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Lipschitz Algebras and Peripherally-multiplicative Maps

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Abstract Let X be a compact metric space and let $\mathrm{Lip}(X)$ be the Banach algebra of all scalar-valued Lipschitz functions on X, endowed with a natural norm. For each $f \in \mathrm{Lip}(X)$, $\sigma_\pi(f)$ denotes the peripheral spectrum of f. We state that any map Φ from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ which preserves multiplicatively the peripheral spectrum:

$$\sigma_\pi(\Phi(f)\Phi(g)) = \sigma_\pi(fg), \quad \forall f, g \in \mathrm{Lip}(X),$$

is a weighted composition operator of the form $\Phi(f) = \tau \cdot (f \circ \varphi)$ for all $f \in \mathrm{Lip}(X)$, where $\tau : Y \to \{-1, 1\}$ is a Lipschitz function and $\varphi : Y \to X$ is a Lipschitz homeomorphism. As a consequence of this result, any multiplicatively spectrum-preserving surjective map between $\mathrm{Lip}(X)$-algebras is of the form above.

Keywords Lipschitz algebra, peripherally-multiplicative map, spectrum-preserving map, peaking function, peripheral spectrum

MR(2000) Subject Classification 46J10, 46J20

1 Introduction

The problem of characterizing the linear maps between Banach algebras which preserve the spectrum has attracted the attention of many mathematicians (see, e.g., [1–5]). A similar problem has been investigated by several authors omitting the condition of linearity, but assuming the more restrictive condition that the maps preserve the spectrum of products of any two elements of the algebra. Following the literature, and for brevity, we shall call these maps multiplicatively spectrum-preserving.

Molnár initiated in [6] this new approach obtaining some characterizations of the multiplicatively spectrum-preserving maps between some known Banach algebras as, for example, the Banach algebra $C(X)$ of all complex-valued continuous functions on a first countable compact Hausdorff space X endowed with the supremum norm. Namely, if for any $f \in C(X)$, $\sigma(f)$ denotes the spectrum of f, which in this case coincides with the range of f, Molnár [6, Theorem 5] proved that any surjective map $\Phi : C(X) \to C(X)$ which preserves multiplicatively the spectrum, that is $\sigma(\Phi(f)\Phi(g)) = \sigma(fg)$ for all $f, g \in C(X)$, is of the form $\Phi(f) = \tau \cdot (f \circ \varphi)$ for all $f \in C(X)$, where $\tau : X \to \{-1, 1\}$ is a continuous function and $\varphi : X \to X$ is a homeomorphism.

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Rao and Roy [7, Main theorem] obtained a generalization of Molnár's theorem for uniform algebras $A \subset C(X)$, assuming that the compact Hausdorff space X is the maximal ideal space of A.

For the case of uniform algebras A on arbitrary compact Hausdorff spaces X, Hatori, Miura and Takagi [8, Theorem 1.1] generalized the result of Rao and Roy by replacing the spectrum $\sigma(f)$ by the range $\operatorname{Ran}(f) = f(X)$, and the aforementioned spectrum multiplicativity condition: $\operatorname{Ran}(\Phi(f)\Phi(g)) = \operatorname{Ran}(fg)$ for all $f, g \in A$. These same authors proved the corresponding general result for unital semi-simple commutative Banach algebras [9, Theorem 3.2].

Recently, Luttman and Tonev [10] have introduced a new point of view by replacing the range $\operatorname{Ran}(f)$ by the peripheral range

$$\operatorname{Ran}_\pi(f) := \{z \in \operatorname{Ran}(f) : |z| = \|f\|_\infty\},$$

or, equivalently (see [10, Lemma 1]), by the peripheral spectrum

$$\sigma_\pi(f) := \Big\{z \in \sigma(f) : |z| = \max_{w \in \sigma(f)} |w|\Big\}.$$

They have proved [10, Corollary 3] that if A and B are uniform algebras on their maximal ideal spaces X and Y respectively, then any surjective map $\Phi : A \to B$ such that $\sigma_\pi(\Phi(f)\Phi(g)) = \sigma_\pi(fg)$ for all $f, g \in A$, has the form $\Phi(f) = \tau \cdot (f \circ \varphi)$ for all $f \in A$, where $\tau : Y \to \{-1, 1\}$ is a function in B and $\varphi : Y \to X$ is a homeomorphism. Recently, further work has been done in this direction (see [11–12]).

The object of this paper is to show that the result of Luttman and Tonev has a natural formulation in the case of algebras of Lipschitz functions.

In order to state our results we introduce some notation. Given a compact metric space (X, d), $\operatorname{Lip}(X)$ will denote the Banach space of all Lipschitz functions f defined on X with values in the set $\mathbb{K}$ of real or complex numbers, equipped with any of the natural norms:

$$\|f\| = \|f\|_\infty + L(f), \quad \text{or} \quad \|f\| = \max\{\|f\|_\infty, L(f)\},$$

where $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$ is the supremum norm of f, and

$$L(f) := \sup\{|f(x) - f(y)|/d(x, y) : x, y \in X,\ x \neq y\}$$

is the Lipschitz constant of f. It is easily seen that the sum norm $\|f\| = \|f\|_\infty + L(f)$ is a Banach algebra norm, that is,

$$\|fg\| \leq \|f\| \|g\| \quad (f, g \in \operatorname{Lip}(X)),$$

but the maximum norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$ is a Banach algebra norm in the weak sense that it satisfies the law

$$\|fg\| \leq 2\|f\| \|g\| \quad (f, g \in \operatorname{Lip}(X)).$$

The algebras $\operatorname{Lip}(X)$ with any one of these norms are known as Lipschitz algebras, and they have been studied extensively (see, for example, [13] for the Lipschitz algebras with the sum norm, and [14] with the maximum norm).

