Fractional Laplacian BVP with nonlinearities having multiple zeroes

Problemas de contorno para el Laplaciano Fraccionario con no linealidades que presentan múltiples ceros

Rubén Fiñana Aránega

Tutor: José Carmona Tapia

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Abstract

In this paper we investigate the existence and multiplicity of weak solutions of a nonlinear elliptic problem involving the fractional Laplacian, $(-\Delta)^s u(x) = \lambda f(u(x))$, $x \in \Omega$, $u(x) = 0$, $x \in \mathbb{R}^N \setminus \Omega$, with $s \in (0,1)$, under some certain conditions of the set Ω (bounded open domain of $\mathbb{R}^N$ with smooth boundary) and of the function f, which has to satisfy a necessary condition about its integral between its zeroes, in order to the solutions exist, what we also prove. Particularly we want to see that there are at least two positive solutions between each two zeroes of f.

To achieve this goal, we base mainly on the work about the fractional Laplacian by Ros-Oton and Serra [17], and some results for the classical Laplacian by P. Hess, [12] and by E. N. Dancer and K. Schmitt, [6] by studying some properties of the solutions of a truncated version of our problem, such as radial symmetry, regularity, bounds with the use of Leray Schauder degree, results involving the Sobolev space $H^s_0(\Omega)$, convergence results for sequences, compact embeddings and some well known results about Lebesgue integration.

Resumen

En este artículo investigamos la existencia y multiplicidad de soluciones débiles de un problema elíptico no lineal que involucra el laplaciano fraccionario, $(-\Delta)^s u(x) = \lambda f(u(x))$, $x \in \Omega$, $u(x) = 0$, $x \in \mathbb{R}^N \setminus \Omega$, con $s \in (0,1)$, bajo ciertas condiciones del conjunto Ω (un dominio abierto, acotado de $\mathbb{R}^N$ con borde suave) y de la función f, que tiene que cumplir una condición necesaria sobre su integral entre sus ceros, para que existan las soluciones, lo que también demostramos. En particular queremos ver que hay al menos dos soluciones positivas entre cada dos ceros de f.

Para ello, nos basamos principalmente en el trabajo sobre el Laplaciano fraccional de Laplacian by Ros-Oton and Serra [17], algunos resultados para el Laplaciano clásico de P. Hess, [12] y de E. N. Dancer y K. Schmitt, [6] estudiando algunas propiedades de las soluciones de una versión truncada de nuestro problema, como simetría radial, regularidad, acotaciones con el uso del grado de Leray Schauder, resultados que involucran el espacio de Sobolev $H_0^s(\Omega)$, resultados de convergencia para sucesiones, embebimientos compactos y algunos resultados conocidos sobre la integración de Lebesgue.

Introduction

This work culminates my training in the Master's degree in Mathematics and is also a consequence of my previous training in the degree in Mathematics and my initiation into research as a collaboration intern in the mathematics department, when I began the study of certain nonlocal operators involving the Fractional Laplacian Operator. My proposal for this TFM is to provide unprecedented results, to the best of my knowledge, about the study of certain boundary problems such that involve the Fractional Laplacian from some well known result about local operators such as the classical Laplacian operator. Particularly we will extend the known multiplicity results in the local case for nonlinearities that have multiplies zeroes. Later, we will make explicit reference to bibliographical references.

We will start by noting that in recent years, the study of nonlocal operators and nonlocal models have increasingly impacted upon different branches of science and technology.

We will focus on the fractional Laplacian operator, $(-\Delta)^s$ for some $s \in (0,1)$. Let us recall that the fractional Laplacian is a pseudo-differential operator of order $2s$ defined according to [15], with the symbol $|\xi|^{2s}$, by:

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u), \tag{1}$$

where $\mathcal{F}$ is the Fourier transformation and $u : \mathbb{R}^N \to \mathbb{R}$ belongs to the Schwartz space of C^∞ functions. This definition may be extended by duality to a larger class of distributions. However we will use the alternative definition, which also is found in [15], motivated by some stochastic processes (Lévy processes):

$$(-\Delta)^s u(x) = c_{N,s} p.v. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \tag{2}$$

with $x \in \mathbb{R}^N$ and where $p.v$ refers to Cauchy principal value and the normalization constant $c_{N,s}$ is given by:

$$c_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1-\cos(y_1)}{|y|^{n+2s}} dy \right)^{-1} > 0.$$

This operator play an important role in many fields of sciences such as fluid dynamics or economy. It is essential in order to model anomalous diffusion, which has been found in several environments. There are many documents where this fact is reflected, for example in [2] the authors have performed experiments and theoretical studies into contaminant transport in aquifers and in [5] is used in study of crime diffusion.

In relation to the study of this operator, the properties of the fractional Laplacian and the regularity of solutions of problems, such as the Dirichlet Problem for this operator, began in the 1960s, when Eskin and Vishik used a factorization property of pseudo-differential symbols to derive important mapping properties of fractional Laplacian, [21].

After that, the work by Caffarelli and Silvestre [4] appeared, based in a well known relation, that the Dirichlet-to-Neumann map of the harmonic extension problem in the upper half-space is given by the square root of the Laplacian, what means that for a smooth bounded function $f : \mathbb{R}^N \to \mathbb{R}$ and for the extension problem:

$$\begin{cases} u(x,0) = f(x) & x \in \mathbb{R}^N, \\ \Delta u(x,y) = 0 & (x,y) \in \mathbb{R}^N \times \mathbb{R}^+, \end{cases}$$

we can realize $(-\Delta)^{1/2}$ as the operator $T : f \to -u_y(x,0)$. They generalize this relation to a similar extension problem for each fractional power of the Laplacian. Then, for any $s \in (0,1)$, this relation connects the nonlocal operator given by (1) or (2) with local operators and allows to use local differential operators for its analysis. After this work nonlocal operators generated a lot of interest, so a lot of recent work has appeared. One of the most important works about the regularity of the solutions to $(-\Delta)^s u(x) = g(x)$ in a bounded domain Ω was performed by Ros-Oton and Serra [17], where they studied the Hölder regularity up to the boundary of solutions to the Dirichlet problem, i.e. $u(x) = 0$ in $\mathbb{R}^N \setminus \Omega$. Their estimates measure in a precise way the singular behavior of solutions near the boundary.

Great attention has also been devoted to symmetry results for equations involving the fractional Laplacian, such those known in the local case since the pioneering works by Serrin in 1971 [18], and Gidas, Ni and Nirenberg ten years later [11], who initiated the study of radial symmetry and monotonicity of positive solutions for non-linear elliptic equations in bounded domains using the moving planes, a method based on the Maximum Principle.

The purpose of this Master thesis is the study of the existence and multiplicity of solutions to the following nonlinear elliptic eigenvalue problem involving the fractional Laplacian:

$$
\begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3}
$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq \max\{2, 3-2s\}$), is a bounded domain with smooth boundary $\partial\Omega$, (it's enough if Ω is a $C^{1,1}$ domain), λ is a nonnegative constant. With respect to the nonlinear term f we assume that it has multiple zeroes. More precisely we assume that f satisfies the following hypotheses:

(A1) $f \in C^{1,1-s}(\mathbb{R}^+, \mathbb{R})$, particularly $f \in C^{0,1}_{loc}(\mathbb{R}^+, \mathbb{R})$.
(A2) $f(0) > 0$, and $f(s) = f(0)$ for $s < 0$.
(A3) For some $a_1, a_2, ..., a_m \in \mathbb{R}$ with $0 < a_1 < ... < a_m$ we have that $f(a_k) = 0$, $k = 1, ..., m$.
(A4) If we define $F(s) = \int_0^s f(\sigma)d\sigma$, then we have, with $k = 2, ..., m$, that $\max\{F(s) : 0 \leq s \leq a_{k-1}\} < F(a_k)$.

Observe that (A3)-(A4) imply, in the case of a unique zero s_k of f in (a_k, a_{k+1}) with $f(s) < 0$ for every $s \in (a_k, s_k)$ and $f(s) > 0$ for every $s \in (s_k, a_{k+1})$, then we have that $\int_d^{a_{k+1}} f(s)ds > 0$, for any $d \leq s_k$.

Problem (3) has been studied in the local case, i.e. for the Laplacian operator. Thus, in the paper by P. Hess, [12], he proved, for large λ, the existence of at least two solutions having uniform norm between each two zeroes of f in which f is positive. E. N. Dancer and K. Schmitt in [6] obtained the optimal sufficient condition in order to have a solution with given uniform norm.

In this work we want to prove similar results to those in [12] and [6] but working with a different kind of operator, the fractional Laplacian. Up to our knowledge, this problem has not been considered for the fractional Laplacian, where the regularity results by Ros-Oton and Serra in [17] will be a key point.

In addition, this work is clearly connected with [10], where I studied for the first time the fractional Laplacian and the fractional porous media equation. In that work I focused on the different concepts of solutions, introducing the elementary concepts, Sobolev Spaces, basic properties of the operator, etc. Therefore, I will try to make the writing of this work as self-contained as possible but avoiding the explicit developments already showed in [10].

The development of our work is organised in 5 Chapters:

- In the first chapter we will make a brief introduction to some fundamental concepts that we will use later, by outstanding the mainly results and by developing some of them. In order to do this, we divide it in 5 sections.

  We start by defining the concept of weak solutions and the functional spaces where we will work. After that, we will describe some notions about the topological Leray-Scahuder degree. The following section is focus on some convergence results about sequences and compact embeddings, that will be essential. In the next section we will deal with variational results by studying some properties of a functional, whose minimum will be a key point in the existence of solutions. Finally, the last section is dedicated to the Maximum principle for the fractional Laplacian, by showing two results about it, that will be helpful to show the uniqueness of the solution to our problem and to show properties such as positivity. In addition to a sub-supersolution method that yields the existence of a maximal solution of a truncated version of (3).
- In the second chapter, in the first section, we will study regularity results about solutions of (3) and in the second section we will focus on the demonstration of a necessary condition of our function f, about the integral of this function between its zeroes, in order to guarantee the existence of solutions of our problem.
- The third chapter is where we will work with the truncated version of (3) to demonstrate a multiplicity result for solutions to our problem by showing in the first section the existence of a solution to this truncated version based on some variational results about the minimum of a truncated functional, as consequence we will guarantee that there is, between each zeroes of our f, a solution of (3). And in the second section we will develop an result of existence of solutions of our truncated problem by using the degree theory.
- In the fourth chapter we will use the previous results about existence and the degree theory developed in the first section of this chapter, to demonstrate that there at least two solution two our problem between two zeroes of our f in the last part of the chapter.
- Finally, in the last section I reach several conclusions about the work, also indicating what I would like to continue developing.

