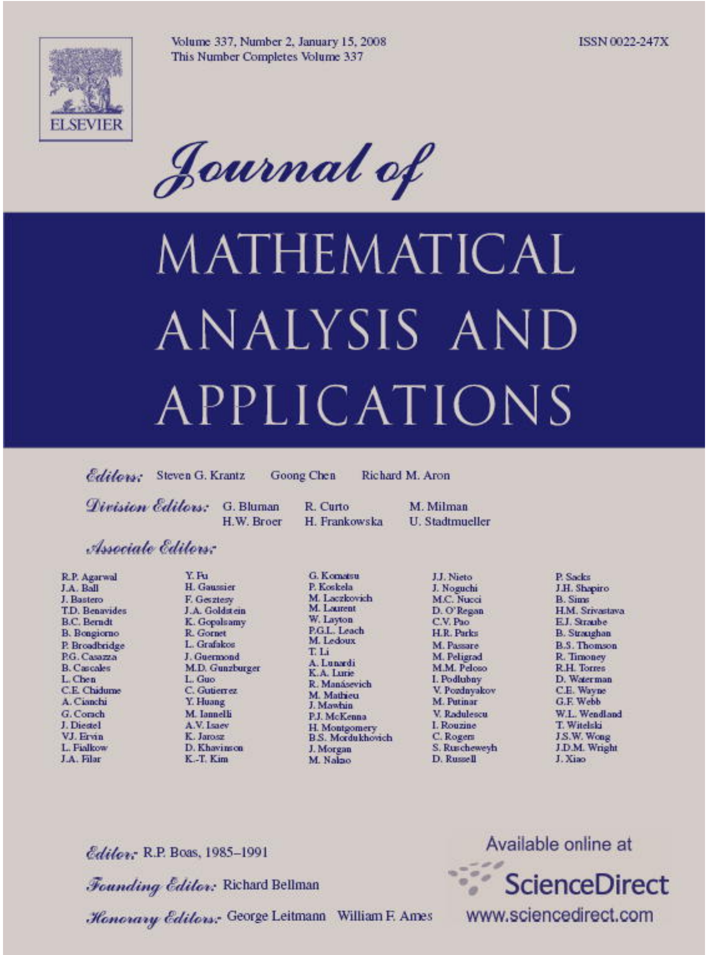
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Disjointness preserving operators between little Lipschitz algebras ☆

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Abstract

Given a real number $\alpha \in (0,1)$ and a metric space (X,d), let $Lip_\alpha(X)$ be the algebra of all scalar-valued bounded functions f on X such that

$$p_\alpha(f) = \sup\{|f(x) - f(y)|/d(x,y)^\alpha \colon x, y \in X, x \neq y\} < \infty,$$

endowed with any one of the norms $\|f\| = \max\{p_\alpha(f), \|f\|_\infty\}$ or $\|f\| = p_\alpha(f) + \|f\|_\infty$. The little Lipschitz algebra $lip_\alpha(X)$ is the closed subalgebra of $Lip_\alpha(X)$ formed by all those functions f such that $|f(x) - f(y)|/d(x,y)^\alpha \to 0$ as $d(x,y) \to 0$. A linear mapping $T : lip_\alpha(X) \to lip_\alpha(Y)$ is called disjointness preserving if $f \cdot g = 0$ in $lip_\alpha(X)$ implies $(Tf) \cdot (Tg) = 0$ in $lip_\alpha(Y)$. In this paper we study the representation and the automatic continuity of such maps T in the case in which X and Y are compact. We prove that T is essentially a weighted composition operator $Tf = h \cdot (f \circ \varphi)$ for some nonvanishing little Lipschitz function h and some continuous map φ. If, in addition, T is bijective, we deduce that h is a nonvanishing function in $lip_\alpha(Y)$ and φ is a Lipschitz homeomorphism from Y onto X and, in particular, we obtain that T is automatically continuous and T^{-1} is disjointness preserving. Moreover we show that there exists always a discontinuous disjointness preserving linear functional on $lip_\alpha(X)$, provided X is an infinite compact metric space.

Keywords: Disjointness preserving map; Lipschitz algebras; Banach–Stone theorem; Automatic continuity; Discontinuous disjointness preserving linear functionals

1. Introduction

Given a metric space (X,d) and a real number $\alpha \in (0,1]$, $Lip_\alpha(X)$ is the Banach space of all scalar-valued functions f on X such that

$$p_\alpha(f) = \sup\{|f(x) - f(y)|/d(x,y)^\alpha \colon x, y \in X, x \neq y\} < \infty$$

and

$$\|f\|_\infty = \sup\{|f(x)| \colon x \in X\} < \infty,$$

endowed with the maximum norm $\|f\| = \max\{p_\alpha(f), \|f\|_\infty\}$ or the sum norm $\|f\| = p_\alpha(f) + \|f\|_\infty$.

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The little Lipschitz space $lip_\alpha(X)$ is then defined to be the closed subspace of $Lip_\alpha(X)$ consisting of those functions f with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) \leqslant \delta$ implies $|f(x) - f(y)| \leqslant \varepsilon d(x, y)^\alpha$.

Under pointwise multiplication, $Lip_\alpha(X)$ is a Banach algebra with respect to the sum norm because it satisfies the law $\|fg\| \leqslant \|f\|\|g\|$, but with the maximum norm $Lip_\alpha(X)$ is a Banach algebra in a weak sense since $\|fg\| \leqslant 2\|f\|\|g\|$. Furthermore $lip_\alpha(X)$ is a subalgebra of $Lip_\alpha(X)$, known as little Lipschitz algebra. Extensive study of Lipschitz algebras $Lip_\alpha(X)$ and $lip_\alpha(X)$ started with de Leeuw [4] and Sherbert [14,15].

Given two algebras with respect to pointwise multiplication, $A(X)$ and $B(Y)$, of scalar-valued functions defined on the nonempty sets X and Y, it is said that a linear map T from $A(X)$ into $B(Y)$ is disjointness preserving if $f \cdot g = 0$ in $A(X)$ implies $(Tf) \cdot (Tg) = 0$ in $B(Y)$.

In the context of spaces $C(X)$ of scalar-valued continuous functions defined on compact Hausdorff topological spaces X equipped with the supremum norm, Beckenstein, Narici and Todd introduced in [1] the concept of disjointness preserving map with the name of separating map. Jarosz obtained in [8] a complete description of the general form of these maps as well as some automatic continuity properties for disjointness preserving linear bijections. These results were extended in [6] and [9] to disjointness preserving linear maps defined between spaces $C_0(X)$ of scalar-valued continuous functions vanishing at infinity on locally compact Hausdorff spaces X.

