. *Let (X,d) be a compact metric space, let $\alpha \in (0,1)$ and let $F \in lip(X, d^\alpha)^*$ with $F \geq 0$. Then $x \in supp(F)$ if and only if for every neighborhood U of x there exists a positive function $f \in Lip(X,d)$ with $coz(f) \subset U$ such that $F(f) > 0$.*

Proof. To prove the "only if" part, let $x \in supp(F)$ and let U be a neighborhood of x. Let V be a neighborhood of x such that $\overline{V} \subset U$. By the definition of $supp(F)$, one has a $g \in lip(X, d^\alpha)$ such that $coz(g) \subset V$ and $F(g) \neq 0$. Since $\overline{coz}(g)$ is closed and $\overline{coz}(g) \subset U$, by the normality of $Lip(X,d)$ there exists $f \in Lip(X,d)$ with $f \geq 0$ such that $coz(f) \subset U$ and $f(z) = 1$ for all $z \in \overline{coz}(g)$. Then

$$- \|g\|_\infty f \leq Re(g) \leq \|g\|_\infty f,$$

$$- \|g\|_\infty f \leq Im(g) \leq \|g\|_\infty f.$$

Since $F \geq 0$ and $Re(g), Im(g) \in lip(X, d^\alpha)$, it follows that

$$- \|g\|_\infty F(f) \leq F(Re(g)) \leq \|g\|_\infty F(f),$$

$$- \|g\|_\infty F(f) \leq F(Im(g)) \leq \|g\|_\infty F(f),$$

and thus $|F(g)| \leq 2\|g\|_\infty F(f)$. This implies $F(f) > 0$ since $F(g) \neq 0$. The proof of the "if" part is trivial. □

After this preparation, we formulate our main result. Recall that a map between metric spaces $h : X \to Y$ is a Lipschitz homeomorphism if h is a bijection such that h and h^{-1} are both Lipschitz.

Theorem 2.4. *Let (X, d_X) and (Y, d_Y) be compact metric spaces and let $\alpha, \beta \in (0,1)$. A bijective linear map $T : lip(X, d_X^\alpha) \to lip(Y, d_Y^\beta)$ is an order isomorphism if and only if there exists a nonvanishing positive function a in $lip(Y, d_Y^\beta)$ and a Lipschitz homeomorphism h from (Y, d_Y^β) onto (X, d_X^α) such that T is of the form*

$$T(f) = a \cdot (f \circ h), \quad \forall f \in lip(X, d_X^\alpha).$$

Moreover,

$$T^{-1}(g) = [1/(a \circ h^{-1})] \cdot (g \circ h^{-1}), \quad \forall g \in lip(Y, d_Y^\beta).$$

Proof. It is straightforward to check that every map T of the form $T(f) = a \cdot (f \circ h)$ for all $f \in lip(X, d_X^\alpha)$ with a, h being as in the statement above, is a linear bijection from $lip(X, d_X^\alpha)$ onto $lip(Y, d_Y^\beta)$ such that T and T^{-1} are both order-preserving.

Let us suppose now that $T : lip(X, d_X^\alpha) \to lip(Y, d_Y^\beta)$ is an order isomorphism. For each $y \in Y$, let δ_y be the linear functional on $lip(Y, d_Y^\beta)$ defined by $\delta_y(f) = f(y)$. The idea to define the function h is to show that for each point $y \in Y$, the set $supp(\delta_y \circ T)$ is a singleton of X, which we shall denote by $h(y)$. We have divided the proof of the "only if" part into five steps.

Step 1. *For each $y \in Y$, $supp(\delta_y \circ T)$ is a singleton.*

PROOF. Let $y \in Y$. Since T is surjective and $lip(Y, d_Y^\beta)$ separates the points of Y, we have $\delta_y \circ T \neq 0$. Let us suppose that $supp(\delta_y \circ T)$ is empty. Then, for each $x \in X$, there exists a neighborhood $U(x)$ of x such that $T(f)(y) = 0$ for all $f \in lip(X, d_X^\alpha)$ satisfying that $coz(f) \subset U(x)$. By the compactness of X, we have $X = \cup_{k=1}^n U(x_k)$ for some natural n. By Lemma 2.2, there exist $f_1, ..., f_n \in Lip(X, d_X)$ such that $\sum_{k=1}^n f_k = 1_X$ and $coz(f_k) \subset U(x_k)$ for all $k \in \{1, ..., n\}$. Then, for any $f \in lip(X, d_X^\alpha)$, we have $f = \sum_{k=1}^n f f_k$, and therefore

$$T(f)(y) = T(\sum_{k=1}^n f f_k)(y) = \sum_{k=1}^n T(f f_k)(y) = 0,$$

since T is linear and $f f_k \in lip(X, d_X^\alpha)$ with $coz(f f_k) \subset coz(f_k) \subset U(x_k)$ for every $k \in \{1, ..., n\}$. Thus $\delta_y \circ T = 0$, which is a contradiction. Hence $supp(\delta_y \circ T)$ is nonempty.