CHAPTER 1

Preliminaries

For convenience of the reader, we provide some preliminary results and comments about the notation, which will be useful in the next chapters. Most of the results presented in the following sections are well known although not all of them has been studied in my curricula. Thus, a deep analysis on each of these concepts and proofs will require a proper Master thesis. Therefore we will focus on our main objective and will prove only the results most directly related with it. We will provide the reader with some references for detailed proofs of each result.

1. Concept of solution and regularity

The fractional Sobolev space where we consider problem (3) is the subspace of

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{u(x)-u(y)}{|x-y|^{N/2+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

given by $H_0^s(\Omega) = \{u \in H^s(\mathbb{R}^N) : u \equiv 0 \text{ a.e. } x \in \mathbb{R}^N \setminus \Omega\}$. This is a Hilbert space with the scalar product

$$\langle u, v\rangle = \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy, \; u, v \in H_0^s(\Omega).$$

The main properties of the fractional Sobolev spaces are well described in [7]. In the following lemma we summarize the main properties related with the concept of solution.

Lemma 1.1. *Assume that $u, v \in H_0^1(\Omega)$ then it is satisfied that:*

(1) $\|u\|^2_{H_0^1(\Omega)} = \|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\mathbb{R}^N)}$.

(2) $\displaystyle\int_\Omega v(-\Delta)^s u = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v = \langle u, v\rangle.$

Now we may define the concept of solution of (3) as follows.

Definition 1.1. *A weak solution of problem (3) is a function u in the space $H_0^s(\Omega)\bigcap L^\infty(\Omega)$ satisfying that:*

$$\frac{c_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dydx=\lambda\int_\Omega f(u(x))v(x)dx,$$

for all $v\in H_0^s(\Omega)$.

We will say that this function u is classical solution of our problem (3), if the fractional Laplacian of u is well defined at any point of Ω and u satisfies the equation in a pointwise sense.

Most of the results concerning to problem (3) require some regularity of solutions to problems with fixed data at the right hand side of the equation. We use the following notation:

Note 1.1. *When $\sigma>0$ is not a integer, we define $C^\sigma(\overline{\Omega}):=C^{k,\alpha}(\overline{\Omega})$, where k is the greatest integer such that $k<\sigma$ and $\sigma=k+\alpha$, $k\in\mathbb{N}$.*

The regularity of solutions established in the following result was proved in [17], Proposition 1.1 for bounded Lipschitz domains satisfying the exterior ball condition (in particular for smooth domains).

Proposition 1.1. *Let Ω be a bounded smooth domain, $g\in L^\infty(\Omega)$, and u be a solution to the problem*

$$\begin{cases}(-\Delta)^s u=g & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^{\mathbb{N}}\setminus\Omega.\end{cases}\tag{4}$$

Then $u\in C^s(\mathbb{R}^{\mathbb{N}})=C^{0,\alpha}(\mathbb{R}^{\mathbb{N}})$ and

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^{\mathbb{N}})}\le C\|g\|_{L^\infty(\Omega)}$$

where C is a constant depending only on Ω and s.

As usual, more regularity on the data g produces more regularity on solutions. Next result is a consequence of that, proved in [17], Proposition 1.4 where the optimal Hölder regularity is established.

Proposition 1.2. *Let Ω be a bounded domain, and $\beta>0$ be such that neither β, $\beta+2s$ is an integer. Let $g\in C^\beta(\Omega)$ and $u\in C^s(\mathbb{R}^N)$ be a solution of (4). Then, $u\in C^{\beta+2s}(\Omega)$.*

2. Leray-Schauder degree

Degree theory is a powerful tool to deal with the existence of solutions of equations in general. Leray-Schauder degree is used in [12] to study the local version of (3). In this section, we are going to make a brief introduction to the Leray-Schauder, by using some definitions and properties that can be consulted in [16], [1], [14].

DEFINITION 1.2. *Let X be a Banach space and denote $\mathcal{D}(X)$ the class of (ϕ, Ω_d, b), where:*

a) *Ω_d is an open subset of X.*
b) *$\phi = I - T$, where I denotes the identity in X and $T \in C(\overline{\Omega}_d, X)$ is a compact map.*
c) *$b \in X \setminus \phi(\partial\Omega_d)$.*

Leray-Schauder topological degree is defined in the following theorem where we summarize its main properties.

THEOREM 1.1. *Let X be a Banach space. There exists a unique function (called Leray-Schauder degree) $deg : \mathcal{D}(X) \to \mathbb{Z}$ such that:*

1. *Normalization property: $deg(I, \Omega_d, b) = 1$ for every $b \in \Omega_d$.*
2. *Additivity property: If Ω_{d1} and Ω_{d2} are open, bounded disjoint subsets in Ω_d and $b \notin \phi(\overline{\Omega}_d \setminus (\Omega_{d_1} \bigcap \Omega_{d_2}))$, then $deg(\phi, \Omega_d, b) = deg(\phi, \Omega_{d_1}, b) + deg(\phi, \Omega_{d_2}, b)$.*
3. *Homotopy property: Let $H \in C([0,1] \times \overline{\Omega}_d, X)$ be a compact homotopy. If $b \in C([0,1], X)$ satisfies $b(t) \notin I - H(t, \partial\Omega_d)$, for every $t \in [0,1]$, then $deg(I - H(t, \cdot), \Omega, b)$ is constant.*
4. *Solution property: If $deg(\phi, \Omega_d, b) \neq 0$, then there exists $x \in \Omega_d$ such that $\phi(x) = b$.*
5. *Excision property: Let $K \subset \Omega_d$ be any compact set such that $b \notin \phi(K)$. Then $deg(\phi, \Omega, b) = deg(\phi, \Omega \setminus K, b)$.*

The proof of this theorem relies on the fact that compact perturbations of the identity operator on Banach spaces can be approximated by finite rank continuous operators. Therefore Leray-Schauder topological degree is the natural extension of the finite dimensional Brouwer degree for continuous functions. Let us recall its definition in the case of regular values of differentiable functions.

DEFINITION 1.3. *A regular value of a differentiable function $f : \mathbb{R}^m \to \mathbb{R}^n$ with $n \leq m$, is a value $c \in \mathbb{R}^n$ such that the differential of f is surjective at every preimage of c, $f^{-1}(c)$, which are not critical points of f.*

Now we deal with the definition of Brouwer degree.

DEFINITION 1.4. *Let consider $\mathcal{R}$ the class of the triples (f_d, Ω_d, b), where Ω_d is a bounded open set in $\mathbb{R}^N$, $f_d \in C(\overline{\Omega}_d, \mathbb{R}^N) \bigcap C^1(\Omega_d, \mathbb{R}^N)$ and $b \notin f(\partial\Omega_d)$ a regular value of f_d in Ω, then we can define the Brouwer degree:*

$$d(f_d, \Omega_d, b) = \sum_{f_d(x)=b} sign(J_{f_d}(x)),$$

where $J_{f_d}(x)$ *is the determinant of the Jacobian of* f_d *at* x*,* $f'_d(x)$*.*

Notice that the set of preimage of b, $\{x \in \Omega : f(x) = b\}$ is finite, due to the fact that b is a regular value and Ω is bounded.

To extend the definition of the Brouwer degree to singular values b of f_d, i.e., $G(f_d) = \{x \in \Omega_d : J_{f_d}(x) = 0\}$, Sard's lemma is used:

Lemma 1.2. *The set of singular values,* $f_d(G(f_d))$*, has zero Lebesgue measure.*

In addition, as a consequence, the class of functions $f_d \in C^\infty(\overline{\Omega}_d, \mathbb{R}^N)$ for which b is a regular value, is dense in the space $C(\overline{\Omega_d}, \mathbb{R}^N)$, thus the degree defined for $\mathcal{R}$ is uniquely extended to a continuous map in the class of (f_d, Ω_d, b) with $f_d \in C(\overline{\Omega}_d, \mathbb{R}^N)$, Ω_d a bounded open set in $\mathbb{R}^N$ and $b \notin f_d(\partial\Omega_d)$.

Let us finally introduce the concept of index for isolated fixed points of a compact operator in a Banach space.

Definition 1.5. *Let* $a \in X$ *be an isolated fixed point of a compact operator* $T : X \to X$*. We define the index of* $\phi = I - T$ *at* a *as:*

$$i(\phi, a) = deg(\phi, B(a, r), 0),$$

which is constant for small $r > 0$*.*

Particularly, we are interested in the next theorem that works with Leray-Schauder index and that is proved in [**16**], Theorem 1.1.

Theorem 1.2. *Let* E *be an infinite dimensional Hilbert space,* $y \in E$*,* Ω *bounded neighborhood of* y*, and* $g \in C^1(\Omega, \mathbb{R})$ *with* $g'(u) = u - T(u)$ *where* T *is compact. Suppose* y *is a local minimum and isolated critical point for* g*. Then* $i(g', y, 0) = 1$*.*

3. Some convergence results for sequences of functions

Since we will use Leray-Schauder topological degree in some Banach space of functions defined in a bounded domain of $\mathbb{R}^N$ some convergence results and compact embedding will be needed. We present in this section some of this results.

3.1. Lebesgue integration results. Now we are going to recall some well-known results about integration that are helpful in our work.