In recent years considerable attention has been given to disjointness preserving maps on other algebras of scalar-valued functions. For instance these maps have been studied on algebras of differentiable functions [10], on Fourier algebras [5,11] and on group algebras of locally compact abelian groups [7].

In this paper we shall study the multiplicative representation and the automatic continuity of disjointness preserving linear maps T from $lip_\alpha(X)$ into $lip_\alpha(Y)$ under the conditions that X, Y are compact and α is in $(0, 1)$.

In Section 2 we shall prove that such a map T is essentially a weighted composition operator. Namely there exist a disjoint union $Y = Y_d \cup Y_0 \cup Y_c$, a nonvanishing function h in $lip_\alpha(Y_c)$ and a continuous map φ from $Y_d \cup Y_c$ into X such that $Tf|_{Y_c} = h \cdot (f \circ \varphi)$ and $Tf|_{Y_0} = 0$ for all $f \in lip_\alpha(X)$. Furthermore Y_0 is closed in Y, Y_d is open in Y and $\varphi(Y_d)$ is a finite set of nonisolated points of X such that, for each $y \in Y_d$, the functional $\delta_y \circ T$ is discontinuous on $lip_\alpha(X)$.

In Section 3, using the main theorem in Section 2, we shall deduce that if T is also bijective, then T is a weighted composition operator $Tf = h \cdot (f \circ \varphi)$ for all $f \in lip_\alpha(X)$, where h is a nonvanishing function in $lip_\alpha(Y)$ and φ is a Lipschitz homeomorphism from Y onto X. Furthermore we shall show that T is automatically continuous and T^{-1} is disjointness preserving. Problems of automatic continuity on Lipschitz algebras has been investigated by Bade, Curtis and Dales [2], and Pavlović [12].

In Section 4 we shall establish the existence of discontinuous disjointness preserving linear functionals on $lip_\alpha(X)$, when X is a compact metric space of infinite cardinality and $\alpha \in (0, 1)$. Our approach depends on the analysis of prime ideals of $lip_\alpha(X)$. Sherbert [15] was the first to study the ideal structure of $lip_\alpha(X)$. The same problem was considered for algebras of continuous functions $C(X)$. Jarosz [8] showed that a discontinuous disjointness preserving linear functional on $C(X)$ exists, provided X is infinite. Later Brown and Wong [3] constructed functionals on $C_0(X)$ with these properties. Such functionals arise from prime ideals of $C_0(X)$ and the authors translated the results into the cozero set ideal setting. In the context of Lipschitz algebras, Pavlović [13] solved the problem of existence of homomorphisms and derivations from $lip_\alpha(X)$ which are discontinuous on every dense subalgebra.

We must point out that the method of proof used in Sections 2 and 3 is due to Jarosz [8]. The same technique has produced similar results in algebras $C_0(X)$ [6,9] and group algebras $L^1(G)$ [7]. On the other hand, the arguments to prove the existence of discontinuous disjointness preserving linear functionals on $lip_\alpha(X)$ come essentially from [3] and [8].

2. Representation

In this section we shall give a complete description of disjointness preserving linear maps between little Lipschitz algebras. We first adopt some notation. For $x \in X$ and $\delta > 0$, we write $B(x, \delta) = \{z \in X\colon\ d(z, x) < \delta\}$.

For $f \in lip_\alpha(X)$, let $\operatorname{coz}(f)$ denote the set of all the points $x \in X$ such that $f(x) \neq 0$ and let $\operatorname{supp}(f)$ denote the closure of $\operatorname{coz}(f)$ in X.

For each $y \in Y$, let δ_y be the linear functional on $lip_\alpha(Y)$ defined by $\delta_y(f) = f(y)$. If T is a linear map from $lip_\alpha(X)$ into $lip_\alpha(Y)$, for each $y \in Y$ let $\mathrm{supp}(\delta_y \circ T)$ be the set of all the points $x \in X$ such that for each $\delta > 0$, there exists a function $f \in lip_\alpha(X)$ with $\mathrm{coz}(f) \subset B(x, \delta)$ such that $Tf(y) \neq 0$.

Through this paper we shall use repeatedly the easily checked fact that $Lip_1(X) \subset lip_\alpha(X)$ for all $\alpha \in (0, 1)$.

If $\alpha \in (0, 1)$, for $x \in X$ and $\delta > 0$, the function $h_{x,\delta} : X \to [0, 1]$ defined by

$$h_{x,\delta}(z) = \max\{0, 1 - d(z, x)/\delta\}$$

belongs to $Lip_1(X) \subset lip_\alpha(X)$ with $h_{x,\delta}(x) = 1$ and $\mathrm{coz}(h_{x,\delta}) = B(x, \delta)$. However, for $\alpha = 1$ the existence of nonzero functions $f \in lip_\alpha(X)$ satisfying $\mathrm{coz}(f) \subset B(x, \delta)$ is not guaranteed. For example, if $X = [0, 1]$ and $d(x, y) = |x - y|$ for all $x, y \in [0, 1]$, every function of $lip_1(X)$ is constant. By this motive we shall concern ourselves with the algebras $lip_\alpha(X)$, for $\alpha \in (0, 1)$.

It is evident that $\mathrm{supp}(\delta_y \circ T) = \emptyset$ if $\delta_y \circ T = 0$. We shall see that this comes true only in this case, when T is a disjointness preserving linear map from $lip_\alpha(X)$ into $lip_\alpha(Y)$, for X, Y compact and $\alpha \in (0, 1)$. To prove this, we shall require partitions of unity formed by functions of $lip_\alpha(X)$. In the next lemma we show that these partitions of unity always exist.

Lemma 2.1. *If X is a compact metric space and $\{B(x_k, \delta_{x_k}):\ k = 1, \ldots, n\}$ is a covering of X, then there are functions $f_1, \ldots, f_n$ in $Lip_1(X)$ such that $f_k(z) \in [0, 1]$ and $\sum_{k=1}^n f_k(z) = 1$ for all $z \in X$ and, furthermore, $\mathrm{coz}(f_k) = B(x_k, \delta_{x_k})$, for each $1 \leqslant k \leqslant n$.*

Proof. For each $k \in \{1, \ldots, n\}$, let f_k be the function on X given by

$$f_k(z) = h_{x_k,\delta_{x_k}}(z) \Big/ \sum_{k=1}^{n} h_{x_k,\delta_{x_k}}(z), \quad \forall z \in X.$$