Recall that a map between metric spaces $\varphi : X \to Y$ is said to be a Lipschitz homeomorphism if φ is a bijection such that φ and φ^{-1} are both Lipschitz.

Given two compact metric spaces X and Y, it is easy to check that any map $\Phi : \operatorname{Lip}(X) \to \operatorname{Lip}(Y)$ defined by $\Phi(f) = \tau \cdot (f \circ \varphi)$ for all $f \in \operatorname{Lip}(X)$, being $\tau : Y \to \{-1, 1\}$ a Lipschitz function and $\varphi : Y \to X$ a Lipschitz homeomorphism, is surjective and preserves multiplicatively the peripheral spectrum, that is,

$$\sigma_\pi(\Phi(f)\Phi(g)) = \sigma_\pi(fg), \quad \forall f, g \in \mathrm{Lip}(X). \tag{1}$$

Following [10], maps that satisfy the condition (1) will be called peripherally-multiplicative maps.

Our main result asserts that the converse is also true. Namely, we prove (Theorem 3.1) that any peripherally-multiplicative surjective map $\Phi : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is a weighted composition operator of the form

$$\Phi(f) = \tau \cdot (f \circ \varphi), \quad \forall\, f \in \mathrm{Lip}(X),$$

where $\tau : Y \to \{-1, 1\}$ is a Lipschitz function and $\varphi : Y \to X$ is a Lipschitz homeomorphism.

In particular, Theorem 3.1 provides a complete description of all the surjective maps from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ which preserve multiplicatively the spectrum (see Corollary 3.4). We can say that this corollary is a Lipschitz version of the aforementioned Molnar's theorem.

Moreover, we show that any unital peripherally-multiplicative surjective map $\Phi : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is an algebra isomorphism of the form $\Phi(f) = f \circ \varphi$ for all $f \in \mathrm{Lip}(X)$, where $\varphi : Y \to X$ is a Lipschitz homeomorphism (Corollary 3.2).

Given a map $\Phi : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$, we say that Φ is multiplicative if $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in \mathrm{Lip}(X)$, and that Φ preserves the peripheral spectrum if $\sigma_\pi(\Phi(f)) = \sigma_\pi(f)$ for all $f \in \mathrm{Lip}(X)$.

We also deduce that any surjective multiplicative map from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ preserving the peripheral spectrum is an algebra isomorphism of the form above (Corollary 3.3).

2 Previous Results

In the first place we shall state some facts concerning Lipschitz functions which will be used later. Recall that a function h in $\mathrm{Lip}(X)$ is said to be a peaking function if for any $z \in X$, either $h(z) = 1$ or $|h(z)| < 1$, and the set $\{z \in X : h(z) = 1\}$ is nonempty. We denote by $F(X)$ the set of all peaking functions of $\mathrm{Lip}(X)$. Observe that $F(X) = \{h \in \mathrm{Lip}(X) : \sigma_\pi(h) = \{1\}\}$. In this paper we shall use frequently the following special class of peaking functions of $\mathrm{Lip}(X)$.

Lemma 2.1 *For each pair $x \in X$ and $\delta > 0$, there exists a Lipschitz function $h_{x,\delta} : X \to [0, 1]$ such that $h_{x,\delta}(x) = 1$ and $h_{x,\delta}(z) < 1$ if $z \neq x$. Furthermore, $h_{x,\delta}(z) = 0$ if $d(z, x) \geq \delta$.*

Proof A trivial verification shows that the function $h_{x,\delta}$ defined on X by

$$h_{x,\delta}(z) = \max\{0, 1 - d(z, x)/\delta\}$$

satisfies the conditions of the statement above.

The following result is a version for Lipschitz functions of a Bishop's theorem (see [15, Theorem 2.4.1]).

Lemma 2.2 *Let $x \in X$ and $f \in \mathrm{Lip}(X)$ with $f(x) \neq 0$. Then there exists a Lipschitz function $h : X \to [0, 1]$ such that $h(x) = 1$, $h(z) < 1$ if $z \neq x$ and $|f(z)h(z)| < |f(x)|$ if $z \neq x$.*

Proof We can suppose, without loss of generality, that $|f(x)| = 1$. Let g be the function defined on X by

$$g(z) = \begin{cases} 1 & \text{if } \ |f(z)| < 1, \\ 2 - |f(z)| & \text{if } \ 1 \leq |f(z)| \leq 2, \\ 0 & \text{if } \ |f(z)| > 2. \end{cases}$$

An easy calculation shows that $g \in \mathrm{Lip}(X)$, $g(x) = 1$, $0 \leq g \leq 1$ y $|fg| \leq 1$. Define $h = gh_{x,1}$. It follows immediately that h verifies the properties cited in the statement.