Theorem 1.3 (Lebesgue's dominated convergence theorem). *Let* $\Omega \subset \mathbb{R}^N$ *be a measurable set and* $f_n \in L^1(\Omega)$ *be a sequence pointwise convergent to a measurable function* f*. If there exists an integrable* *function g, such that $|f_n(x)| \le g(x)$ for every $n \in \mathbb{N}$ and a.e. $x \in \Omega$, then $f \in L^1(\Omega)$ and*

$$\int_\Omega f = \lim_{n\to\infty} \int_\Omega f_n.$$

THEOREM 1.4 (Fubini's Theorem). *Assume that $f \in L^1(A \times B)$. Then*

$$\int_{A\times B} f(x,y) = \int_A \left(\int_B f(x,y)dy \right) dx = \int_B \left(\int_A f(x,y)dx \right) dy.$$

Moreover if $f(x,y) = g(x)h(y)$, then it is satisfied that:

$$\int_{A\times B} f(x,y) = \int_A g(x)dx \int_B h(y)dy.$$

3.2. Compact embeddings. We first recall the classical compact embedding of Hölder continuous functions in the space of continuous functions

THEOREM 1.5 (Arzelà-Ascoli Theorem). *Let $\Omega \subset R^n$, $\alpha \in (0,1)$, and let $\{f_i\}_{i\in\mathbb{N}}$ be any sequence of functions f_i satisfying*

$$\|f_i\|_{C^{0,\alpha}(\overline{\Omega})} \le C,$$

where C is a constant. Then, there exists a subsequence f_{i_j} which converges uniformly to a function $f \in C^{0,\alpha}(\overline{\Omega})$.

The next compact embedding involves the fractional Sobolev space $H_0^s(\Omega)$ which is compactly embedded in some Lebesgue spaces. This is a consequence of the general result established in [7], Corollary 7.2 for more general fractional Sobolev spaces in bounded extension domains which is valid for $H_0^s(\Omega)$ in the case of a bounded domain with smooth boundary.

THEOREM 1.6. *Let $s \in (0,1)$, $2s < N$, $q \in \left[1, \frac{2N}{N-2s}\right)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Then $H_0^s(\Omega)$ is compactly embedded in $L^q(\Omega)$.*

4. Variational results

In this section, taking into account the variational structure of the problem in Definition 1.1, we will summarize the main abstract variational results and we will prove those related to our problem.

Observe, that given $g \in L^\infty(\Omega)$, the functional $\Phi : H_0^s(\Omega) \to \mathbb{R}$ given by:

$$\Phi(u) = \frac{1}{2}\|u\|^2_{H_0^s(\Omega)} - \int_\Omega g(x)u,$$

satisfies that $\Phi \in C^1(H^s_0(\Omega), \mathbb{R})$ and critical points of Φ are weak solutions to (4).

Let us recall that as a direct consequence of the Weierstrass theorem we have following result (see [1]).

THEOREM 1.7. *Assume that H is a Hilbert space and $J \in C^1(H, \mathbb{R})$ is coercive and weak lower semicontinuous. Then J is bounded from below and has a global minimum.*

The first important result deals with the existence of solution to (4) as a minimum of the functional Φ.

THEOREM 1.8. *For every $g \in L^\infty(\Omega)$ there exists $u \in H^s_0(\Omega)$ a solution to (4) which is a minimum of the functional Φ.*

PROOF. First we observe that Φ is a weak lower semicontinuous function, i.e., if $u_n \rightharpoonup u$ in $H^s_0(\Omega)$, then

$$\liminf \Phi(u_n) \geq \Phi(u).$$

Indeed, let u_n be a sequence such that $u_n \rightharpoonup u$ in $H^s_0(\Omega)$ then, using the compact embedding (Theorem 1.6) u_n strongly converges to u in $L^2(\Omega)$ and, using also that $\|\cdot\|^2_{H^s_0(\Omega)}$ is weak lower semicontinuous we have

$$\liminf \Phi(u_n) = \frac{1}{2} \liminf \|u_n\|^2_{H^s_0(\Omega)} - \int_\Omega g(x)u \geq \Phi(u).$$

Now we prove that Φ is a coercive function. Indeed,

$$\frac{1}{2}\|u\|^2_{H^s_0(\Omega)} - \lambda \int_\Omega g(x)u \geq \frac{1}{2}\|u\|^2_{H^s_0(\Omega)} - C\|u\|_{H^s_0(\Omega)}$$

which implies that $\lim\limits_{\|u\|_{H^s_0(\Omega)} \to +\infty} \Phi(u) = +\infty$.

Thus we can use Theorem 1.7 and we finished the proof. □

REMARK 1.1. *The previous proof is also valif for the funtional*

$$\Phi_k(u) = \frac{1}{2}\|u\|^2_{H^s_0(\Omega)} + \frac{k}{2}\|u\|^2_{L^2(\Omega)} - \int_\Omega g(x)u.$$

Thus, for every $k > 0$, we have the existence of solution for the problem

$$\begin{cases} (-\Delta)^s u + ku = g & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^{\mathbb{N}} \setminus \Omega. \end{cases} \tag{5}$$

5. Maximum principles and Symmetry

In this subsection we will show some maximum principles for the fractional Laplacian which are essential in order to deal with problem (3).

In Theorem 2.2 of [13] it is shown that classical Harnack's inequality is not true in the nonlocal case without having into account the values outside Ω. Thus, the nonlocal case of the maximum principle, weak or strong, does not admit an analogous formulation to the local case. The result in [13] proves the existence of a s-harmonic function in $B_1(0)$ which is positive in $B_1(0) \setminus \{0\}$ and $u(0) = 0$.

However, most of the techniques to deal with problem (3) in the local case deeply involves the maximum principle and the strong maximum principle. We give here the precise formulation we will use in this manuscript.

Firstly we have the result proved in [20], Proposition 2.2.8.

PROPOSITION 1.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, let u be a lower semicontinuous function in $\overline{\Omega}$, such that $(-\Delta)^s u \geq 0$ in Ω and $u \geq 0$ in $\mathbb{R}^N \setminus \Omega$. Then $u \geq 0$ in $\mathbb{R}^N$. Moreover, if $u(x) = 0$, for one point x inside Ω, then $u = 0$ in the whole $\mathbb{R}^N$.*

REMARK 1.2. *Observe that this result immediately implies uniqueness of solution of (4). Moreover, the uniqueness is also true for (5) as a consequence of the next remark.*

REMARK 1.3. *Observe that for $k \geq 0$ and $w \in L^2_{loc}(\mathbb{R}^N)$ such that $\dfrac{w(x) - w(y)}{|x - y|^{N/2+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$ and*

$$\begin{cases} (-\Delta)^s w + kw \geq 0 & \Omega, \\ w \geq 0 & \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6}$$

we have that $w \geq 0$. Indeed, multiplying (6) by $w^- = \min\{w, 0\}$ and integrating we have, with $\Omega^- = \{x \in \mathbb{R}^N : w(x) < 0\} \subset \Omega$, that

$$
\begin{aligned}
0 &\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(w(x)-w(y))(w^-(x)-w^-(y))}{|x-y|^{N+2s}}dydx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N\setminus\Omega^-} \frac{(w(x)-w(y))(w^-(x))}{|x-y|^{N+2s}}dydx \\
&\quad + \int_{\mathbb{R}^N}\int_{\Omega^-} \frac{(w(x)-w(y))(w^-(x)-w^-(y))}{|x-y|^{N+2s}}dydx \\
&= \int_{\Omega^-}\int_{\mathbb{R}^N\setminus\Omega^-} \frac{(w^-(x)-w(y))(w^-(x))}{|x-y|^{N+2s}}dydx \\
&\quad + \int_{\mathbb{R}^N\setminus\Omega^-}\int_{\Omega^-} \frac{(w(x)-w^-(y))(-w^-(y))}{|x-y|^{N+2s}}dydx \\
&\quad + \int_{\Omega^-}\int_{\Omega^-} \frac{(w^-(x)-w^-(y))(w^-(x)-w^-(y))}{|x-y|^{N+2s}}dydx \\
&\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(w^-(x)-w^-(y))(w^-(x)-w^-(y))}{|x-y|^{N+2s}}dydx \geq 0
\end{aligned}
$$

In particular, $\|w^-\|_{H_0^s(\Omega)} = 0$ and $w \geq 0$.

Remark 1.4. *Using condition (A2) we can assure that any solution to (3) is non-negative. Indeed, we may take $v = u^-$ as test function in Definition 1.1 and arguing as in Remark 1.3 we obtain that $u^- \equiv 0$.*

Remark 1.5. *Observe that given a nontrivial function $w \geq 0$ in $\mathbb{R}^N$, for every $x_0 \in \mathbb{R}^N$ such that $w(x_0) = 0$ and for every $k \in \mathbb{R}$ we have that*

$$(-\Delta)^s w(x_0) + kw(x_0) = c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{-w(y)}{|x_0-y|^{N+2s}}dy < 0.$$

In particular, nontrivial and non-negative functions $w : \mathbb{R}^N \to \mathbb{R}$ for which $(-\Delta)^s w + kw \geq 0$ in $\Omega \subset \mathbb{R}^N$ are strictly positive in Ω.

The previous remarks allow us to prove nonexistence results for nonlinear problems, we include here the proof in the next lemma.

Lemma 1.3. *Let $g \in C^{0,1}_{loc}(\mathbb{R}, \mathbb{R})$ and assume that $g(s_0) \leq 0$ for some $s_0 > 0$. Assume also that $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$ is a weak solution of*

$$\begin{cases} (-\Delta)^s u = \lambda g(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $\|u\|_\infty \neq s_0$.

PROOF. We argue by contradiction and assume that $\|u\|_\infty = s_0$. Since $g \in C^{0,1}_{loc}(\mathbb{R},\mathbb{R})$ we can take $k > 0$ such that $|g(x)-g(y)| \leq k|x-y|$ for every $x, y \in [0, s_0]$, particularly for $0 < y < x < s_0$ we have that $g(x) - g(y) \geq -k(x-y)$, i.e. we have that $g(s) + ks$ is an increasing function in $[0, s_0]$.