Put $f = \sum_{k=1}^n h_{x_k,\delta_{x_k}}$. If $z \in X$, then $d(z, x_j) < \delta_{x_j}$ for some $j \in \{1, \ldots, n\}$, therefore $h_{x_j,\delta_{x_j}}(z) = 1 - d(z, x_j)/\delta_{x_j} > 0$ and so $f(z) \geqslant h_{x_j,\delta_{x_j}}(z) > 0$. Hence f_k is well defined. Since $f \in Lip_1(X)$ is never zero, by the compactness of X it is bounded away from zero and then it is immediate that $1/f \in Lip_1(X)$. Hence $f_k \in Lip_1(X)$. A trivial verification shows that f_k satisfies the conditions of the statement. □

Theorem 2.2. *Let X and Y be compact metric spaces, let $\alpha \in (0, 1)$, and let T be a disjointness preserving linear map from $lip_\alpha(X)$ into $lip_\alpha(Y)$. Then there exist a disjoint union $Y = Y_c \cup Y_0 \cup Y_d$ where Y_0 is closed in Y and Y_d is open in Y, a continuous map $\varphi : Y_c \cup Y_d \to X$ such that*

$$\varphi(y) \notin \mathrm{supp}(f) \quad \Rightarrow \quad Tf(y) = 0, \quad \forall f \in lip_\alpha(X),$$

and a nonvanishing function h in $lip_\alpha(Y_c)$ such that

$$Tf(y) = h(y) f\big(\varphi(y)\big), \quad \forall f \in lip_\alpha(X),\ \forall y \in Y_c,$$
$$Tf(y) = 0, \quad \forall f \in lip_\alpha(X),\ \forall y \in Y_0.$$

Furthermore $\varphi(Y_d)$ is a finite set of nonisolated points of X and, for each $y \in Y_d$, the functional $\delta_y \circ T$ is discontinuous on $lip_\alpha(X)$.

Proof. We divide the set Y into three disjoint parts: its null part

$$Y_0 = \{y \in Y:\ \delta_y \circ T = 0\},$$

its nonnull continuous part

$$Y_c = \{y \in Y:\ \delta_y \circ T \text{ is nonzero and continuous}\}$$

and its discontinuous part

$$Y_d = \{y \in Y:\ \delta_y \circ T \text{ is discontinuous}\}.$$

We now prove the theorem into a series of steps.

Step 1. For each $y \in Y_c \cup Y_d$, $\operatorname{supp}(\delta_y \circ T)$ is nonempty and, in fact, contains exactly one point.

Proof. Let y be in $Y_c \cup Y_d$ and suppose $\operatorname{supp}(\delta_y \circ T)$ is empty. Let $x \in X$. Since $x \notin \operatorname{supp}(\delta_y \circ T)$, there exists a scalar $\delta_x > 0$ such that $Tf(y) = 0$ for every function $f \in lip_\alpha(X)$ with $\operatorname{coz}(f) \subset B(x, \delta_x)$. Since $\{B(x, \delta_x)\colon x \in X\}$ is an open covering of X, $X = \bigcup_{k=1}^n B(x_k, \delta_{x_k})$ for some natural n. By Lemma 2.1 there exist functions $f_1, \dots, f_n$ in $lip_\alpha(X)$ such that $\sum_{k=1}^n f_k(z) = 1$ for all $z \in X$ and $\operatorname{coz}(f_k) = B(x_k, \delta_{x_k})$, for each $k \in \{1, \dots, n\}$. Then, for all $f \in lip_\alpha(X)$, we have

$$Tf(y) = T\left(\sum_{k=1}^n f f_k\right)(y) = \sum_{k=1}^n T(f f_k)(y) = 0,$$

since $f f_k \in lip_\alpha(X)$ and $\operatorname{coz}(f f_k) \subset \operatorname{coz}(f_k) = B(x_k, \delta_{x_k})$ for all k in $\{1, \dots, n\}$. This means that $\delta_y \circ T = 0$ and thus $y \in Y_0$, a contradiction. Hence $\operatorname{supp}(\delta_y \circ T)$ is nonempty.

Suppose $\operatorname{supp}(\delta_y \circ T)$ has two distinct points $x_1, x_2 \in X$ and let $\delta = d(x_1, x_2)/2 > 0$. By the definition of $\operatorname{supp}(\delta_y \circ T)$ there are functions $f_1, f_2 \in lip_\alpha(X)$ with $\operatorname{coz}(f_1) \subset B(x_1, \delta)$ and $\operatorname{coz}(f_2) \subset B(x_2, \delta)$ such that $Tf_1(y) \neq 0 \neq Tf_2(y)$. Clearly $f_1 \cdot f_2 = 0$ which implies $(Tf_1) \cdot (Tf_2) = 0$, but $Tf_1(y)Tf_2(y) \neq 0$, a contradiction. Therefore $\operatorname{supp}(\delta_y \circ T)$ has an unique point. □

Step 1 motivates the following:

Definition 2.1. Let φ be the map from $Y_c \cup Y_d$ into X defined by

$$\{\varphi(y)\} = \operatorname{supp}(\delta_y \circ T).$$

Step 2. If $y \in Y_c \cup Y_d$, $f \in lip_\alpha(X)$ and $\varphi(y) \notin \operatorname{supp}(f)$, then $Tf(y) = 0$.

Proof. If $\varphi(y) \notin \operatorname{supp}(f)$, there exists $\delta > 0$ such that $f(z) = 0$ if $d(z, \varphi(y)) < \delta$. Since $\{\varphi(y)\} = \operatorname{supp}(\delta_y \circ T)$, there exists a function $g \in lip_\alpha(X)$ with $g(z) = 0$ if $d(z, \varphi(y)) \geqslant \delta$ such that $Tg(y) \neq 0$. Then $f(z)g(z) = 0$ for all $z \in X$ which implies $Tf(y)Tg(y) = 0$. Hence $Tf(y) = 0$. □

Step 3. The mapping $\varphi : Y_c \cup Y_d \to X$ is continuous.

Proof. Let y be in $Y_c \cup Y_d$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $Y_c \cup Y_d$ converging to y. By the compactness of X we can suppose, taking a subsequence if it is necessary, that $(\varphi(y_n))_{n\in\mathbb{N}}$ converges to a point x in X. Suppose $x \neq \varphi(y)$. Put $\delta = d(x, \varphi(y)) > 0$ and since $\{\varphi(y)\} = \operatorname{supp}(\delta_y \circ T)$ there exists a function $f \in lip_\alpha(X)$ with $f(z) = 0$ if $d(z, \varphi(y)) \geqslant \delta/3$ such that $Tf(y) \neq 0$.