The next lemma provides a method to identify two Lipschitz functions with the aid of the peripheral spectrum and the peaking functions.

Lemma 2.3 *Let $f, g \in \mathrm{Lip}(X)$.*

i) *If $\|fh\|_\infty \leq \|gh\|_\infty$ for all $h \in F(X)$, then $|f| \leq |g|$.*

ii) *If $\|fh\|_\infty = \|gh\|_\infty$ for all $h \in F(X)$, then $|f| = |g|$.*

iii) *If $\sigma_\pi(fh) = \sigma_\pi(gh)$ for all $h \in F(X)$, then $f = g$.*

Proof i) Suppose there exists a point $x \in X$ such that $|f(x)| > |g(x)|$. Let ε be a real number such that $|f(x)| > \varepsilon > |g(x)|$. By the continuity of g at x, there exists a $\delta > 0$ such that $|g(z)| < \varepsilon$ if $d(z, x) < \delta$. Consider the function $h_{x,\delta} \in F(X)$ defined at Lemma 2.1. Clearly $|gh_{x,\delta}| < \varepsilon$ and $\|fh_{x,\delta}\|_\infty \geq |f(x)h_{x,\delta}(x)| = |f(x)| > \varepsilon$. Then $\|gh_{x,\delta}\|_\infty \leq \varepsilon < \|fh_{x,\delta}\|_\infty$ and this proves i).

ii) follows immediately from i).

iii) Given $h \in F(X)$, it is clear that $\sigma_\pi(fh) = \sigma_\pi(gh)$ implies $\|fh\|_\infty = \|gh\|_\infty$. Hence $|f| = |g|$ by ii). We now check that $f = g$. Let $x \in X$. If $f(x) = 0$ or $g(x) = 0$, it is clear that $f(x) = 0 = g(x)$. If $f(x) \neq 0 \neq g(x)$, by Lemma 2.2 there exists a function $h \in F(X)$ with $h(x) = 1$ such that $h(z) < 1$, $|f(z)h(z)| < |f(x)|$ and $|g(z)h(z)| < |g(x)|$ for all $z \neq x$. From this, we deduce that $\sigma_\pi(fh) = \{f(x)\}$ and $\sigma_\pi(gh) = \{g(x)\}$. Since $\sigma_\pi(fh) = \sigma_\pi(gh)$, we have $f(x) = g(x)$.

3 The Main Theorem and Some Consequences

Theorem 3.1 *Let $\Phi : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a surjective mapping such that*

$$\sigma_\pi(\Phi(f)\Phi(g)) = \sigma_\pi(fg), \quad \forall f, g \in \mathrm{Lip}(X).$$

Then there exist a Lipschitz function $\tau : Y \to \{-1, 1\}$ and a Lipschitz homeomorphism $\varphi : Y \to X$ such that

$$\Phi(f) = \tau \cdot (f \circ \varphi), \quad \forall f \in \mathrm{Lip}(X).$$

Proof We begin with the following fact:

Remark 3.1 Φ is uniform norm multiplicative, that is,

$$\|\Phi(f)\Phi(g)\|_\infty = \|fg\|_\infty, \quad \forall f, g \in \mathrm{Lip}(X),$$

and Φ preserves the uniform norm, that is

$$\|\Phi(f)\|_\infty = \|f\|_\infty, \quad \forall f \in \mathrm{Lip}(X).$$

Proof Let $f, g \in \mathrm{Lip}(X)$. The first equality follows from $\sigma_\pi(\Phi(f)\Phi(g)) = \sigma_\pi(fg)$ since $|z| = \|f\|_\infty$ for all $z \in \sigma_\pi(f)$. Using the first equality we obtain

$$\|\Phi(f)\|_\infty^2 = \left\|\Phi(f)^2\right\|_\infty = \left\|f^2\right\|_\infty = \|f\|_\infty^2,$$

which implies $\|\Phi(f)\|_\infty = \|f\|_\infty$.

We distinguish two cases:

Unital Case Assume Φ is unital, that is, $\Phi(1_X) = 1_Y$ where 1_X and 1_Y denote the functions constantly equal 1 on X and on Y, respectively. We divide the proof of this case into a sequence of steps.

Step 1 Φ preserves the peripheral spectrum, that is,

$$\sigma_\pi(\Phi(f)) = \sigma_\pi(f), \quad \forall f \in \mathrm{Lip}(X).$$

Proof Let $f \in \mathrm{Lip}(X)$. Since $\Phi(1_X) = 1_Y$ and Φ is peripherally-multiplicative, we have $\sigma_\pi(\Phi(f)) = \sigma_\pi(\Phi(f)1_Y) = \sigma_\pi(\Phi(f)\Phi(1_X)) = \sigma_\pi(f1_X) = \sigma_\pi(f)$.

Step 2 Φ is injective.

Proof Let $f, g \in \mathrm{Lip}(X)$ and suppose $\Phi(f) = \Phi(g)$. Then, for any $h \in F(X)$, we have $\Phi(f)\Phi(h) = \Phi(g)\Phi(h)$. Consequently $\sigma_\pi(\Phi(f)\Phi(h)) = \sigma_\pi(\Phi(g)\Phi(h))$, and since Φ is peripherally-multiplicative, it follows that $\sigma_\pi(fh) = \sigma_\pi(gh)$. Then $f = g$ by Lemma 2.3.