Now, we take $w = s_0 - u \in L^2_{loc}(\mathbb{R}^N)$ and $\dfrac{w(x)-w(y)}{|x-y|^{N/2+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$. We claim that

$$\begin{cases} (-\Delta)^s w + \lambda k w \geq 0 & \Omega, \\ w \geq 0 & \mathbb{R}^N \setminus \Omega. \end{cases}$$

Indeed, by definition we have $w(x) \in [0, s_0]$ for a.e. $x \in \mathbb{R}^N$ and, using that $g(s_0) \leq 0$, $g(s) + ks$ is non decreasing in $[0, s_0]$ and that $(-\Delta)^s s_0 = 0$, we have that:

$$\begin{aligned} (-\Delta)^s w + \lambda k w =& - (-\Delta)^s u + \lambda k s_0 - \lambda k u \\ &\geq g(s_0) + \lambda k s_0 - (g(u) + \lambda k u) \geq 0. \end{aligned}$$

Thus we have $w = s_0 - u > 0$ in $\mathbb{R}^N \setminus \Omega$, see Remark 1.5, which contradicts that $\|u\|_\infty = s_0$.

□

An important result, consequence of the maximum principle is the existence of solution between a sub and a super solution in the sense of the following definition.

DEFINITION 1.6. *We say that* $\overline{u} \in L^2_{loc}(\mathbb{R}^N)$ *with* $\dfrac{\overline{u}(x)-\overline{u}(y)}{|x-y|^{N/2+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$ *is a supersolution to problem (3) if it satisfies*

$$\begin{cases} (-\Delta)^s \overline{u} \geq \lambda f(\overline{u}) & in\ \Omega, \\ \overline{u} \geq 0 & in\ \mathbb{R}^N \setminus \Omega, \end{cases}$$

i.e., $\overline{u} \geq 0$ *in* $\mathbb{R}^N \setminus \Omega$ *and for every* $v \in H^s_0(\Omega)$ *it is satisfied that*

$$\frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dydx \geq \lambda \int_\Omega f(u(x))v(x)dx,$$

Analogously we say that $\underline{u}$ *is a subsolution if weakly satisfies*

$$\begin{cases} (-\Delta)^s \underline{u} \leq \lambda f(\underline{u}) & in\ \Omega, \\ \underline{u} \leq 0 & in\ \mathbb{R}^N \setminus \Omega. \end{cases}$$

We include now the result of existence of solution between a subsolution and a supersolution. We include here the proof for convenience of the reader.

Theorem 1.9. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded smooth domain and let $\underline{u}$ and $\overline{u}$, with $\overline{u} \geq \underline{u} \geq 0$ and $\overline{u} \in L^\infty(\Omega)$, be a subsolution and a supersolution respectively of the problem (3). Then there exists a solution $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$ of this problem in $[\underline{u}, \overline{u}]$.*

Proof. As in the proof of Lemma 1.3, we can choose $k > 0$ such that $f(s) + ks$ is an increasing function in the interval $[0, \|\overline{u}\|_\infty]$. Now let us consider the sequence $\{u_n\} \in H_0^s(\Omega)$ definded inductively as:

$$\begin{cases} (-\Delta)^s u_{n+1} + \lambda k u_{n+1} = \lambda f(u_n) + \lambda k u_n & \text{in } \Omega, \\ u_{n+1} = 0 & \text{in } \mathbb{R}^{\mathbb{N}} \setminus \Omega, \end{cases}$$

and $u_0 = \underline{u}$. We will prove, through several steps, that this sequence is well defined and converges to a solution.

Step 1. We claim that $0 \leq u_n \leq \overline{u}$ for every $n \in \mathbb{N} \cup \{0\}$. This implies, using Theorem 1.8 (see also Remark 1.1) that u_{n+1} is well defined. In order to prove that we are going to use induction.

(1) By hypothesis it is satisfied that $0 \leq u_0 = \underline{u} \leq \overline{u}$.
(2) Now we suppose that $0 \leq u_n \leq \overline{u}$ for some n and let us prove that $0 \leq u_{n+1} \leq \overline{u}$. Since $\lambda f(u_n) + \lambda k u_n \geq \lambda f(0) + \lambda k 0 \geq 0$ we can argue as in Remark 1.3 and obtain that $0 \leq u_{n+1}$. Observe that

$$(-\Delta)^s u_{n+1} + \lambda k u_{n+1} = \lambda f(u_n) + \lambda k u_n,$$
$$(-\Delta)^s \overline{u} + \lambda k \overline{u} \geq \lambda f(\overline{u}) + \lambda k \overline{u}.$$

So, subtracting these two expressions above

$$(-\Delta)^s (\overline{u} - u_{n+1}) + \lambda k (\overline{u} - u_{n+1}) \geq 0, \text{ in } \Omega.$$

Here we have used that $u_n \in [0, \|\overline{u}\|_\infty]$ and that $f(s) + ks$ is increasing in that interval. On the other hand

$$\overline{u} - u_{n+1} \geq 0, \text{ in } \mathbb{R}^{\mathbb{N}} \setminus \Omega.$$

Therefore, Remark 1.3 yields that $u_{n+1} \leq \overline{u}$ in $\mathbb{R}^N$.

Step 2. We claim that $u_n \leq u_{n+1}$. In order to prove that we are going to use induction again.

(1) Firstly, let us prove that $u_0 \leq u_1$. Observe that

$$(-\Delta)^s u_1 + \lambda k u_1 = \lambda f(\underline{u}) + \lambda k \underline{u},$$
$$(-\Delta)^s \underline{u} + \lambda k \underline{u} \leq \lambda f(\underline{u}) + \lambda k \underline{u}.$$

So, subtracting these two expressions above

$$(-\Delta)^s (u_1 - \underline{u}) + \lambda k (u_1 - \underline{u}) \geq 0, \text{ in } \Omega.$$

And on other hand $u_1 - \underline{u} \geq 0$, in $\mathbb{R}^{\mathbb{N}} \setminus \Omega$. Therefore, Remark 1.3 yields that $u_1 \geq \underline{u}$ in $\mathbb{R}^N$.

(2) Now we suppose that $u_n \leq u_{n+1}$ and let us prove that $u_{n+1} \leq u_{n+2}$. We have that

$$(-\Delta)^s u_{n+2} + \lambda k u_{n+2} = \lambda f(u_{n+1}) + \lambda k u_{n+1},$$

$$(-\Delta)^s u_{n+1} + \lambda k u_{n+1} = \lambda f(u_n) + \lambda k u_n.$$

Again by subtracting these two expressions above

$$(-\Delta)^s (u_{n+2} - u_{n+1}) + \lambda k (u_{n+2} - u_{n+1}) = \lambda[f(u_{n+1}) + k u_{n+1} - (f(u_n) + k u_n)] \geq 0,$$

in Ω. Thanks to the fact that $f(u) + k(u)$ is increasing and to our assumption $u_n \leq u_{n+1}$. And by using again Remark 1.3, we have that $u_{n+2} \leq u_{n+1}$.

Then, we have that $\{u_n\}$ is an increasing sequence.
Step 3. To finish the proof we are going to prove that our sequence converges to a solution of (3). First of all, we know that $\{u_n\}$ is an increasing sequence and is bounded between $\underline{u}$ and $\overline{u}$, then we can guarantee that $u_n \to u$ a.e. Moreover

$$\langle u_{n+1}, v\rangle = \lambda \int_\Omega (f(u_n) + k(u_n - u_{n+1}))v, \tag{7}$$

for every $v \in H_0^s(\Omega)$. Taking $v = u_{n+1}$ and using that f is continuous and u_n bounded we deduce that $\|u_n\|_{H_0^s(\Omega)}$ is bounded. Thus, up to a subsequence, we have that u_n weakly converges in $H_0^s(\Omega)$. Moreover, using the compact embedding in Theorem 1.6 we may assume strong convergence in $L^2(\Omega)$ and a.e. in Ω. In particular u is the weak limit of u_n and

$$\langle u_{n+1}, v\rangle \to \langle u, v\rangle$$

$$\lambda \int_\Omega (f(u_n) + k(u_n - u_{n+1}))v \to \lambda \int_\Omega f(u)v,$$

for every $v \in H_0^s(\Omega)$. Here we have used Theorem 1.3 for the last limit.

Thus, taking into account (7) we have that u is a solution of (3). □

Remark 1.6. *Observe that the solution obtained in the above proof is minimal in the sense that any other solution is greater. Indeed, given $v \in H_0^s(\Omega) \cap L^\infty(\Omega)$ solution of (3) in $[\underline{u}, \overline{u}]$ we can perform the same proof with $v = \overline{u}$ and the solution u obtained in this proof satisfies $u \leq v$. Even more, taking $u_0 = \overline{u}$ and arguing similarly we can assure the existence of a maximal solution. However we do not know if the maximal and the minimal solutions are different.*

Now we deal with another important property for solutions when the domain presents some symmetry properties and which is deduced from the maximum principle. Thus, it is possible to deduce radial symmetry of solutions in a ball. More precisely, in [8], Theorem 1.1 it is proved the next result in the unit ball B.

THEOREM 1.10. *Assume that* $f : [0, \infty) \to \mathbb{R}$ *is a locally Lipschitz function and* $g : [0, 1) \to \mathbb{R}$ *is a non-increasing function. Assume also that* u *is a classical solution to*

$$\begin{cases} (-\Delta)^s u = \lambda f(u) + g(|x|) & \text{in } B, \\ u = 0 & \text{in } \mathbb{R}^{\mathbb{N}} \setminus B. \end{cases}$$

Then u *is radially symmetric and strictly decreasing in* $r = |x|$ *for* $r \in (0, 1)$.

Observe that the previous result is also true for a ball of any ratio R instead of 1.

It would be useful, in case of radial symmetry, to have an expression of the spatial fractional Laplacian only in terms of the ratio. Next result is proved in [9], Theorem 1.1 and give us such expression.