On the other hand, since $(\varphi(y_n))_{n\in\mathbb{N}}$ converges to x, there exists a natural m such that $d(\varphi(y_n), x) < \delta/3$ if $n \geqslant m$. Fix $n \geqslant m$. It is easy to see that $d(z, \varphi(y_n)) < \delta/3$ implies $d(z, \varphi(y)) > \delta/3$. Then $f(z) = 0$ if $d(z, \varphi(y_n)) < \delta/3$. This means that $\varphi(y_n) \notin \operatorname{supp}(f)$ and then $Tf(y_n) = 0$ by Step 2. Since n was arbitrary, it follows that $Tf(y_n) = 0$ for all $n \geqslant m$ and thus $Tf(y) = 0$, a contradiction. □

Step 4. For each $y \in Y_c \cup Y_d$ define

$$J_y = \{f \in lip_\alpha(X)\colon \varphi(y) \notin \operatorname{supp}(f)\}$$

and

$$M_y = \{f \in lip_\alpha(X)\colon f(\varphi(y)) = 0\}.$$

Then J_y is a dense subspace of M_y.

Proof. Let $y \in Y_c \cup Y_d$. Observe that J_y is the set formed by all the functions in $lip_\alpha(X)$ vanishing at a neighborhood of $\varphi(y)$. Clearly J_y and M_y are vector subspaces of $lip_\alpha(X)$ and $J_y \subset M_y$.

To prove that J_y is dense in M_y, let $f \in M_y$ and $\varepsilon > 0$. Since $f \in lip_\alpha(X)$, there is a number δ in $(0, 1)$ such that $d(z, w) \leqslant 2\delta$ implies $|f(z) - f(w)| \leqslant \varepsilon d(z, w)^\alpha$.

Define the functions $\rho_y : X \to \mathbb{R}$ and $h : \mathbb{R} \to [0,1]$ by

$$\rho_y(z) = d\big(z, \varphi(y)\big), \quad \forall z \in X,$$

and

$$h(t) = \begin{cases} 0 & \text{if } t \leqslant \delta/2, \\ (2/\delta)t - 1 & \text{if } \delta/2 < t < \delta, \\ 1 & \text{if } t \geqslant \delta, \end{cases}$$

respectively. Then $\rho_y \in Lip_1(X)$ and h is Lipschitz on the interval $[0, \operatorname{diam}(X)]$, where $\operatorname{diam}(X)$ denotes the diameter of X, since it is piecewise differentiable with bounded derivative. This implies that $h \circ \rho_y \in Lip_1(X)$ and thus $g = (h \circ \rho_y) \cdot f \in lip_\alpha(X)$. Moreover $g(z) = 0$ if $d(z, \varphi(y)) \leqslant \delta/2$. Hence $g \in J_y$. We conclude by showing that $p_\alpha(f - g) \leqslant 5\varepsilon$ and $\|f - g\|_\infty \leqslant \varepsilon$. If $z, w \in B(\varphi(y), \delta)$, we have

$$\big|\rho_y(z) - \rho_y(w)\big| \leqslant d(z,w) \leqslant (2\delta)^{1-\alpha} d(z,w)^\alpha,$$
$$\big|f(z) - f(w)\big| \leqslant \varepsilon d(z,w)^\alpha$$

and

$$\big|f(z)\big| = \big|f(z) - f\big(\varphi(y)\big)\big| \leqslant \varepsilon d\big(z, \varphi(y)\big)^\alpha \leqslant \varepsilon\delta^\alpha.$$

Therefore $p_\alpha(\rho_y|_{B(\varphi(y),\delta)}) \leqslant (2\delta)^{1-\alpha}$, $p_\alpha(f|_{B(\varphi(y),\delta)}) \leqslant \varepsilon$ and $\|f|_{B(\varphi(y),\delta)}\|_\infty \leqslant \varepsilon\delta^\alpha$. Since the Lipschitz constant of $1 - h$ as an element of $Lip_1(\mathbb{R})$ is $\leqslant 2/\delta$ and $\|1 - h\|_\infty \leqslant 1$, restricting all functions to $B(\varphi(y), \delta)$ and using $f - g = ((1 - h) \circ \rho_y) \cdot f$, we obtain

$$\begin{aligned} p_\alpha(f - g) &\leqslant p_\alpha\big((1-h) \circ \rho_y\big)\|f\|_\infty + \big\|(1-h) \circ \rho_y\big\|_\infty p_\alpha(f) \\ &\leqslant (2/\delta)(2\delta)^{1-\alpha}\varepsilon\delta^\alpha + \varepsilon = \big(2^{2-\alpha} + 1\big)\varepsilon \leqslant 5\varepsilon. \end{aligned}$$

Since $f(z) - g(z) = 0$ if $d(z, \varphi(y)) \geqslant \delta$, to complete the estimate of $p_\alpha(f - g)$ we must bound $|(f - g)(z) - (f - g)(w)|/d(z,w)^\alpha$ when $d(z, \varphi(y)) < \delta$ and $d(w, \varphi(y)) \geqslant \delta$. Writing $a = d(z, \varphi(y))$, we get

$$\begin{aligned} \big|(f-g)(z) - (f-g)(w)\big|/d(z,w)^\alpha &= \big|(1-h)(a)\big|\big|f(z)\big|/d(z,w)^\alpha \\ &\leqslant (2 - 2a/\delta)\varepsilon\delta^\alpha/(\delta - a)^\alpha = 2\varepsilon\big((\delta - a)/\delta\big)^{1-\alpha} < 2\varepsilon. \end{aligned}$$

An easy calculation yields $\|f - g\|_\infty \leqslant \varepsilon\delta < \varepsilon$. □

Step 5. There exists a nonvanishing function h in $lip_\alpha(Y_c)$ such that

$$Tf(y) = h(y) f\big(\varphi(y)\big), \quad \forall f \in lip_\alpha(X),\ \forall y \in Y_c.$$

Proof. Let $y \in Y_c$. Since $\delta_y \circ T$ is a nonzero continuous linear functional on $lip_\alpha(X)$, $\ker(\delta_y \circ T)$ is a proper closed subspace of $lip_\alpha(X)$. We have that $J_y \subset \ker(\delta_y \circ T)$ by Step 2 and therefore $M_y \subset \ker(\delta_y \circ T)$ by Step 4, this is, $\ker \delta_{\varphi(y)} \subset \ker(\delta_y \circ T)$. Since both are maximal subspaces of $lip_\alpha(X)$, it follows that $\ker \delta_{\varphi(y)} = \ker(\delta_y \circ T)$. Consequently there exists a nonzero scalar $h(y)$ such that $\delta_y \circ T = h(y)\delta_{\varphi(y)}$ and therefore $Tf(y) = h(y) f(\varphi(y))$ for all $f \in lip_\alpha(X)$. Clearly $h = T1_X|_{Y_c}$, where 1_X denotes the function constantly equal 1 on X. Since $T1_X \in lip_\alpha(Y)$, we conclude that h is in $lip_\alpha(Y_c)$. □

Step 6. Y_0 is closed in Y and Y_d is open in Y.