Step 3 $\Phi|_{F(X)}$ is a bijection from $F(X)$ onto $F(Y)$.

Proof In view of Step 2, it suffices to show that $\Phi(F(X)) = F(Y)$. If $f \in F(X)$, then $\sigma_\pi(f) = \{1\}$, hence $\sigma_\pi(\Phi(f)) = \{1\}$ by Step 1 and thus $\Phi(f) \in F(Y)$. Conversely, if $g \in F(Y)$, then there exists $f \in \mathrm{Lip}(X)$ such that $g = \Phi(f)$ by the surjectivity of Φ, therefore $\sigma_\pi(f) = \sigma_\pi(\Phi(f)) = \sigma_\pi(g) = \{1\}$ and so $f \in F(X)$.

Step 4 Φ is homogeneous.

Proof Let $\lambda \in \mathbb{K}$ and $f \in \mathrm{Lip}(X)$. For every function $h \in F(X)$,

$$\sigma_\pi(\Phi(\lambda f)\Phi(h)) = \sigma_\pi(\lambda f h) = \lambda\sigma_\pi(fh) = \lambda\sigma_\pi(\Phi(f)\Phi(h)) = \sigma_\pi(\lambda\Phi(f)\Phi(h)).$$

It follows that $\sigma_\pi(\Phi(\lambda f)g) = \sigma_\pi(\lambda\Phi(f)g)$ for all $g \in F(Y)$ by Step 3. Then Lemma 2.3 gives $\Phi(\lambda f) = \lambda\Phi(f)$.

Step 5 If $f, g \in \mathrm{Lip}(X)$, then $|f| \leq |g|$ if and only if $|\Phi(f)| \leq |\Phi(g)|$.

Proof Assume $|f| \leq |g|$ and let $k \in F(Y)$. Then $k = \Phi(h)$ for some $h \in F(X)$ by Step 3. Clearly $\|fh\|_\infty \leq \|gh\|_\infty$. Using Remark 3.1, we obtain

$$\|\Phi(f)\Phi(h)\|_\infty = \|fh\|_\infty \leq \|gh\|_\infty = \|\Phi(g)\Phi(h)\|_\infty,$$

and thus $\|\Phi(f)k\|_\infty \leq \|\Phi(g)k\|_\infty$. Since k is arbitrary, we infer that $|\Phi(f)| \leq |\Phi(g)|$ by Lemma 2.3.

To prove the another implication observe that the mapping Φ^{-1} has the same properties as Φ. Applying the above-proved implication to Φ^{-1}, we obtain that for $h, k \in \mathrm{Lip}(Y)$, $|h| \leq |k|$ implies $\left|\Phi^{-1}(h)\right| \leq \left|\Phi^{-1}(k)\right|$. In particular, for $f, g \in \mathrm{Lip}(X)$, we obtain $|f| \leq |g|$ when $|\Phi(f)| \leq |\Phi(g)|$.

Step 6 For each $x \in X$, let $F_x(X)$ be the set of all functions $f \in F(X)$ such that $f(x) = 1$, that is,

$$F_x(X) := \{f \in \mathrm{Lip}(X) : \sigma_\pi(f) = \{1\},\ f(x) = 1\}.$$

As well, for each $y \in Y$ we define the set

$$F_y(Y) := \{f \in \mathrm{Lip}(Y) : \sigma_\pi(f) = \{1\},\ f(y) = 1\}.$$

Let $x \in X$.

i) If $f \in F_x(X)$, then $\Phi(f) \in F_y(Y)$ for some point $y \in Y$.

ii) There exists a point $z \in Y$ such that $\Phi(f)(z) = 1$ for every $f \in F_x(X)$.

iii) If $k \in F_z(Y)$ and $k = \Phi(h)$ for some $h \in \mathrm{Lip}(X)$, then $h \in F_x(X)$.

iv) If $w \in Y$ and $\Phi(f)(w) = 1$ for all $f \in F_x(X)$, then $w = z$.

Proof i) If $f \in F_x(X)$, we have $f \in F(X)$. Then Step 3 yields $\Phi(f) \in F(Y)$, that is, $\sigma_\pi(\Phi(f)) = \{1\}$. Hence $\Phi(f)(y) = 1$ for some $y \in Y$ and thus $\Phi(f) \in F_y(Y)$.

ii) For each $f \in F_x(X)$, the set $P(f) := \{y \in Y : \Phi(f)(y) = 1\}$ is nonempty by i) and closed in Y. We shall prove that the family $\{P(f) : f \in F_x(X)\}$ has the finite intersection property. Let $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in F_x(X)$. Let us define $g = f_1 \cdots f_n \in \mathrm{Lip}(X)$ and let us check that $g \in F_x(X)$. Clearly $g(x) = 1$ and $\|g\|_\infty = 1$, hence $1 \in \sigma_\pi(g)$. Let $w \in X$ such that $g(w) \in \sigma_\pi(g)$. Suppose there exists some $j \in \{1, \ldots, n\}$ with $f_j(w) \neq 1$. Since $\sigma_\pi(f_j) = \{1\}$, we have $|f_j(w)| < 1$ and thus $|g(w)| \leq |f_j(w)| < 1$, a contradiction. Therefore, $f_j(w) = 1$ for all $j \in \{1, \ldots, n\}$ and so $g(w) = 1$. Hence $\sigma_\pi(g) = \{1\}$ and $g \in F_x(X)$.