On the other hand, let us suppose that x_1 and x_2 are two distinct points of $supp(\delta_y \circ T)$. Let U_1 and U_2 be disjoint neighborhoods of x_1 and x_2, respectively. Since $\delta_y \circ T \geq 0$ ($f \geq 0$ implies $T(f) \geq 0$ since T is order-preserving linear, and so $(\delta_y \circ T)(f) = T(f)(y) \geq 0$), Lemma 2.3 gives us two positive functions $f_1, f_2 \in Lip(X, d_X)$ such that $coz(f_1) \subset U_1$, $coz(f_2) \subset U_2$, $T(f_1)(y) > 0$ and $T(f_2)(y) > 0$. Let $k = \min\{T(f_1), T(f_2)\}$. It is easily seen that $k \in lip(Y, d_Y^\beta)$ with $0 \leq k \leq T(f_1)$ and $0 \leq k \leq T(f_2)$. Furthermore, k is nonzero since $k(y) > 0$. Let $j = T^{-1}(k)$. Since T^{-1} is order-preserving linear, we have $0 \leq j \leq f_1$ and $0 \leq j \leq f_2$. But j is also nonzero, hence there exists a point $x \in X$ such that $0 < j(x) \leq f_1(x)$ and $0 < j(x) \leq f_2(x)$. This implies $x \in U_1 \cap U_2$, which is impossible. □

Step 1 permits us to define a mapping $h : Y \to X$ such that $h(y) = supp(\delta_y \circ T)$ for any $y \in Y$. Following the literature, we call h the support map of T.

Step 2. *If $y \in Y$, $f \in lip(X, d_X^\alpha)$ with $f \geq 0$ and $h(y) \notin \overline{coz}(f)$, then $T(f)(y) = 0$.*

PROOF. Since $h(y) \notin \overline{coz}(f)$, there exists a neighborhood U of $h(y)$ such that $coz(f) \subset X \backslash U$. On the other hand, since $h(y) = supp(\delta_y \circ T)$, Lemma 2.3 gives us a positive function $g \in Lip(X, d_X)$ such that $coz(g) \subset U$ and $T(g)(y) > 0$. It follows that $fg = 0$.

To obtain a contradiction, assume $Tf(y) \neq 0$. Since $f \geq 0$, we have $T(f) \geq 0$. Define $k = \min\{T(f), T(g)\}$. Clearly, k is a nonzero function in $lip(Y, d_Y^\beta)$ such that $0 \leq k \leq T(f)$ and $0 \leq k \leq T(g)$. Set $h = T^{-1}(k)$. Since T^{-1} is order-preserving linear, it follows that $0 \leq h \leq f$ and $0 \leq h \leq g$. Since h is nonzero, there exists some $x \in X$ for which $0 < h(x) \leq f(x)$ and $0 < h(x) \leq g(x)$. In consequence, we have $f(x)g(x) > 0$, contrary to $fg = 0$. □

Step 3. $\ker \delta_{h(y)} \subset \ker(\delta_y \circ T)$ *for every* $y \in Y$.

Proof. Let $y \in Y$ and suppose that $\ker \delta_{h(y)}$ is not contained in $\ker(\delta_y \circ T)$. Then we can find a positive function $f \in lip(X, d_X^\alpha)$ such that $f(h(y)) = 0$, but $T(f)(y) > 0$. Choose a positive real number λ such that $\lambda T(1_X)(y) < T(f)(y)$. Therefore

$$\max \{T(f)(y), T(\lambda 1_X)(y)\} = T(f)(y).$$

On the other hand, since f is continuous at $h(y)$ and $f(h(y)) = 0$, there exists a neighborhood U of $h(y)$ such that $f(x) < \lambda$ for all $x \in U$, and thus $\max \{f, \lambda 1_X\} = \lambda 1_X$ on U. Then Step 2 yields

$$T(\max \{f, \lambda 1_X\})(y) = T(\lambda 1_X)(y).$$

Taking into account that

$$T(\max \{f, \lambda 1_X\}) = \max \{T(f), T(\lambda 1_X)\},$$

it follows that $T(\lambda 1_X)(y) = T(f)(y)$, a contradiction. □

Step 4. *There is a function* a *in* $lip(Y, d_Y^\beta)$ *with* $a(y) > 0$ *for all* $y \in Y$ *such that*

$$T(f)(y) = a(y)f(h(y)), \quad \forall f \in lip(X, d_X^\alpha),\ \forall y \in Y.$$