Proof. Y_0 is closed in Y since $Y_0 = \bigcap_{f \in lip_\alpha(X)} \ker(Tf)$. To prove that Y_d is open in Y, let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $Y \backslash Y_d$ which converges to a point y in Y. Given $f \in lip_\alpha(X)$, we have

$$\begin{aligned} \big|Tf(y_m)\big| &\leqslant \sup\big\{\big|Tf(y_n)\big|:\ y_n \in Y_0 \cup Y_c\big\} = \sup\big\{\big|Tf(y_n)\big|:\ y_n \in Y_c\big\} \\ &= \sup\big\{\big|h(y_n) f\big(\varphi(y_n)\big)\big|:\ y_n \in Y_c\big\} \leqslant \|h\|_\infty \|f\|_\infty \leqslant \|h\|_\infty \|f\| \end{aligned}$$

for all natural m. By the continuity of Tf in Y, it follows that $|Tf(y)| \leqslant \|h\|_\infty \|f\|$, that is, $|(\delta_y \circ T)(f)| \leqslant \|h\|_\infty \|f\|$. Hence the linear functional $\delta_y \circ T$ is continuous on $lip_\alpha(X)$ and so $y \in Y \backslash Y_d$. This proves that $Y \backslash Y_d$ is closed in Y. With other words, Y_d is open in Y. □

Step 7. $\varphi(Y_d)$ is a finite set of nonisolated points of X.

Proof. To prove that $\varphi(Y_d)$ is finite, suppose that there exists a sequence $(\varphi(y_n))_{n\in\mathbb{N}}$ of distinct elements of X such that $y_n \in Y_d$ for all $n \in \mathbb{N}$. As X is compact we can find a subsequence of $(\varphi(y_n))_{n\in\mathbb{N}}$, which we shall follow denoting by $(\varphi(y_n))_{n\in\mathbb{N}}$, and a sequence $(B(\varphi(y_n), 2\delta_n))_{n\in\mathbb{N}}$ of pairwise disjoint open balls of X with $\delta_n \in (0, 1)$, for each $n \in \mathbb{N}$. For each natural n let g_n be the function defined on X by

$$g_n(x) = \max\bigl\{0, \delta_n - d\bigl(x, B\bigl(\varphi(y_n), \delta_n\bigr)\bigr)\bigr\}.$$

The function g_n is in $lip_\alpha(X)$ with $\|g_n\| < 2$ since $\|g_n\|_\infty \leqslant \delta_n < 1$ and $p_\alpha(g_n) \leqslant 1$. Clearly $g_n(x) = \delta_n$ when $x \in B(\varphi(y_n), \delta_n)$ and an easy calculation gives us that $\operatorname{coz}(g_n) \subset B(\varphi(y_n), 2\delta_n)$.

On the other hand, since the linear functional $\delta_{y_n} \circ T$ is discontinuous on $lip_\alpha(X)$, there exists a function h_n in $lip_\alpha(X)$ with $\|h_n\| \leqslant 1$ such that $|Th_n(y_n)| \geqslant n^3/\delta_n$ for all $n \in \mathbb{N}$.

For each natural n, let us define the function $f_n = (1/n^2)g_n h_n$. Clearly $f_n \in lip_\alpha(X)$ and since $g_n(x) = \delta_n$ for all $x \in B(\varphi(y_n), \delta_n)$, we have $f_n - (1/n^2)\delta_n h_n = 0$ on $B(\varphi(y_n), \delta_n)$. Hence $Tf_n(y_n) = (1/n^2)\delta_n Th_n(y_n)$ by Step 2 and thus $|Tf_n(y_n)| \geqslant n$.

Since $\|f_n\| < 4/n^2$ for all $n \in \mathbb{N}$ and $lip_\alpha(X)$ is complete, we can define the function $f = \sum_{n=1}^{+\infty} f_n \in lip_\alpha(X)$. As the sequence $(B(\varphi(y_n), 2\delta_n))_{n\in\mathbb{N}}$ is pairwise disjoint and $\operatorname{coz}(f_n) \subset B(\varphi(y_n), 2\delta_n)$ for all $n \in \mathbb{N}$, it follows that $\varphi(y_m) \notin \operatorname{supp}(f_n)$ for all $n \neq m$. Using Step 2 we have

$$\bigl|Tf(y_m)\bigr| = \left|Tf_m(y_m) + T\left(\sum_{n=1,\, n\neq m}^{+\infty} f_n\right)(y_m)\right| = \bigl|Tf_m(y_m)\bigr| \geqslant m,$$

for every $m \in \mathbb{N}$, which is a contradiction since $Tf \in lip_\alpha(Y)$ is bounded. This proves that $\varphi(Y_d)$ is finite.

We now shall show that $\varphi(Y_d)$ is a subset of nonisolated points of X. Let $y \in Y_d$ and suppose that $\varphi(y)$ is an isolated point of X. Then there is a scalar $\delta > 0$ such that $B(\varphi(y), \delta) = \{\varphi(y)\}$. If $f(\varphi(y)) = 0$, then $\varphi(y) \notin \operatorname{supp}(f)$ and by Step 2 we have $Tf(y) = 0$. With other words, $\ker \delta_{\varphi(y)} \subset \ker(\delta_y \circ T)$ and therefore $\delta_y \circ T = \beta\delta_{\varphi(y)}$ for some nonzero scalar β. Then the nonzero linear functional $\delta_y \circ T$ is continuous on $lip_\alpha(X)$ and so $y \in Y_c$, a contradiction. □

3. Automatic continuity

In this section we shall prove that any disjointness preserving linear bijection between little Lipschitz algebras is automatically continuous. Recall that a map between metric spaces $\varphi : X \to Y$ is a Lipschitz homeomorphism if φ is a bijection such that φ and φ^{-1} are both Lipschitz.

Theorem 3.1. *Let X and Y be compact metric spaces, let $\alpha \in (0, 1)$, and let T be a disjointness preserving linear bijection from $lip_\alpha(X)$ onto $lip_\alpha(Y)$. Then T is a weighted composition operator*

$$Tf(y) = h(y) f\bigl(\varphi(y)\bigr), \quad \forall f \in lip_\alpha(X),\ \forall y \in Y,$$

where h is a nonvanishing function in $lip_\alpha(Y)$ and φ is a Lipschitz homeomorphism from Y onto X. In particular, T is automatically continuous and T^{-1} is disjointness preserving.