Clearly $|g| \leq |f_i|$ for all $i \in \{1, \ldots, n\}$. By Step 5, $|\Phi(g)| \leq |\Phi(f_i)|$ for all $i \in \{1, \ldots, n\}$. By i), $\Phi(g)(y) = 1$ for some point $y \in Y$. Then, for each $i \in \{1, \ldots, n\}$, Step 3 gives $\sigma_\pi(\Phi(f_i)) = \{1\}$, and therefore $\Phi(f_i)(y) = 1$. This proves our claim. By the compactness of Y it follows that there exists a point $z \in X$ such that $\Phi(f)(z) = 1$ for all $f \in F_x(X)$.

iii) From Step 3 we deduce that $h \in F(X)$. It remains to show that $h(x) = 1$. Evidently $h_{x,1} \in F_x(X)$, and therefore $\Phi(h_{x,1})(z) = 1$ by ii). Let $\lambda = \Phi(h_{x,1})\Phi(h)$. Clearly $\lambda \in \mathrm{Lip}(Y)$ and so $\lambda = \Phi(\mu)$ for a certain function $\mu \in \mathrm{Lip}(X)$. Since $|\Phi(\mu)| \leq \min\{|\Phi(h_{x,1})|, |\Phi(h)|\}$, from Step 5 we infer that $|\mu| \leq \min\{h_{x,1}, |h|\}$. As $\sigma_\pi(\mu) = \sigma_\pi(\Phi(\mu)) = \sigma_\pi(\lambda)$ and $\lambda(z) = \Phi(h_{x,1})(z)\Phi(h)(z) = \Phi(h_{x,1})(z)k(z) = 1 = \|\lambda\|_\infty$, we see that $\mu(w) = 1$ for a suitable $w \in X$. Since $\max\{h_{x,1}, |h|\} \leq 1$, we deduce that $h_{x,1}(w) = |h(w)| = 1$. Hence $h(w) = 1$ since $h \in F(X)$. Clearly $h_{x,1}(w) = 1$ implies $w = x$, and thus $h(x) = 1$.

iv) Suppose, contrary to our claim, that $\Phi(f)(w) = 1$ for all $f \in F_x(X)$ and that $w \neq z$. Consider the function $h_{z,1} \in F(Y)$. Clearly $h_{z,1}(z) = 1$ and $h_{z,1}(w) < 1$. By Step 3, $\Phi(g) = h_{z,1}$ for some $g \in F(X)$. As $h_{z,1} \in F_z(Y)$, we have $g \in F_x(X)$ by iii). Thus $\Phi(g)(w) = 1$ by hypothesis, this is $h_{z,1}(w) = 1$, a contradiction.

Taking into account Step 6, we can consider the following mapping:

Definition 3.1 *Let $\psi : X \to Y$ be the function which sends the point $x \in X$ to the unique point $\psi(x) \in Y$ such that $\Phi(f)(\psi(x)) = 1$ for all $f \in F_x(X)$.*

Remark 3.2 Observe (see assertion iii) of Step 6) that ψ satisfies the following property: If $k \in F_{\psi(x)}(Y)$ and $k = \Phi(h)$ for some $h \in \mathrm{Lip}(X)$, then $h \in F_x(X)$.

Step 7 $|f(x)| = |\Phi(f)(\psi(x))|$ for all $f \in \mathrm{Lip}(X)$ and $x \in X$.

Proof Let $f \in \mathrm{Lip}(X)$ and $x \in X$. Assume first $f(x) \neq 0$. By Lemma 2.2 there exists a function $h \in F_x(X)$ such that $\|fh\|_\infty = |f(x)|$. Since $\|\Phi(f)\Phi(h)\|_\infty = \|fh\|_\infty$ by Remark 3.1, we have $|\Phi(f)(\psi(x))\Phi(h)(\psi(x))| \leq |f(x)|$. As $h \in F_x(X)$, the definition of ψ gives $\Phi(h)(\psi(x)) = 1$ and so $|\Phi(f)(\psi(x))| \leq |f(x)|$.

Assume now $f(x) = 0$. Let $\varepsilon > 0$. By the continuity of f at x we have

$$\exists\ \delta > 0 : d(z,x) < \delta \quad \Rightarrow \quad |f(z)| < \varepsilon.$$

Take the function $h_{x,\delta} \in F_x(X)$. Given $z \in X$, we see that

$$d(z,x) < \delta \quad \Rightarrow \quad |f(z)h_{x,\delta}(z)| \le |f(z)| < \varepsilon,$$

$$d(z,x) \ge \delta \quad \Rightarrow \quad |f(z)h_{x,\delta}(z)| = 0 < \varepsilon.$$