THEOREM 1.11. *Let* $s \in (0, 1)$. *For every radial function* $u \in C^2(\Omega)$ *satisfying*

$$\int_0^{+\infty} \frac{|u(r)|}{(1+r)^{N+2s}} r^{N-1} dr < +\infty,$$

we have that

$$(-\Delta)^s u(r) = \frac{c_{N,s}}{r^{2s}} \int_1^{+\infty} \left(u(r) - u(r\tau) + \left(u(r) - u\left(\frac{r}{\tau}\right)\right) \tau^{-N+2s} \right) H(\tau) d\tau,$$

where $r = |x|$ *and* H *is the positive and continuous function defined, for* $\tau > 1$, *by:*

$$H(\tau) = \frac{2\pi\alpha_n \tau}{(\tau^2-1)^{1+2s}} \int_0^{\pi} \operatorname{sen}^{N-2} \left(\sigma \frac{(\sqrt{\tau^2 - \operatorname{sen}^2\sigma} + \cos\sigma)^{1+2s}}{\sqrt{\tau^2 - \operatorname{sen}^2\sigma}} \right) d\sigma,$$

where $\alpha_n = \dfrac{\pi^{\frac{N-3}{2}}}{\Gamma(\frac{N-1}{2})}$ *and* Γ *denotes the Euler's gamma function.*

CHAPTER 2

Necessary condition for the existence of solution

In this chapter we include one of the main contributions of this work which is a necessary condition for the existence of a solution of the original problem in terms of the area enclosed by f between its zeros. The result was known for the Laplacian operator (see [6]) although the proof in the case of the fractional Laplacian is not a simple adaptation of the previous one since no direct relation is available, as in the case of the Laplacian, between the fractional Laplacian of a radial solution and the one-dimensional fractional Laplacian in terms of the radius.

In the first section we summarize the main regularity properties of the solutions to (3) and in the second section we prove the necessary condition.

1. Regularity of solutions to (3)

We prove in this section that solutions of (3) are actually $C^2(\Omega)$ functions.

Lemma 2.1. *Assume (A1), (A2) and that $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (3). Then:*

(1) $u \in C(\overline{\Omega}) \bigcap C^{2+s}(\Omega)$.
(2) *u is positive.*

Proof. (1) $u \in C(\overline{\Omega}) \bigcap C^{2+s}(\Omega)$.

In order to prove it, we are going to use Proposition 1.1. It's clear that our space Ω and our function $g(x) = \lambda f(u(x))$ satisfy hypotheses of Proposition 1.1 (because we have a bounded domain Ω with smooth boundary, it's enough if is a $C^{1,1}$ domain, λ is a constant and $f(u) \in L^\infty(\Omega)$ since $f \in C(\mathbb{R})$ and $u \in H_0^s(\Omega) \bigcap L^\infty(\Omega)$). Thus, our solution $u \in C^s(\mathbb{R}^{\mathbb{N}})$, what implies that $u \in C(\mathbb{R}^N)$ particularly $u \in C(\overline{\Omega})$.

In addition, we know that $f \in C^{2-s}(\mathbb{R}^+)$, $u \in C^s(\mathbb{R}^N)$, so we can be sure that $f(u) \in C^s(\Omega)$ and by Proposition 1.2 with $\beta = s$ we have $u \in C^{3s}(\Omega)$. Therefore, $f(u) \in C^{\beta_1}(\Omega)$ for $\beta_1 = \min\{2-s, 3s\}$, if $\beta_1 = 3s$ then, applying again Proposition 1.2, we have that $u \in C^{5s}(\Omega)$. Arguing inductively we have that $f(u) \in C^{\beta_n}(\Omega)$ for $\beta_n = \min\{2-s, (2n+1)s\}$. Thus, after a finite number of steps, $\beta_n = 2-s$,

i.e. $f(u) \in C^{2-s}(\Omega)$ and by using one last time Proposition 1.2, with $\beta = 2 - s$, we have $u \in C^{2+s}(\Omega)$.
(2) u is positive as was shown in Remark 1.3 (see also Remark 1.4). □

2. Necessary condition.

We proved in Lemma 1.3 that if $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$ is a solution of (3) with $\|u\|_\infty = \gamma$ then $f(\gamma) > 0$. In this section we prove that actually a stronger condition is necessary. More precisely we need $\int_d^\gamma f(s)ds > 0$ for every $d < \gamma$. This was first observed in [6] for the Laplacian operator.

THEOREM 2.1. *Let Ω be a bounded domain with smooth boundary and f a function satisfying $(A1), (A2)$ and $f(a) = 0$ for some $a > 0$. Assume that $u \in H_0^s(\Omega) \cap L^\infty(\Omega)$ is a solution of (3) with $\|u\|_\infty = \gamma \in (0, a)$, then*

$$\int_d^\gamma f(s)ds > 0, \text{ for every } d \in (0,\gamma). \tag{8}$$

PROOF. We are going to argue by contradiction, assume that there exists $d^* < \gamma$ such that

$$\int_{d^*}^\gamma f(s)ds \leq 0. \tag{9}$$

Let B be a ball in $\mathbb{R}^N$, with radius R, centered at the origin such that $\overline{\Omega} \subset B$. We consider the following problem:

$$\begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } B, \\ u = 0 & \text{in } \mathbb{R}^N \setminus B. \end{cases} \tag{10}$$

Now let define the function $\alpha(x)$:

$$\alpha(x) = \begin{cases} u(x) & \text{in } \overline{\Omega}, \\ 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Observe that α is a sub-solution of (10) since:

$$\begin{cases} (-\Delta)^s\alpha(x) = (-\Delta)^s u(x) = \lambda f(u(x)) = \lambda f(\alpha(x)) & \text{in } \Omega, \\ (-\Delta)^s\alpha(x) = 0 < \lambda f(0) = \lambda f(\alpha(x)) & \text{in } B \setminus \Omega, \\ \alpha(x) = 0 & \text{in } \mathbb{R}^N \setminus B. \end{cases}$$

On the other hand, if we consider $\beta(x) = a$ we have that $\beta(x)$ is a super-solution of of (10):

$$\begin{cases} (-\Delta)^s\beta(x) = 0 = \lambda f(a) = f(\beta(x)) & \text{in } B, \\ \beta(x) > 0 & \text{in } \mathbb{R}^N \setminus B. \end{cases}$$

Hence, by using Theorem 1.9, problem (10) admits a solution $v \in H_0^s(B) \cap L^\infty(B)$ such that $v(x) \in (\alpha(x), \beta(x))$ a.e. $x \in \overline{B}$, so $\|v\|_\infty \in [\gamma, a)$.

It follows from the symmetry result of Theorem 1.10, (applied to (10) with $g = 0$) that v is a radially symmetric solution. Now, thanks to the regularity of v that we have seen before in Lemma 2.1, we have that $v \in C^{2+s}(B)$.

Then, we can write $v(x) = w(\|x\|) = w(\rho)$ and by Theorem 1.11, it solves the following problem:

$$\begin{cases} \frac{c_{N,s}}{\rho^{2s}} \int_1^\infty \left(w(\rho) - w(\rho\tau) + (w(\rho) - w(\rho/\tau))\, \tau^{-N+2s}\right) H(\tau) d\tau \\ \qquad\qquad\qquad\qquad = \lambda f(w(\rho)),\ \rho \in (0, R) \\ w(R) = 0. \end{cases}$$

Where R is the radius of B. Then, multiplying by $w'(\rho)\rho^{1-N}$ we have:

$$\frac{c_{N,s}}{\rho^{2s-1+N}} \int_1^\infty \left(w(\rho) - w(\rho\tau) + (w(\rho) - w(\rho/\tau))\, \tau^{-N+2s}\right) H(\tau) d\tau w'(\rho) \\ = \lambda \rho^{1-N} f(w(\rho)) w'(\rho).$$

Recall that $\gamma = \|v\|_\infty = v(0) = w(0)$ (since v is radially non-increasing). Let us choose also $r < R$ such that $d^* = w(r) < \gamma$. Now, we integrate for $\rho \in (0, r)$, performing the change of variable $\sigma = w(\rho)$, we have that:

$$\int_0^r \frac{c_{N,s}}{\rho^{2s-1+N}} \int_1^\infty \left(w(\rho) - w(\rho\tau) + (w(\rho) - w(\rho/\tau))\, \tau^{-n+2s}\right) w'(\rho)\tau \\ H(\tau) d\tau d\rho = -\lambda \int_{w(r)}^{w(0)} f(\sigma) d\sigma = -\lambda \int_{d^*}^{\gamma} f(\sigma) d\sigma \geq 0.$$

Here we have used our assumption (9).

Now, we claim that the integral in the left hand side is negative, so we have a contradiction.

First of all, using Theorem 1.4 we have that the left hand side is equal to

$$c_{N,s} \int_1^\infty \int_0^r \frac{\left(w(\rho) - w(\rho\tau) + (w(\rho) - w(\rho/\tau))\, \tau^{-n+2s}\right)}{\rho^{2s-1+N}} w'(\rho) d\rho H(\tau) d\tau.$$

Now we argue as in Section 3 of [9] and we have that

$$w(\rho) - w(\rho\tau) + (w(\rho) - w(\rho/\tau))\, \tau^{-N+2s} = \\ -\frac{1}{2}\rho^2 \frac{(\tau-1)^2(\tau^{N+2-2s}+1)}{\tau^{N+2-2s}} \left(w''(\rho) + \frac{2\tau(\tau^{N+1-2s}-1)}{(1+\tau^{N+2-2s})(\tau-1)} \frac{w'(\rho)}{\rho}\right) \\ + o(\tau-1)^2.$$

Then, we can write that

$$sign\left[\int_0^r \frac{\left(w(\rho)-w(\rho\tau)+\left(w(\rho)-w(\rho/\tau)\right)\tau^{-N+2s}\right)w'(\rho)}{\rho^{2s-1+N}}d\rho\right]=$$

$$-\,sign\left[\int_0^r \rho^{-2s+3-N}\left(\frac{1}{2}\frac{(\tau-1)^2(\tau^{N+2-2s}+1)}{\tau^{N+2-2s}}w''(\rho)\right.\right.$$
$$\left.\left.+\frac{(\tau-1)(\tau^{N+1-2s}-1)}{\tau^{N+2-2s}}\frac{w'(\rho)}{\rho}\right)w'(\rho)d\rho\right].$$