Proof. We adopt the notations used in Theorem 2.2 and divide the proof in several claims.

Claim 1. *Y_0 is empty and Y_c is compact.*

Proof. Suppose $y \in Y_0$. Then $Tf(y) = 0$ for all $f \in lip_\alpha(X)$. Consider the function $h_{y,1}$ in $lip_\alpha(Y)$. By the surjectivity of T there exists $f \in lip_\alpha(X)$ such that $Tf = h_{y,1}$. Then $h_{y,1}(y) = Tf(y) = 0$, but $h_{y,1}(y) = 1$, a contradiction. Hence $Y_0 = \emptyset$.

Since Y_d is open in Y by Theorem 2.2, we have that $Y_0 \cup Y_c = Y \backslash Y_d$ is closed in Y. As $Y_0 = \emptyset$, it follows that Y_c is closed in the compact space Y and thus Y_c is compact. □

Claim 2. *$\varphi(Y_c)$ is dense in X.*

Proof. We first prove that $\varphi(Y_c \cup Y_d)$ is dense in X. Suppose on the contrary that there exists a point $x \in X$ such that $\delta = d(x, \varphi(Y_c \cup Y_d)) > 0$. Consider the function $h_{x,\delta/2}$ in $lip_\alpha(X)$. Since $\mathrm{coz}(h_{x,\delta/2}) = B(x, \delta/2)$, it is immediate that $\mathrm{supp}(h_{x,\delta/2}) \subset \{z \in X\colon d(z,x) \leqslant \delta/2\}$. From this it is easily seen that $\varphi(y) \notin \mathrm{supp}(h_{x,\delta/2})$ for all $y \in Y_c \cup Y_d$ and then Theorem 2.2 gives us $Th_{x,\delta/2}(y) = 0$ for all $y \in Y_c \cup Y_d$. Since also $Th_{x,\delta/2}(y) = 0$ for all $y \in Y_0$, we have $Th_{x,\delta/2} = 0$. By the linearity and injectivity of T, it follows that $h_{x,\delta/2} = 0$, but $h_{x,\delta/2}(x) = 1$, a contradiction.

We now show that $\overline{\varphi(Y_c \cup Y_d)} = \overline{\varphi(Y_c)}$. It suffices to prove that $\varphi(Y_d) \subset \overline{\varphi(Y_c)}$ since $\varphi(Y_d)$ is a finite subset of X by Theorem 2.2. Let $x \in \varphi(Y_d)$ and suppose $B(x, \varepsilon) \cap \varphi(Y_c) = \emptyset$ for some $\varepsilon > 0$. Put $\varepsilon' = \min\{\varepsilon, d(x, \varphi(Y_d)\backslash\{x\})\} > 0$ and clearly $(B(x, \varepsilon')\backslash\{x\}) \cap \varphi(Y_c \cup Y_d) = \emptyset$. Since x is a nonisolated point of X by Theorem 2.2, we can choose a point y in $B(x, \varepsilon'/2)\backslash\{x\}$. Let $\varepsilon'' = d(x, y) > 0$. We have $B(y, \varepsilon'') \subset B(x, \varepsilon')\backslash\{x\}$. Then $B(y, \varepsilon'') \cap \varphi(Y_c \cup Y_d) = \emptyset$ and this contradicts the density of $\varphi(Y_c \cup Y_d)$ in X.

Since $\varphi(Y_c \cup Y_d)$ is dense in X and $\overline{\varphi(Y_c \cup Y_d)} = \overline{\varphi(Y_c)}$, the claim follows. □

Claim 3. *Y_d is empty and T is continuous.*

Proof. To prove that Y_d is empty, suppose there is a point z in Y_d. Then $\delta = d(z, Y_c) > 0$ because $z \notin Y_c$ and Y_c is closed in Y by Claim 1. Consider $h_{z,\delta} \in lip_\alpha(Y)$ and by the surjectivity of T, there exists some $f \in lip_\alpha(X)$ such that $Tf = h_{z,\delta}$. Then $Tf(y) = h_{z,\delta}(y) = 0$ for all $y \in Y_c$. It follows that $f(\varphi(y)) = 0$ for all $y \in Y_c$ by Theorem 2.2. Hence $f(x) = 0$ for all $x \in X$ since $\overline{\varphi(Y_c)} = X$ by Claim 2. Therefore $f = 0$ and so $Tf = 0$, but $Tf(z) = h_{z,\delta}(z) = 1$, a contradiction.

To prove that T is continuous, let $f \in lip_\alpha(X)$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of points of $lip_\alpha(X)$ converging to f such that $(Tf_n)_{n\in\mathbb{N}}$ converges to a function g in $lip_\alpha(Y)$. Since Y_d is empty, the linear functional $\delta_y \circ T$ is continuous on $lip_\alpha(X)$, for every $y \in Y$. In consequence, $(Tf_n(y) - Tf(y))_{n\in\mathbb{N}} = ((\delta_y \circ T)(f_n - f))_{n\in\mathbb{N}}$ converges to 0 for every $y \in Y$. Since the convergence in the norm of $lip_\alpha(Y)$ implies pointwise convergence, we have that $(Tf_n(y))_{n\in\mathbb{N}}$ converges to $g(y)$ for every $y \in Y$. Therefore $Tf(y) = g(y)$ for all $y \in Y$ and thus $Tf = g$. Hence T has closed graph, so it must be continuous by the closed graph theorem. □

Claim 4. *$Y = Y_c$ and $Tf(y) = h(y) f(\varphi(y))$ for all $f \in lip_\alpha(X)$ and $y \in Y$.*

Proof. $Y_0 = Y_d = \emptyset$ by Claims 1 and 3. Hence $Y = Y_c$ and the claim follows from Theorem 2.2. □

Claim 5. *T^{-1} is disjointness preserving.*

Proof. Let g_1, g_2 be in $lip_\alpha(Y)$ such that $g_1 \cdot g_2 = 0$. There exist $f_1, f_2 \in lip_\alpha(X)$ such that $g_1 = Tf_1 = h \cdot (f_1 \circ \varphi)$ and $g_2 = Tf_2 = h \cdot (f_2 \circ \varphi)$. Then $h^2 \cdot (f_1 \circ \varphi) \cdot (f_2 \circ \varphi) = 0$ and so $(f_1 \cdot f_2) \circ \varphi = 0$ since h is nonvanishing. Since $\varphi(Y)$ is dense in X by Claims 2 and 4, $f_1 \cdot f_2 = 0$ as we wanted to show. □

Claim 6. *φ is a Lipschitz homeomorphism from Y onto X.*

Proof. To show that $\varphi : Y \to X$ is injective, let y, z be in Y with $y \neq z$ and suppose $\varphi(y) = \varphi(z)$. Put $\delta = d(y, z) > 0$. Since $h_{y,\delta} \in lip_\alpha(Y)$ and T is surjective, $Tf = h_{y,\delta}$ for some $f \in lip_\alpha(X)$. By Claim 4 we have

$$h_{y,\delta}(t) = Tf(t) = h(t) f\bigl(\varphi(t)\bigr), \quad \forall t \in Y.$$

In particular, $1 = h_{y,\delta}(y) = h(y) f(\varphi(y))$ and $0 = h_{y,\delta}(z) = h(z) f(\varphi(z))$. Since h is nonvanishing, it follows that $f(\varphi(y)) = 1/h(y)$ and $f(\varphi(z)) = 0$. As $\varphi(y) = \varphi(z)$, we have $1/h(y) = 0$ which is absurd.