Therefore, $\|fh_{x,\delta}\|_\infty < \varepsilon$. By Remark 3.1 we have $\|\Phi(f)\Phi(h_{x,\delta})\|_\infty = \|fh_{x,\delta}\|_\infty$, so

$$|\Phi(f)(\psi(x))\Phi(h_{x,\delta})(\psi(x))| < \varepsilon,$$

and since $\Phi(h_{x,\delta})(\psi(x)) = 1$ by the definition of ψ, we obtain $|\Phi(f)(\psi(x))| < \varepsilon$. Since ε was arbitrary, we have $\Phi(f)(\psi(x)) = 0$ and thus $\Phi(f)(\psi(x)) = 0 = f(x)$. In any case we have proved that

$$|\Phi(f)(\psi(x))| \le |f(x)|.$$

To prove the contrary inequality we first suppose $\Phi(f)(\psi(x)) \ne 0$. Again, by Lemma 2.2, there is a function $g \in F_{\psi(x)}(Y)$ such that $\|\Phi(f)g\|_\infty = |\Phi(f)(\psi(x))|$. With Φ being surjective there exists an $h \in \mathrm{Lip}(X)$ such that $\Phi(h) = g$, and since $g \in F_{\psi(x)}(Y)$, it follows that $h \in F_x(X)$ by Remark 3.2. Using Remark 3.1 it follows that

$$|f(x)| = |f(x)h(x)| \le \|fh\|_\infty = \|\Phi(f)\Phi(h)\|_\infty = |\Phi(f)(\psi(x))|.$$

If $\Phi(f)(\psi(x)) = 0$, we reason as above to obtain $f(x) = 0 = \Phi(f)(\psi(x))$. This proves Step 7.

Step 8 ψ is a homeomorphism from X onto Y.

Proof We first prove the continuity of ψ. Let $x \in X$ and let $\{x_n\}$ be a sequence in X converging to x. Suppose that the sequence $\{\psi(x_n)\}$ does not converge to $\psi(x)$. Then there exist a scalar $\varepsilon > 0$ and a subsequence $\{\psi(x_{\sigma(n)})\}$ such that

$$d(\psi(x_{\sigma(n)}), \psi(x)) \ge \varepsilon, \quad \forall\, n \in \mathbb{N}.$$

Consider $h_{\psi(x),\varepsilon} \in \mathrm{Lip}(Y)$ and let $g \in \mathrm{Lip}(X)$ be such that $\Phi(g) = h_{\psi(x),\varepsilon}$. Since $|g(x_{\sigma(n)})| = |\Phi(g)(\psi(x_{\sigma(n)}))|$ for all $n \in \mathbb{N}$ by Step 7, and

$$\Phi(g)(\psi(x_{\sigma(n)})) = h_{\psi(x),\varepsilon}(\psi(x_{\sigma(n)})) = 0, \quad \forall\, n \in \mathbb{N},$$

we get $g(x_{\sigma(n)}) = 0$ for all $n \in \mathbb{N}$. This implies $g(x) = 0$, but, on the other hand, it is clear that

$$|g(x)| = |\Phi(g)(\psi(x))| = |h_{\psi(x),\varepsilon}(\psi(x))| = 1$$

again by Step 7. We arrive at a contradiction and so ψ is continuous at x.

Let us see now that ψ is surjective. Assume there is a point $y \in Y \backslash \psi(X)$ and let $\varepsilon = d(y, \psi(X))$. Since $\psi(X)$ is closed in Y, $\varepsilon > 0$. Clearly the Lipschitz function $h_{y,\varepsilon}$ is nonzero and $h_{y,\varepsilon}(\psi(z)) = 0$ for all $z \in X$. Let $g \in \mathrm{Lip}(X)$ be such that $\Phi(g) = h_{y,\varepsilon}$. Step 7 gives

$$|g(z)| = |\Phi(g)(\psi(z))| = |h_{y,\varepsilon}(\psi(z))| = 0, \quad \forall\, z \in X.$$

Since Φ is homogeneous by Step 4, $h_{y,\varepsilon} = \Phi(g) = 0$, which is impossible.

We assert that ψ is injective. Let $x, w \in X$ be such that $\psi(x) = \psi(w)$. Assume $x \ne w$. Clearly $h_{x,1}(x) = 1$ and $h_{x,1}(w) < 1$. By Step 7 we obtain $|h_{x,1}(z)| = |\Phi(h_{x,1})(\psi(z))|$ for all $z \in X$. In particular, since $\psi(x) = \psi(w)$ it follows that $h_{x,1}(x) = h_{x,1}(w)$, which is absurd.

Finally, since X is compact, Y is Hausdorff and $\psi : X \to Y$ is a continuous bijection, we conclude that ψ is a homeomorphism.

Step 9 $f(x) = \Phi(f)(\psi(x))$ for all $f \in \mathrm{Lip}(X)$ and $x \in X$.

Proof Let $f \in \mathrm{Lip}(X)$ and $x \in X$. If $f(x) = 0$, Step 7 gives $f(x) = 0 = \Phi(f)(\psi(x))$. If $f(x) \neq 0$, by Lemma 2.2 there exists $h \in F_x(X)$ such that $|f(z)h(z)| < |f(x)|$ for all $z \neq x$. Since $f(x) \in \sigma_\pi(fh) = \sigma_\pi(\Phi(f)\Phi(h))$, there is a point $y \in Y$ such that $f(x) = \Phi(f)(y)\Phi(h)(y)$. By the surjectivity of ψ there exists a $w \in X$ such that $y = \psi(w)$. Hence $f(x) = \Phi(f)(\psi(w))\Phi(h)(\psi(w))$. By Step 7 we have $|\Phi(f)(\psi(w))\Phi(h)(\psi(w))| = |f(w)h(w)|$. Therefore, $|f(x)| = |f(w)h(w)|$. This implies $w = x$ and, consequently, $f(x) = \Phi(f)(\psi(x))\Phi(h)(\psi(x))$. Since $\Phi(h)(\psi(x)) = 1$ by the definition of ψ, we conclude that $f(x) = \Phi(f)(\psi(x))$.