Proof. Let $f \in lip(X, d_X^\alpha)$ and $y \in Y$. Set $g = f - f(h(y))1_X$. Since $g \in lip(X, d_X^\alpha)$ and $g(h(y)) = 0$, Step 3 gives $T(g)(y) = 0$, that is, $T(f)(y) = T(1_X)(y)f(h(y))$. Define $a = T(1_X)$. Then $T(f)(y) = a(y)f(h(y))$. Clearly, $a \in lip(Y, d_Y^\beta)$ and $a \geq 0$.

We claim that $a(y) > 0$ for all $y \in Y$. If $a(y) = 0$ for some $y \in Y$, we have $T(f)(y) = 0$ for all $f \in lip(X, d_X^\alpha)$. Because of the surjectivity of T, it follows that $g(y) = 0$ for all $g \in lip(Y, d_Y^\beta)$, which contradicts that $lip(Y, d_Y^\beta)$ separates the points of Y. □

Step 5. *The support map* h *of* T *is a Lipschitz homeomorphism from* (Y, d_Y^β) *to* (X, d_X^α).

Proof. We begin by proving that h is bijective. Since T^{-1} is also an order isomorphism of $lip(Y, d_Y^\beta)$ onto $lip(X, d_X^\alpha)$, from what has already been proved we deduce that there exist $b \in lip(X, d_X^\alpha)$ with $b(x) > 0$ for all $x \in X$ and $k : X \to Y$ such that

$$T^{-1}(g) = b \cdot (g \circ k), \quad \forall g \in lip(Y, d_Y^\beta).$$

For any $f \in lip(X, d_X^\alpha)$, we have

$$f = T^{-1}(Tf) = T^{-1}(a \cdot (f \circ h)) = b \cdot (a \circ k) \cdot ((f \circ h) \circ k).$$

Taking above $f = 1_X$, we have $b = 1/(a \circ k)$. We check that $h \circ k = I_X$, where I_X denotes the identity function on X. If it were not true, there would exist a point $x_0 \in X$ for which $(h \circ k)(x_0) = x_1 \neq x_0$. Taking $f \in lip(X, d_X^\alpha)$ such that $f(x_0) = 0$ and $f(x_1) = 1$, we would have

$$0 = f(x_0) = b(x_0)a(k(x_0))f(x_1) = b(x_0)a(k(x_0)) > 0,$$

which is impossible. In the same manner, we can see that $k \circ h = I_Y$. So h is bijective and $k = h^{-1}$ and $b = 1/(a \circ h^{-1})$.

We now prove that h is Lipschitz. Fix a pair of distinct points p, q of Y and choose a real number γ strictly between α and 1. Define $f_{pq} : X \to \mathbb{R}$ by

$$f_{pq}(x) = \frac{d_X^\gamma(x, h(q)) - d_X^\gamma(x, h(p))}{2d_X^{\gamma-\alpha}(h(p), h(q))}, \quad \forall x \in X.$$

We claim that $f_{pq} \in lip(X, d^\alpha)$ with $\|f_{pq}\|_\alpha = 1 + (1/2)d_X^\alpha(h(q), h(p))$. First, we have $\|f_{pq}\|_\infty = (1/2)d_X^\alpha(h(q), h(p))$ since

$$|f_{pq}(x)| \leq (1/2)d_X^\alpha(h(q), h(p)) = |f_{pq}(h(p))|, \quad \forall x \in X.$$