Since $\overline{\varphi(Y)} = X$ by Claims 2 and 4, φ is continuous by Theorem 2.2 and Y is compact, it follows that $\varphi(Y) = X$ and so $\varphi : Y \to X$ is surjective.

We claim that φ is Lipschitz. For $y_1, y_2 \in Y$ define

$$f_{y_1y_2}(z) = \max\{2d(\varphi(y_1), z)^\alpha - d(\varphi(y_1), \varphi(y_2))^\alpha, 0\}$$

for all $z \in X$. Clearly $f_{y_1y_2} \in Lip_\alpha(X)$ with $\|f_{y_1y_2}\| \leqslant 2(1 + \mathrm{diam}(X)^\alpha)$. To show $f_{y_1y_2} \in lip_\alpha(X)$, define $\rho_{y_1} \in Lip_1(X)$ by $\rho_{y_1}(z) = d(\varphi(y_1), z)$ and $h_{y_1y_2} : \mathbb{R} \to \mathbb{R}$ by

$$h_{y_1y_2}(t) = \max\{2t^\alpha - d(\varphi(y_1), \varphi(y_2))^\alpha, 0\}.$$

Then $h_{y_1y_2}$ is Lipschitz on the interval $[0, \mathrm{diam}(X)]$ since it is piecewise differentiable with bounded derivative. Since $f_{y_1y_2} = h_{y_1y_2} \circ \rho_{y_1}$, this implies that $f_{y_1y_2} \in Lip_1(X)$ and thus we obtain the desired conclusion.

Therefore the set $\{f_{y_1y_2}\colon\ y_1, y_2 \in Y\}$ is bounded in $lip_\alpha(X)$. Since the linear map T is continuous by Claim 3, the set $\{Tf_{y_1y_2}\colon\ y_1, y_2 \in Y\}$ is bounded in $lip_\alpha(Y)$. Hence there is a constant $\gamma > 0$ such that $\|Tf_{y_1y_2}\| \leqslant \gamma$ for all $y_1, y_2 \in Y$ and so $p_\alpha(Tf_{y_1y_2}) \leqslant \gamma$ for all $y_1, y_2 \in Y$. Given $y_1, y_2 \in Y$, we have

$$|Tf_{y_1y_2}(y_1) - Tf_{y_1y_2}(y_2)| \leqslant \gamma d(y_1, y_2)^\alpha.$$

It is clear that

$$Tf_{y_1y_2}(y_1) = h(y_1) f_{y_1y_2}(\varphi(y_1)) = 0,$$
$$Tf_{y_1y_2}(y_2) = h(y_2) f_{y_1y_2}(\varphi(y_2)) = h(y_2) d(\varphi(y_1), \varphi(y_2))^\alpha,$$

and so $|Tf_{y_1y_2}(y_1) - Tf_{y_1y_2}(y_2)| = |h(y_2)| d(\varphi(y_1), \varphi(y_2))^\alpha$. Hence

$$d(\varphi(y_1), \varphi(y_2))^\alpha \leqslant (\gamma/\beta) d(y_1, y_2)^\alpha$$

with $\beta = \min\{|h(y)|\colon\ y \in Y\} > 0$, which is our claim.

To prove that φ^{-1} is Lipschitz, we have that T^{-1} is continuous by the inverse mapping theorem and T^{-1} is a disjointness preserving linear bijection of $lip_\alpha(Y)$ onto $lip_\alpha(X)$ by Claim 5. By the above proved,

$$T^{-1}g = j \cdot (g \circ k), \quad \forall g \in lip_\alpha(Y),$$

where j is a nonvanishing function in $lip_\alpha(X)$ and k is a Lipschitz bijection of X onto Y. The proof is finished if we prove that $k = \varphi^{-1}$. Given $f \in lip_\alpha(X)$, we have

$$f = T^{-1}(Tf) = T^{-1}(h \cdot (f \circ \varphi)) = j \cdot (h \circ k) \cdot ((f \circ \varphi) \circ k).$$

Let I_X be the identity function on X. Now we claim that $\varphi \circ k = I_X$, since otherwise $(\varphi \circ k)(x_0) = x_1 \neq x_0$ for some $x_0 \in X$ and, taking $h_{x_1,\delta}$ in $lip_\alpha(X)$ with $\delta = d(x_1, x_0)$, we have

$$0 = h_{x_1,\delta}(x_0) = j(x_0) h(k(x_0)) h_{x_1,\delta}(x_1) = j(x_0) h(k(x_0)) \neq 0$$

which is absurd. Similarly $k \circ \varphi = I_Y$, where I_Y denotes the identity on Y. □

4. Discontinuous disjointness preserving linear functionals

In this section we shall state the existence of discontinuous disjointness preserving linear functionals of $lip_\alpha(X)$ under the condition that X is an infinite compact metric space and α is a real parameter in $(0, 1)$. For $\alpha = 1$, $lip_\alpha(X)$ may be finite dimensional, which we want to avoid. In the next lemma we obtain some surely known properties of ideals in $lip_\alpha(X)$. For the sake of completeness we give the proof.

Lemma 4.1. *Let X be a compact metric space and let $\alpha \in (0, 1)$. For each $x \in X$, define*

$$J_x = \{f \in lip_\alpha(X)\colon\ x \notin \mathrm{supp}(f)\}$$

and

$$M_x = \{f \in lip_\alpha(X)\colon\ f(x) = 0\}.$$

(1) *J_x is an ideal and M_x is a maximal ideal. Furthermore every maximal ideal of $lip_\alpha(X)$ is of the form M_x for some $x \in X$.*
(2) *M_x is the unique maximal ideal containing J_x.*
(3) *J_x is the intersection of all the prime ideals containing J_x.*
(4) *There exists a prime ideal P such that $J_x \subset P \subsetneq M_x$.*

Proof. (1) Since J_x is the set formed by all functions in $lip_\alpha(X)$ vanishing at a neighborhood of x, it is immediate that J_x is an ideal.