Step 10 Φ is linear and multiplicative.

Proof Φ is homogeneous by Step 4. We check that Φ is additive and multiplicative. Let $f, g \in \mathrm{Lip}(X)$. By Step 9, for each $x \in X$, we have

$$f(x) = \Phi(f)(\psi(x)), \quad g(x) = \Phi(g)(\psi(x))$$

and

$$(f+g)(x) = \Phi(f+g)(\psi(x)), \quad (fg)(x) = \Phi(fg)(\psi(x)).$$

Therefore,

$$\Phi(f+g)(\psi(x)) = (f+g)(x) = f(x) + g(x) = \Phi(f)(\psi(x)) + \Phi(g)(\psi(x))$$

and

$$\Phi(fg)(\psi(x)) = (fg)(x) = f(x)g(x) = \Phi(f)(\psi(x))\Phi(g)(\psi(x)).$$

We infer that $\Phi(f+g) = \Phi(f) + \Phi(g)$ and $\Phi(fg) = \Phi(f)\Phi(g)$, since ψ is surjective by Step 8.

Step 11 Φ is an isomorphism from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$.

Proof Φ is surjective by hypothesis, injective by Step 2 and linear by Step 10. Since Φ is a linear bijection between the Banach spaces $\mathrm{Lip}(X)$ and $\mathrm{Lip}(Y)$, if we show that Φ is continuous, then Φ will be an isomorphism by the inverse mapping theorem.

First we observe that the mapping $\|f\|_0 = \|\Phi^{-1}(f)\|$, where $\|\cdot\|$ is one of the two canonical norms defined on $\mathrm{Lip}(X)$, is a complete norm on $\mathrm{Lip}(Y)$, because Φ^{-1} is a linear bijection from $\mathrm{Lip}(Y)$ onto $\mathrm{Lip}(X)$ and $\|\cdot\|$ is a complete norm on $\mathrm{Lip}(X)$. Moreover, for all $f \in \mathrm{Lip}(Y)$ we have $\|f\|_\infty = \|\Phi^{-1}(f)\|_\infty \leq \|\Phi^{-1}(f)\| = \|f\|_0$ by Step 1 (observe that Φ^{-1} has the same properties as Φ).

To prove the continuity of Φ, let us check first that the identity I on $\mathrm{Lip}(Y)$ is a continuous map from $(\mathrm{Lip}(Y), \|\cdot\|_0)$ onto $(\mathrm{Lip}(Y), \|\cdot\|)$. Let $\{f_n\}$ be a sequence in $\mathrm{Lip}(Y)$ such that $\lim_{n\to\infty} \|f_n\|_0 = 0$ and suppose that $\lim_{n\to\infty} \|f_n - f\| = 0$ for some function $f \in \mathrm{Lip}(Y)$. Since $\|f\|_\infty \leq \|f - f_n\|_\infty + \|f_n\|_\infty \leq \|f_n - f\| + \|f_n\|_0$ for all $n \in \mathbb{N}$, taking limits it follows that $f = 0$. Hence I has closed graph and therefore I is continuous by the closed graph theorem.

On the other hand, Φ is continuous from $(\mathrm{Lip}(X), \|\cdot\|)$ onto $(\mathrm{Lip}(Y), \|\cdot\|_0)$ since $\|\Phi(f)\|_0 = \|f\|$ for all $f \in \mathrm{Lip}(X)$. From the above we deduce that the map Φ from $(\mathrm{Lip}(X), \|\cdot\|)$ into $(\mathrm{Lip}(Y), \|\cdot\|)$ is continuous.

Step 12 There exists a Lipschitz homeomorphism $\varphi : Y \to X$ such that

$$\Phi(f) = f \circ \varphi, \quad \forall f \in \mathrm{Lip}(X).$$

Proof Define $\varphi = \psi^{-1}$. Step 9 gives us the equality

$$\Phi(f) = f \circ \varphi, \quad \forall f \in \mathrm{Lip}(X).$$

We now show that φ is Lipschitz. For each $x \in X$, let f_x be the function

$$f_x(z) = d(z, x), \quad \forall\, z \in X.$$

Clearly $f_x \in \mathrm{Lip}(X)$ and $\|f_x\| \leq 1 + \mathrm{diam}(X)$, where $\mathrm{diam}(X)$ denotes the diameter of X.