Let $a(\tau)=\frac{1}{2}\frac{(\tau-1)^2(\tau^{N+2-2s}+1)}{\tau^{N+2-2s}}$ and $b(\tau)=\frac{(\tau-1)(\tau^{N+1-2s}-1)}{\tau^{N+2-2s}}$ which are positive functions for $\tau>1$ and observe that

$$\int_0^r \rho^{-2s+3-N}\left(a(\tau)\frac{d}{d\rho}(w'(\rho))^2+b(\tau)\frac{(w'(\rho))^2}{\rho}\right)d\rho$$
$$=b(\tau)\int_0^r (w'(\rho))^2\rho^{-2s+2-N}d\rho$$
$$+\,a(\tau)\int_0^r \rho^{-2s+3-N}\frac{d}{d\rho}\left((w'(\rho))^2\right)d\rho$$
$$=b(\tau)\int_0^r (w'(\rho))^2\rho^{-2s+2-N}d\rho$$
$$+\,a(\tau)\int_0^r \frac{d}{d\rho}\left(\rho^{-2s+3-N}(w'(\rho))^2\right)d\rho$$
$$-\,a(\tau)\int_0^r \frac{d}{d\rho}\left(\rho^{-2s+3-N}\right)(w'(\rho))^2d\rho$$
$$=b(\tau)\int_0^r (w'(\rho))^2\rho^{-2s+2-N}d\rho$$
$$+\,a(\tau)\left(r^{-2s+3-N}(w'(r))^2\right)$$
$$+\,(N+2s-3)a(\tau)\int_0^r\left(\rho^{-2s+2-N}\right)(w'(\rho))^2d\rho>0.$$

Thus, we have the claim proved which leads to a contradiction concluding the proof. □

REMARK 2.1. *Observe that conditions* $(A3)$ *and* $(A4)$ *implies that condition* (8) *is satisfied for every* γ *such that* $\gamma<a_k$ *and* $|\gamma-a_k|$ *small enough.*

CHAPTER 3

Existence and multiplicity of solutions with uniform norm separated by zeroes of f

In this chapter we prove our first multiplicity result for solutions to (3). Let us illustrate it in the case of non-negative nonlinearity f. In this case $(A4)$ is always satisfied for any finite family $a_1, \dots, a_m$ of zeroes of f. In this chapter we will prove that for large λ there exists at least a solution $u_k \in H_0^s(\Omega) \cap L^\infty(\Omega)$ such that

$$a_{k-1} < \|u_k\|_\infty < a_k,\ k = 2, \dots, m.$$

Actually, when f changes sign, $(A3)-(A4)$ implies that a_k is a zero of the function f such that f is positive at the left hand side of a_k (i.e. if f changes sign in (a_{k-1}, a_k) then f has at least another zero in this interval) and previous inequality can be improved.

In the first section we will prove the existence of u_k as a minimum of a functional whose critical points give us solutions with $L^\infty(\Omega)$ norm less that a_{k+1}. In the second section we will prove the existence of solution using degree theory this will be useful in the next chapter to prove multiplicity even with norms between zeroes of f.

1. Existence of minimum of a truncated functional

In this section, for every k, we consider a problem whose solutions are those of (3) with $L^\infty(\Omega)$ norm less than a_k. Then we will prove the existence of solution of this truncated problem as a minimum of a truncated functional. More precisely we consider the problem

$$\begin{cases} (-\Delta)^s u = \lambda f_k(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{11}$$

where

$$f_k(s) = \begin{cases} f(0), & s \le 0, \\ f(s), & 0 \le s \le a_k, \\ 0, & a_k \le s. \end{cases}$$

LEMMA 3.1. *Assume $(A1) - A(4)$. Then $u \in H_0^s(\Omega)$ is a solution to (3) with $u \in L^\infty(\Omega)$ and $\|u\|_\infty < a_k$ if and only if u is a solution to (11).*

Proof. The proof is straightforward taking into account that, using Proposition 1.1 and Lemma 1.3, solutions of (11) are bounded and have $L^\infty(\Omega)$ norm less than a_k. $\square$

Now we prove the existence of solution to (11) for every $k > 1$ which is a minimum for the functional $\phi_k : H_0^s(\Omega) \longrightarrow \mathbb{R}$ is defined by:

$$\phi_k(u) = \frac{1}{2}\|u\|^2_{H_0^s(\Omega)} - \lambda \int_\Omega F_k(u(x))dx,$$

where $F_k(s) = \displaystyle\int_0^s f_k(\sigma)d\sigma$.

Theorem 3.1. *Assume that $(A1) - (A4)$ are satisfied. Then $\phi_k \in C^1(H_0^s(\Omega), \mathbb{R})$ and its critical points are weak solutions to (11). Moreover ϕ_k attains its minimum and as a consequence (3) admits at least a solution $u_k \in H_0^s(\Omega) \cap L^\infty(\Omega)$ with $\|u_k\|_\infty < a_k$.*

Proof. It is clear from definition that $\phi_k \in C^1(H_0^s(\Omega), \mathbb{R})$ and that $\phi_k' : H_0^s(\Omega) \to (H_0^s(\Omega))'$ is given by

$$\langle \phi_k'(u), v\rangle = \langle u, v\rangle - \lambda \int_\Omega f_k(u)v,$$

for every $u, v \in H_0^s(\Omega)$. In particular, critical points of ϕ_k are weak solutions to (11).

In order to prove that ϕ_k admits a critical point which is a minimum we will use Theorem 1.7.

First we prove that ϕ_k is a weak lower semicontinuous function, i.e., if $u_n \rightharpoonup u$ in $H_0^s(\Omega)$, then

$$\liminf \phi_k(u_n) \geq \phi_k(u).$$

Let u_n be a sequence such that $u_n \rightharpoonup u$ in $H_0^s(\Omega)$ and observe that

$$\phi_k(u_n) = \frac{1}{2}\|u_n\|^2_{H_0^s(\Omega)} - \lambda \int_\Omega F_k(u_n(x))dx.$$

Since, by using Theorem 1.6, we know that $H_0^s(\Omega)$ is compactly embedded in $L^2(\Omega)$ we may assume that u_n strongly converges to u in $L^1(\Omega)$. Thus we have:

$$\int_\Omega |F_k(u_n) - F_k(u)| \leq \max_{r\in[0,a_k]} |f(r)| \int_\Omega |u_k - u| \to 0.$$

On the other hand, since the norm is always weak lower semicontinuous we have that

$$\liminf \frac{1}{2}\|u_n\|^2_{H_0^s(\Omega)} \geq \frac{1}{2}\|u\|^2_{H_0^s(\Omega)}.$$

In particular,

$$\liminf \left\{ \frac{1}{2}\|u_n\|^2_{H^s_0(\Omega)} - \lambda \int_\Omega F_k(u_n(x))dx \right\} \geq \frac{1}{2}\|u\|^2_{H^s_0(\Omega)} - \lambda \int_\Omega F_k(u(x))dx.$$

Now we claim that ϕ_k is coercive, i.e. $\lim_{\|u\|_{H^s_0(\Omega)}\to+\infty} \phi_k(u) = +\infty$. Observe that $F_k(s)$ is upper bounded:

$$F_k(s) = sf(0) < 0, \text{ for } s < 0,$$
$$F_k(s) \leq \max_{r\in[0,a_k]} |f(r)| \cdot \min\{s, a_k\}, \text{ for } 0 \leq s.$$

Then, for some constant $C > 0$

$$\frac{1}{2}\|u\|^2_{H^s_0(\Omega)} - \lambda \int_\Omega F_k(u(x))dx \geq \frac{1}{2}\|u\|^2_{H^s_0(\Omega)} - C$$

whose limit when $\|u\|_{H^s_0(\Omega)} \to \infty$ is $+\infty$, thus

$$\lim_{\|u\|_{H^s_0(\Omega)}\to+\infty} \left(\frac{1}{2}\|u\|^2_{H^s_0(\Omega)} - \lambda \int_\Omega F_k(u(x))dx \right) = +\infty.$$

Thus, we can apply Theorem 1.7 and we have that ϕ_k has a minimum, i.e. there exists $\hat{v} \in H^s_0(\Omega)$:

$$\phi_k(\hat{v}) = \inf\{\phi_k(v) : v \in H^s_0(\Omega)\},$$

and the proof is finished. □

REMARK 3.1. *Let us recall the proof of Theorem 1.7 in this particular case. Assume that u_n is a sequence such that $\phi_k(u_n) \to \inf\{\phi_k(v) : v \in H^s_0(\Omega)\}$. Due to the fact that ϕ_k is coercive, then u_n is bounded, and as consequence of that $H^s_0(\Omega)$ is a Banach reflexive space, we can find a weakly convergent subsequence of u_n, that we will denote again as u_n, such that $u_n \rightharpoonup \hat{v}$, then by using that ϕ_k is weak lower semicontinuous,*

$$\phi_k(\hat{v}) \leq \liminf \phi_k(u_n) = \inf\{\phi_k(v) : v \in H^s_0(\Omega)\}.$$

Thus, $\phi_k(\hat{v}) = \inf\{\phi_k(v) : v \in H^s_0(\Omega)\}$.

In the next result we prove the main multiplicity result of this chapter.

THEOREM 3.2. *Assume that $(A1) - A(4)$ are satisfied. There exist $\lambda_0 > 0$ such that, for every $\lambda > \lambda_0$, $\inf \phi_k < \inf \phi_{k-1}$ for $k = 2, \ldots, m$. As a consequence (3) admits at least m different solutions, $v_1, \ldots, v_m$ for $\lambda > \lambda_0$ with*

$$0 < \|v_1\|_\infty < a_1 < \|v_2\|_\infty < a_2 < \cdots < a_{m-1} < \|v_m\|_\infty < a_m.$$

PROOF. Let $v_{k-1} \in H_0^s(\Omega) \cap L^\infty(\Omega)$ with $\|v_{k-1}\|_\infty < a_{k-1}$ and $\phi_{k-1}(v_{k-1}) = \inf\{\phi_{k-1}(v) : v \in H_0^s(\Omega)\}$ given by Theorem 3.1. Observe that

$$\begin{aligned} F(v_{k-1}) =& F(a_k) - (F(a_k) - F(v_{k-1})) \\ \leq& F(a_k) - (F(a_k) - \max\{F(s) : 0 \leq s \leq a_{k-1}\}). \end{aligned}$$

Let us denote $\alpha = F(a_k) - \max\{F(s) : 0 \leq s \leq a_{k-1}\}$, $\alpha > 0$ due to (A4). Thus, we have that

$$\int_\Omega F(v_{k-1}) \leq \int_\Omega F(a_k) - \alpha|\Omega|. \tag{12}$$

Now, for $\delta > 0$ let us define the set $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$ which clearly satisfies that $|\Omega_\delta| \to 0$ when $\delta \to 0$. Let us consider also $w_\delta \in C_0^\infty(\Omega)$ with $0 \leq w_\delta(x) \leq a_k$ for every $x \in \Omega_\delta$ and $w_\delta(x) = a_k$ for every $x \in \Omega \setminus \Omega_\delta$. Since $w_\delta \in H_0^s(\Omega)$ (see [19, Lemma 5.1]) we have that $\phi_k(w_\delta) \geq \inf \phi_k$.