Clearly δ_x is a nonzero linear functional on $lip_\alpha(X)$ and M_x is the kernel of δ_x. Hence M_x is a maximal subspace of $lip_\alpha(X)$. In fact, M_x is an ideal since δ_x is, in addition, multiplicative. Besides M_x must be a maximal ideal, since M_x cannot be properly contained in a proper subspace, then much less in a proper ideal.

Now suppose M is a maximal ideal. If M is not contained in M_x for all $x \in X$, then for each $x \in X$ there exists some $f_x \in M$ such that $f_x(x) \neq 0$. Since f_x is continuous, there exists some $\delta_x > 0$ such that $f_x(z) \neq 0$ for all $z \in B(x, \delta_x)$. The family $\{B(x, \delta_x)\colon x \in X\}$ is clearly an open covering of the compact metric space X and so there exist $B(x_1, \delta_{x_1}), \dots, B(x_n, \delta_{x_n})$ such that $X = \bigcup_{k=1}^n B(x_k, \delta_{x_k})$. Consider the function $f \in lip_\alpha(X)$ defined by $f = \sum_{k=1}^n f_{x_k} \bar{f}_{x_k}$. Since M is an ideal, it is clear that $f \in M$. Moreover, if $x \in X$, then there exists some $j \in \{1, \dots, n\}$ such that $x \in B(x_j, \delta_{x_j})$ and thus

$$f(x) = \sum_{k=1}^n f_{x_k}(x) \bar{f}_{x_k}(x) = \sum_{k=1}^n \left| f_{x_k}(x) \right|^2 \geqslant \left| f_{x_j}(x) \right|^2 > 0.$$

Since $f \in lip_\alpha(X)$ is never zero, then f is bounded away from zero by compactness. It is easily seen that $1/f = h \circ f \in lip_\alpha(X)$, where $h(t) = 1/t$ for all $t \in f(X)$ is Lipschitz on $f(X)$ since it is differentiable with bounded derivative. Hence f is invertible in $lip_\alpha(X)$. This contradicts the fact that a proper ideal does not contain invertible elements. Thus $M \subset M_x$ for some $x \in X$. Then $M = M_x$ since M is maximal.

(2) Clearly $J_x \subset M_x$. Suppose that $J_x \subset M_y$ for some $y \in X \backslash \{x\}$. Let $\varepsilon = (1/2)d(x, y) > 0$ and put $\delta = d(B(x, \varepsilon), y)$. Since $d(x, y) > \varepsilon$, we have $\delta > 0$. The function $h_{y,\delta}$, which is in $lip_\alpha(X)$, satisfies $h_{y,\delta}(y) = 1$ and $h_{y,\delta}(z) = 0$ if $d(z, x) < \varepsilon$. Hence $h_{y,\delta} \in J_x \backslash M_y$, a contradiction.

(3) Let f be in $lip_\alpha(X)$. If there exists a prime ideal P containing J_x but not f, then f cannot belong to J_x since f does not belong to P. Conversely, let $f \notin J_x$. We will use Zorn's lemma to produce a prime ideal in $lip_\alpha(X)$ containing J_x, but not f. Let $\mathcal{F}$ denote the set of all ideals I in $lip_\alpha(X)$ such that $J_x \subset I$ and $f^n \notin I$ for all natural n. Clearly $J_x \in \mathcal{F}$ and thus $\mathcal{F} \neq \emptyset$. Partially order $\mathcal{F}$ by inclusion. To show that Zorn's lemma applies to $\mathcal{F}$, let $\mathcal{C}$ be any nonempty chain contained in $\mathcal{F}$ and let $Q = \bigcup \mathcal{C}$. One easily sees that Q is in $\mathcal{F}$, and Q is an upper bound for $\mathcal{C}$. By Zorn's lemma, $\mathcal{F}$ has a maximal member, say P. We assert that P is prime. Assume that $g_1 \notin P$ and $g_2 \notin P$. For $i = 1, 2$ the set $\{p + hg_i\colon p \in P,\ h \in lip_\alpha(X)\}$ is an ideal containing strictly P and therefore to J_x. Because of the maximality of P, there exist $n, m \in \mathbb{N}$ such that $f^n = p_1 + h_1 g_1$ and $f^m = p_2 + h_2 g_2$ for some $p_1, p_2 \in P$ and $h_1, h_2 \in lip_\alpha(X)$. A simple calculation shows that

$$f^{n+m} = p_1 p_2 + p_1 h_2 g_2 + p_2 h_1 g_1 + h_1 h_2 g_1 g_2.$$

If $g_1 g_2 \in P$, then $f^{n+m} \in P$ and this contradicts that $P \in \mathcal{F}$. Consequently $g_1 g_2 \notin P$ and P is prime.

(4) We have seen that $J_x \subset M_x$. However $J_x \neq M_x$. For instance, the function on X given by $f(z) = d(z, x)$ is in M_x, but not in J_x. Then by (3) there exists a prime ideal P which contains to J_x, but not to M_x. Let M_y be any maximal ideal containing P. Since $J_x \subset M_y$, we have $y = x$ by (2). Hence $P \subsetneq M_x$. □

Proposition 4.2. *Let X be an infinite compact metric space and let $\alpha \in (0, 1)$. There exists a discontinuous disjointness preserving linear functional on $lip_\alpha(X)$.*

Proof. For each $x \in X$ there is a prime ideal P of $lip_\alpha(X)$ such that $J_x \subset P \subsetneq M_x$ by Lemma 4.1. Let φ be any linear functional of $lip_\alpha(X)$ vanishing on P but not on M_x. We shall see that φ is discontinuous and disjointness preserving. Since $J_x \subset P \subset \ker \varphi$ and $\overline{J_x} = M_x$ (see Step 4 of Theorem 2.2), it follows that $M_x \subset \overline{\ker \varphi}$, but M_x is not contained in $\ker \varphi$. Hence $\ker \varphi$ cannot be closed in $lip_\alpha(X)$ and thus φ is discontinuous. To prove that φ preserves disjointness, let f, g be in $lip_\alpha(X)$ such that $f \cdot g = 0$. Since P is a prime ideal of $lip_\alpha(X)$, one of f and g belongs to P. As $P \subset \ker \varphi$, one of $\varphi(f)$ and $\varphi(g)$ must be zero. Thus $\varphi(f) \cdot \varphi(g) = 0$. □

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