Therefore, the set $\{f_x : x \in X\}$ is bounded in $\mathrm{Lip}(X)$. Since the map Φ is linear continuous by Steps 10 and 11, the set $\{\Phi(f_x) : x \in X\}$ is also bounded in $\mathrm{Lip}(Y)$. Hence there exists a constant $\alpha > 0$ such that $\|\Phi(f_x)\| \leq \alpha$ for all $x \in X$. It follows that $\left\|\Phi(f_{\varphi(y)})\right\| \leq \alpha$ for all $y \in Y$ and this implies that

$$L(\Phi(f_{\varphi(y)})) := \sup\left\{\left|\Phi(f_{\varphi(y)})(z) - \Phi(f_{\varphi(y)})(w)\right| / d(z,w) : z, w \in Y,\ z \neq w\right\} \leq \alpha$$

for all $y \in Y$. Let $y, z \in Y$ with $y \neq z$. We have

$$\left|\Phi(f_{\varphi(y)})(y) - \Phi(f_{\varphi(y)})(z)\right| \leq \alpha d(y, z).$$

A simple calculation shows:

$$\Phi(f_{\varphi(y)})(y) = f_{\varphi(y)}(\varphi(y)) = d(\varphi(y), \varphi(y)) = 0,$$

$$\Phi(f_{\varphi(y)})(z) = f_{\varphi(y)}(\varphi(z)) = d(\varphi(z), \varphi(y)),$$

and thus $d(\varphi(z), \varphi(y)) \leq \alpha\ d(z, y)$. Hence φ is Lipschitz.

We see at once that the inverse function Φ^{-1} comes given by

$$\Phi^{-1}(f) = f \circ \psi, \quad \forall\, f \in \mathrm{Lip}(Y).$$

Since Φ^{-1} is also continuous, we can proceed analogously as with φ showing that ψ is also Lipschitz.

Put $\tau = 1_Y$. Evidently τ is a Lipschitz function from Y into $\{-1, 1\}$ and

$$\Phi(f) = \tau \cdot (f \circ \varphi), \quad \forall\, f \in \mathrm{Lip}(X).$$

This finishes the proof in the unital case.

Non-unital Case Let us assume now $\Phi(1_X) \neq 1_Y$. Consider then the function $\Psi : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ defined by

$$\Psi(f) = \Phi(1_X)\Phi(f), \quad \forall\, f \in \mathrm{Lip}(X).$$

Next we prove that $\Phi(1_X)^2 = 1_Y$. Let $k \in F(Y)$ and let $h \in \mathrm{Lip}(X)$ be such that $\Phi(h) = k$. Using Remark 3.1, we obtain

$$\begin{aligned} \|1_Y k\|_\infty &= \|k\|_\infty = \|\Phi(h)\|_\infty = \|h\|_\infty \\ &= \|1_X h\|_\infty = \|\Phi(1_X)\Phi(h)\|_\infty = \|\Phi(1_X)k\|_\infty . \end{aligned}$$

This implies $|\Phi(1_X)| = 1_Y$ by Lemma 2.3, and therefore $\left|\Phi(1_X)^2\right| = 1_Y$. Since

$$\sigma_\pi(\Phi(1_X)^2) = \sigma_\pi((1_X)^2) = \sigma_\pi(1_X) = \{1\},$$

we deduce that $\Phi(1_X)^2 = 1_Y$. Then we obtain that Ψ is unital:

$$\Psi(1_X) = \Phi(1_X)^2 = 1_Y,$$

and that Ψ is peripherally-multiplicative:

$$\sigma_\pi(\Psi(f)\Psi(g)) = \sigma_\pi(\Phi(1_X)^2\Phi(f)\Phi(g)) = \sigma_\pi(fg), \quad \forall\, f, g \in \mathrm{Lip}(X).$$

Clearly Ψ is surjective. Hence, applying the unital case, there is a Lipschitz homeomorphism $\varphi : Y \to X$ such that $\Psi(f) = f \circ \varphi$ for all $f \in \mathrm{Lip}(X)$. Hence

$$\Phi(f) = \Phi(1_X) \cdot (f \circ \varphi), \quad \forall\, f \in \mathrm{Lip}(X).$$

Define $\tau = \Phi(1_X)$. Clearly τ is a Lipschitz function from Y into $\{-1, 1\}$ and

$$\Phi(f) = \tau \cdot (f \circ \varphi), \quad \forall\, f \in \mathrm{Lip}(X).$$

This completes the proof of our theorem.

The proof of Theorem 3.1 contains the following fact.

Corollary 3.2 *Any unital peripherally-multiplicative surjective map* $\Phi : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ *is an algebra isomorphism of the form*

$$\Phi(f) = f \circ \varphi, \quad \forall f \in \mathrm{Lip}(X),$$

where $\varphi : Y \to X$ *is a Lipschitz homeomorphism.*

Since a multiplicative map preserving the peripheral spectrum is peripherally-multiplicative and unital, we obtain:

Corollary 3.3 *Any peripheral spectrum-preserving surjective multiplicative map* $\Phi : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ *is an algebra isomorphism of the form as in Corollary* 3.2.

Since elements with equal spectra have equal peripheral spectra, we also determine the form of all surjective maps between Lip(X)-algebras which preserve multiplicatively the spectrum. Namely, from Theorem 3.1 we deduce:

Corollary 3.4 *Any multiplicatively spectrum-preserving surjective map* Φ *from* $\mathrm{Lip}(X)$ *onto* $\mathrm{Lip}(Y)$ *is a weighted composition operator*

$$\Phi(f) = \tau \cdot (f \circ \varphi), \quad \forall f \in \mathrm{Lip}(X),$$

where $\tau : Y \to \{-1, 1\}$ *is a Lipschitz function and* $\varphi : Y \to X$ *is a Lipschitz homeomorphism.*

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