We claim that, for δ small enough and λ big enough, we also have that $\phi_{k-1}(v_{k-1}) > \phi_k(w_\delta)$. Observe that

$$\begin{aligned} \int_\Omega F(w_\delta) &= \int_{\Omega\setminus\Omega_\delta} F(a_k) + \int_{\Omega_\delta} F(w_\delta) \\ &= \int_\Omega F(a_k) - \int_{\Omega_\delta} (F(a_k) - F(w_\delta)) \\ &\geq \int_\Omega F(a_k) - 2C|\Omega_\delta|, \end{aligned}$$

with $C = \max\{|F(s)| : 0 \leq s \leq a_m\}$. Therefore, by using (12), we have that

$$\int_\Omega F(w_\delta) \geq \int_\Omega F(v_{k-1}) + \alpha|\Omega| - 2C|\Omega_\delta|.$$

If we choose a δ such that $\mu = \alpha|\Omega| - 2C|\Omega_\delta| > 0$, we have that

$$\begin{aligned} \phi_k(w_\delta) - \phi_{k-1}(v_{k-1}) =& \frac{1}{2}\left(\|w_\delta\|^2_{H_0^s(\Omega)} - \|v_{k-1}\|^2_{H_0^s(\Omega)}\right) \\ & - \lambda \int_\Omega (F(w_\delta) - F(v_{k-1})) \\ \leq& \frac{1}{2}\|w_\delta\|^2_{H_0^s(\Omega)} - \lambda\mu < 0, \end{aligned}$$

this last inequality is true for λ sufficiently large. Then $\phi_k(w_\delta) < \phi_{k-1}(v_{k-1})$ and we have the claim proved. In particular,

$$\inf \phi_k \leq \phi_k(w_\delta) < \phi_{k-1}(v_{k-1}),$$

and the proof is finished. Let us observe that this leads to the existence of multiple solutions to (3) for large λ since the solutions v_k given by Theorem 3.1 for $k = 2, \ldots, m$ satisfy

$$\|v_k\|_\infty > a_{k-1}.$$

Indeed, otherwise $\|v_k\|_\infty < a_{k-1}$ which implies

$$\phi_{k-1}(v_k) = \phi_k(v_k) = \inf \phi_k < \inf \phi_{k-1} \leq \phi_{k-1}(v_k),$$

which is a contradiction. □

2. Solutions using degree theory

In this section we show an alternative proof of the existence of solution to (11) using degree theory. Moreover, we will use the degree computations in the next chapter to improve the multiplicity results.

First we will show how solutions to (11) are the fixed points of a compact map. More precisely, for every $u \in C(\overline{\Omega})$ let us define $T_k(u) = w$ as the solution to the problem

$$\begin{cases} (-\Delta)^s w = \lambda f_k(u) & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{13}$$

This function is well defined because the solution is unique for each k and u (see Remark 1.2). Moreover, $T_k : C(\overline{\Omega}) \to C(\overline{\Omega}) \cap H_0^s(\Omega)$ thanks to Proposition 1.1. Therefore, if $u \in H_0^s(\Omega)$ is a solution of (11) (i.e. it satisfies that $\phi_k'(u) = 0$) then, using Proposition 1.1, we have that $u \in C(\overline{\Omega})$ and $T_k(u) = u$. Conversely, if $T_k(u) = u$, then $u \in H_0^s(\Omega)$ and it is a weak solution of (11) (i.e. $\phi_k'(u) = 0$).

Proposition 3.1. *$T_k \in C(C(\overline{\Omega}), C(\overline{\Omega}) \cap H_0^s(\Omega))$ and is a compact map. As a consequence Leray-Schauder degree applies to $I - T_k$.*

Proof. In order to demonstrate that, let us consider a bounded sequence $u_n \in C(\overline{\Omega})$, and we are going to show that $T_k(u_n)$ has a subsequence that converge uniformly in $C(\overline{\Omega})$ (i.e. it is a compact function), and that if there exists $u \in C(\overline{\Omega})$ such that $u_n \to u \in C(\overline{\Omega})$, then $T_k(u_n) \to T_k(u) \in H_0^s(\Omega) \cap C(\overline{\Omega})$ (i.e. it is a continuous function).

Let $w_n = T_k(u_n)$, i.e. w_n is the solution to the problem

$$\begin{cases} (-\Delta)^s w_n = \lambda f_k(u_n) & \text{in } \Omega, \\ w_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{14}$$

Moreover, thanks to Proposition 1.1, w_n is bounded in $C^{0,\alpha}(\overline{\Omega})$ (what also would mean that w_n is equicontinuous), so we can apply Arzelà-Ascoli, Theorem 1.5. Then, the sequence $w_n = T_k(u_n)$ has a subsequence that uniformly converges to w in $C(\overline{\Omega})$

$$w_n \to w. \tag{15}$$

Therefore, we know then that T_k is a compact function.

The next step is to prove the continuity of the function T_k. Let consider again the same sequence $u_n \in C(\overline{\Omega})$ such that $u_n \to u \in C(\overline{\Omega})$. Then, by using Theorem 1.3, we get that:

$$\int_\Omega f_k(u_n)\varphi \to \int_\Omega f_k(u)\varphi, \forall \varphi \in H_0^s(\Omega). \tag{16}$$

Once we have that, we go back to w_n, due to the fact that w_n solves (14):

$$\langle w_n, \varphi\rangle_{H_0^s(\Omega)} = \lambda \int_\Omega f_k(u_n)\varphi, \ \forall \varphi \in H_0^s(\Omega).$$

If we use $\varphi = w_n$, we have that:

$$\begin{aligned}\|w_n\|^2_{H_0^s(\Omega)} =& \lambda \int_\Omega f_k(u_n)w_n \\ &\leq \lambda \max\{|f(s)| : 0 < s < a_m\} \int_\Omega |w_n| \\ &\leq \lambda \max\{|f(s)| : 0 < s < a_m\}|\Omega|^{\frac{1}{2}} C_1 \|w_n\|_{H_0^s(\Omega)},\end{aligned}$$

where we have used the continuous embedding $H_0^s(\Omega) \subset L^2(\Omega)$, i.e. $\|w_n\|_{L^2(\Omega)} \leq C_1\|w_n\|_{H_0^s(\Omega)}$. Then

$$\|w_n\|_{H_0^s(\Omega)} \leq \lambda \max\{|f(s)| : 0 < s < a_m\}|\Omega|^{\frac{1}{2}} C_1.$$

This implies that there exists $w \in H_0^s(\Omega)$ such that, up to a subsequence, $w_n \rightharpoonup w$ in $H_0^s(\Omega)$, due also to the fact that $H_0^s(\Omega)$ is reflexive. This implies that

$$\langle w_n, \varphi\rangle_{H_0^s(\Omega)} \to \langle w, \varphi\rangle_{H_0^s(\Omega)}.$$

Therefore, thanks to (16), we can affirm that

$$\langle w, \varphi\rangle_{H_0^s(\Omega)} = \lambda \int_\Omega f_k(u)\varphi.$$

In particular, $w = T_k(u) \in C(\overline{\Omega}) \cap H_0^s(\Omega)$. Moreover, since w is uniquely determided, any subsequece of w_n weakly convergent has w as weak limit. This implies that the whole sequence w_n weakly converges to w (observe also that since w_n is bounded in $H_0^s(\Omega)$ any subsequence has a subsequence weakly convergent).

This proves the continuity of T_k and we finish the proof. □

Once we know that we can apply the Leray-Scauder degree, we can demonstrate the following property.

PROPOSITION 3.2. *For every $\lambda > 0$ there exists $R \equiv R(\lambda) > 0$ such that*

$$deg(I - T_k, B_R(0), 0) = 1,$$

where $B_R(0)$ denotes the open ball in $C(\overline{\Omega})$ with radius R and with center at the origin.

REMARK 3.2. *Observe that Proposition 3.2 gives, using the solution property of Theorem 1.1, another proof of the existence of solution to (11).*

PROOF. Let us consider the homotopy $H(t, \cdot)$ defined by

$$H(t, \cdot) = tT_k(\cdot) \text{ with } t \in [0, 1].$$

Observe that $H \in C([0, 1] \times C(\overline{\Omega}), C(\overline{\Omega}))$ and is compact due to Proposition 3.1.

We claim that for some $R > 0$ large, $deg(I - H(t, \cdot), B_R(0), 0)$ is well-defined, i.e. the equation $u - H(t, u) = 0$ has no solution in $\partial B_R(0)$.

Indeed, assume that $\|u\|_\infty = R$ and $u = tT_k(u)$, then

$$\begin{cases} (-\Delta)^s u = \lambda t f_k(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

and by Proposition 1.1 we have that

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^{\mathbb{N}})} \leq C \|\lambda t f_k(u)\|_{L^\infty(\Omega)} < B$$

for some positive constant B depending only on λ (not depending on u nor t). Then

$$R = ||u||_\infty \leq ||u||_{C^{0,\alpha}(\mathbb{R}^{\mathbb{N}})} < B.$$

In particular we can take any $R \geq B$ and we have $deg(H(t, \cdot), B_R(0), 0)$ well-defined.