On the other hand, $p_\alpha(f_{pq}) = 1$ because

$$\begin{aligned} &\frac{|f_{pq}(z) - f_{pq}(w)|}{d_X^\alpha(z, w)} \\ &= \frac{|d_X^\gamma(z, h(q)) - d_X^\gamma(z, h(p)) + d_X^\gamma(w, h(p)) - d_X^\gamma(w, h(q))|}{2d_X^{\gamma-\alpha}(h(p), h(q))d_X^\alpha(z, w)} \\ &\leq \frac{2\min\{d_X^\gamma(z, w), d_X^\gamma(h(p), h(q))\}}{2d_X^{\gamma-\alpha}(h(p), h(q))d_X^\alpha(z, w)} \\ &= \min\left\{\frac{d_X^\alpha(h(p), h(q))}{d_X^\alpha(z, w)}, \frac{d_X^{\gamma-\alpha}(z, w)}{d_X^{\gamma-\alpha}(h(p), h(q))}\right\} \leq 1 \end{aligned}$$

for all $z, w \in X$ with $z \neq w$, and

$$\frac{f_{pq}(h(p)) - f_{pq}(h(q))}{d_X^\alpha(h(p), h(q))} = \frac{d_X^\gamma(h(p), h(q)) - (-d_X^\gamma(h(q), h(p)))}{2d_X^{\gamma-\alpha}(h(p), h(q))d_X^\alpha(h(p), h(q))} = 1.$$

Hence $f_{pq} \in Lip(X, d_X^\alpha)$. Indeed, $f_{pq} \in lip(X, d_X^\alpha)$ since, given $\varepsilon > 0$, define $\delta = d_X(h(p), h(q))\varepsilon^{1/(\gamma-\alpha)}$ and then $0 < d_X(z, w) < \delta$ implies

$$|f_{pq}(z) - f_{pq}(w)| \leq \frac{d_X^{\gamma-\alpha}(z, w)}{d_X^{\gamma-\alpha}(h(p), h(q))} d_X^\alpha(z, w) < \varepsilon d_X^\alpha(z, w).$$

This proves our claim. Consequently, $\|f_{pq}\|_\alpha \leq 1+(1/2)diam(X)^\alpha$ for all $p, q \in Y$ with $p \neq q$, and thus $\{f_{pq} : p, q \in Y,\ p \neq q\}$ is bounded in $lip(X, d_X^\alpha)$. It follows that $\{T(f_{pq}) : p, q \in Y,\ p \neq q\}$ is bounded in $lip(Y, d_Y^\beta)$ since the linear map T is continuous by Lemma 2.1. Hence there exists a constant $\tau > 0$ such that

$$p_\beta(T(f_{pq})) \leq \|T(f_{pq})\|_\beta \leq \tau, \quad \forall p, q \in Y,\ p \neq q.$$

In consequence, for any $p, q \in Y$ with $p \neq q$, we have

$$|T(f_{pq})(p) - T(f_{pq})(q)| \leq \tau d_Y^\beta(p, q). \tag{1}$$

A trivial verification yields

$$T(f_{pq})(p) = a(p) f_{pq}(h(p)) = a(p) d_X^\alpha(h(q), h(p))/2,$$

$$T(f_{pq})(q) = a(q) f_{pq}(h(q)) = -a(q) d_X^\alpha(h(q), h(p))/2,$$

and thus

$$|T(f_{pq})(p) - T(f_{pq})(q)| = (a(p) + a(q))\,(1/2) d_X^\alpha(h(p), h(q)). \tag{2}$$

Substituting (2) into (1), we obtain

$$(a(p) + a(q))\,(1/2) d_X^\alpha(h(p), h(q)) \leq \tau d_Y^\beta(p, q). \tag{3}$$

Let $\rho = \min\{a(y) : y \in Y\} > 0$. From (3), we deduce

$$d_X^\alpha(h(p), h(q)) \leq (\tau/\rho) d_Y^\beta(p, q),$$

which is the desired conclusion.

Finally, we see that h^{-1} is also Lipschitz. We have that the map

$$T^{-1}(g) = b \cdot (g \circ k), \quad \forall g \in lip(Y, d_Y^\beta)$$

is an order isomorphism of $lip(Y, d_Y^\beta)$ onto $lip(X, d_X^\alpha)$, which is continuous by Lemma 2.1. As above we can prove that k is a Lipschitz bijection of X onto Y, but $k = h^{-1}$ as was shown. □

In this way, the proof of Theorem 2.4 is complete. □

Next, as a direct consequence of Theorem 2.4, we deduce a Banach–Stone type result for little Lipschitz algebras.

Corollary 2.5. *Let (X, d_X) and (Y, d_Y) be compact metric spaces and let $\alpha, \beta \in (0, 1)$. The following are equivalent:*

1. *$lip(X, d_X^\alpha)$ is order isomorphic to $lip(Y, d_Y^\beta)$.*
2. *(X, d_X^α) is Lipschitz homeomorphic to (Y, d_Y^β).*

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