Now we can use the homotopy property of the degree in Theorem 1.1 to deduce that $deg(I - H(t, \cdot), B_R(0), 0)$ is constant for $t \in [0, 1]$, in particular

$$deg(I - T_k(.), B_R(0), 0) = deg(I, B_R(0), 0) = 1,$$

where the last equality follows from the normalization property in Theorem 1.1. □

NOTE 3.1. *Note that solutions to* (11) *are also bounded in* $H_0^s(\Omega)$. *Indeed, if* u *is such a solution then*

$$\|u\|_{H_0^s(\Omega)}^2 = \lambda \int_\Omega f_k(u)u \leq \lambda \max\{|f(s)| : 0 < s < a_m\}|\Omega|\|u\|_\infty \leq C.$$

CHAPTER 4

Existence and multiplicity of solutions with uniform norm between zeroes of f

In this chapter we prove, as in [**12**] for the Laplacian operator, that for large λ, apart of the solution given by Theorem 3.1, which have uniform norm in the interval (a_{k-1}, a_k) (see Theorem 3.2), there exists another different solution with uniform norm in this interval. More precisely we prove the following result.

Theorem 4.1. *Assume* $(A1), (A2), (A3), (A4)$. *Then there exists* $\overline{\lambda} > 0$ *such that for every* $\lambda > \overline{\lambda}$ *problem* (3) *has at least* $2m - 1$ *classical positive solutions* $\hat{u}_1, u_2, \hat{u}_2, \ldots, u_m, \hat{u}_m$ *with* $0 < \|\hat{u}_1\|_\infty < a_1$ *and* $a_{k-1} < \|u_k\|_\infty$, $\|\hat{u}_k\|_\infty < a_k$ *for* $k = 2, \ldots, m$. *In addition* $\hat{u}_1 < \hat{u}_2 < \ldots < \hat{u}_m$ *with the natural ordering in* $C(\overline{\Omega})$.

With the same notation of the previous chapter, let us recall that solutions of problem (3) with uniform norm less than a_k correspond with the critical points of ϕ_k. Let K_k be the set of critical points of ϕ_k:

$$K_k := \{u \in H_0^s(\Omega) : \phi_k'(u) = 0\}.$$

Due to the continuity of ϕ_k', then K_k is a closed set and thanks to the Note 3.1, it is clear that K_k is a bounded set. Moreover, we have that $K_{k-1} \subseteq K_k$ and due to the fact that if u is a critical point of ϕ_k', then, it is a solution of (11) and by Proposition 1.1, $u \in C(\overline{\Omega})$ we can write

$$K_k := \{u \in C(\overline{\Omega}) : u = T_k(u)\}.$$

The key idea for the proof of Theorem 4.1 resides in some Leray-Schauder degree computations, performed in the first section, in some neighbourhood of K_k. These will lead us to conclude the proof in the second section.

1. Leray-Schauder degree computations

Let $U_\varepsilon(K_{k-1})$ be the ε-neighborhood of K_{k-1} in $C(\overline{\Omega})$, for $k \in [2, m]$, with $\varepsilon > 0$, then we have an interesting lemma about the Leray-Schauder degree of $I - T_k$ in this neighborhood.

LEMMA 4.1. *There exists* $\varepsilon_{k-1} > 0$ *such that, for every* $0 < \varepsilon < \varepsilon_{k-1}$*, we have that*

$$deg(I - T_k, U_\varepsilon(K_{k-1}), 0) = 1.$$

PROOF. Let us consider the ball $B_R(0)$ with R large enough such that $U_\varepsilon(K_{k-1}) \subset B_R(0)$. Observe that there is no critical point of ϕ_{k-1} in $B_R(0) \setminus \overline{U_\varepsilon(K_{k-1})}$ (or equivalently no fixed point of T_{k-1}). Therefore

$$deg(I - T_{k-1}, B_R(0) \setminus \overline{U_\varepsilon(K_{k-1})}, 0) = 0.$$

Let use now the additivity property of the Leray-Schauder degree in Theorem 1.1 and Proposition 3.2, to calculate $deg(I - T_{k-1}, U_\varepsilon(K_{k-1}), 0)$ for every $\varepsilon > 0$.

$$\begin{aligned} deg(I - T_{k-1}, U_\varepsilon(K_{k-1}), 0) =& deg(I - T_{k-1}, U_\varepsilon(K_{k-1}), 0) \\ &+ deg(I - T_{k-1}, B_R(0) \setminus \overline{U_\varepsilon(K_{k-1})}, 0) \\ =& deg(I - T_{k-1}(.), B_R(0), 0) = 1. \end{aligned}$$

Now we define the compact operator $H_1 \in C([0, 1] \times C(\overline{\Omega}), C(\overline{\Omega}))$ given by

$$H_1(t, u) = tT_{k-1}(u) + (1 - t)T_k(u),\ u \in C(\overline{\Omega}).$$

We claim that there exists $\varepsilon_{k-1} > 0$ such that

$$tT_{k-1}(w) + (1 - t)T_k(w) \neq w,$$

for every $t \in [0, 1]$, $w \in \partial U_\varepsilon(K_{k-1})$, and $0 < \varepsilon < \varepsilon_{k-1}$. Otherwise, for every $\varepsilon = \frac{1}{n}$ with $n \in \mathbb{N}$ we find $t_n \in [0, 1]$ and $w_n \in H^s_0(\Omega)$, which satisfies that $dist(w_n, K_{k-1}) = 1/n$, such that

$$t_nT_{k-1}(w_n) + (1 - t_n)T_k(w_n) = w_n.$$

Then, applying $(-\Delta)^s$ in both sides, w_n satisfies

$$\begin{cases} (-\Delta)^s w_n = \lambda[t_n f_{k-1}(w_n) + (1 - t_n)f_k(w_n)] & \text{in } \Omega, \\ w_n = 0 & \text{in } \mathbb{R}^N/\Omega. \end{cases}$$

Observe also that

$$t_n f_{k-1}(s) + (1 - t_n)f_k(s) = \begin{cases} f(0) \geq 0, & s \leq 0, \\ f(s), & 0 \leq s \leq a_{k-1}, \\ (1 - t_n)f(s), & a_{k-1} \leq s \leq a_k, \\ 0, & a_k \leq s. \end{cases}$$

Moreover, we can guarantee that $t_n f_{k-1}(s) + (1-t_n)f_k(s) \in C^{0,1}_{loc}(\mathbb{R}, \mathbb{R})$, so we can apply Lemma 1.3 to get that $\|w_n\|_\infty \neq a_k$. In addition we know from Lemma 2.1, that $w_n > 0$. Even more, since $t_n f_{k-1}(s) + (1 - t_n)f_k(s) = 0$ for every $s \geq a_k$, it is clear that $w_n \in [0, a_k)$, and $\|w_n\|_\infty \leq a_k$.

On other hand, if $\|w_n\|_\infty \leq a_{k-1}$, $t_n f_{k-1}(w_n) + (1-t_n)f_k(w_n) = f(w_n)$ and w_n will be a solution to our original problem (3), so $w_n \in K_{k-1}$ by Lemma 3.1. But it is a contradiction because $dist(w_n, K_{k-1}) = 1/n$, then $\|w_n\|_\infty > a_{k-1}$.

Moreover, as we have done previously, we can use Proposition 1.1 in order to have that $w_n \in C^{0,\alpha}(\overline{\Omega})$ and bounded in $C^{0,\alpha}(\overline{\Omega})$, so we can apply Arzelà-Ascoli Theorem 1.5. Thus w_n has a sub-sequence that we denote as w_n, such that converges uniformly to v in $C(\overline{\Omega})$, due to the fact that $dist(w_n, K_{k-1}) = 1/n$ and K_{k-1} is closed, we have that $v \in K_{k-1}$. This implies that $\|v\|_\infty < a_{k-1}$. Then $\|w_n\|_\infty \to \|v\|_\infty$, and due to the fact that $\|w_n\|_\infty > a_{k-1}$ we get the contradiction:

$$a_{k-1} > \|v\|_\infty = \lim \|w_n\|_\infty \geq a_{k-1}.$$

Hence the claim is proved. This guaranties that we have well defined $deg(I - H_1(t,\cdot), U_\varepsilon(K_{k-1}), 0)$ and it is constant in t, using the homotopy property of the Leray-Schauder degree in Theorem 1.1. In particular

$$deg(I - T_k, U_\varepsilon(K_{k-1}), 0) = deg(I - T_{k-1}, U_\varepsilon(K_{k-1}), 0) = 1.$$

This allows us to conclude the proof. ☐

2. Proof of Theorem 4.1 completed

In this section we complete the proof of the main result in this chapter with the multiplicity of solutions with uniform norm between zeroes of f. More precisely, let us consider $v_k \in K_k$ as the minimum of ϕ_k given by Theorem 3.1. We know that $K_{k-1} \subset K_k$ and for large λ we have, as a consequence of Theorem 3.2, that $v_k \notin K_{k-1}$ and that $a_{k-1} < \|v_k\|_\infty < a_k$.

Now we prove the multiplicity of solutions in $K_k \setminus K_{k-1}$.

Proposition 4.1. *For each k=2,...,m, there exists $\lambda_k > 0$ such that for all $\lambda > \lambda_k$ there are at least two different solution of (3), v_k, $\hat{v}_k$ in $K_k \setminus K_{k-1}$.*

Proof. We take λ_0 given by Theorem 3.2 and we have that $v_k \notin K_{k-1}$ for every $\lambda > \lambda_0$ and every $k = 2, \ldots, m$.

Now, we are going to consider two cases for v_k:

(1) If v_k is not a isolated critical point of ϕ_k, then there are a lot of points in $K_k \setminus K_{k-1}$.
(2) If v_k is an isolated critical point, we can use Theorem 1.2 to obtain that

$$deg(\phi_k', B_\varepsilon(v_k), 0) = deg(I - T_k, B_\varepsilon(v_k), 0) = 1, \tag{17}$$

for small $\varepsilon > 0$. Thus we choose $0 < \varepsilon < \varepsilon_k$ such that $U_\varepsilon(K_{k-1}) \bigcap B_\varepsilon(v_k) = \emptyset$. Once we have that ε, let us calculate: $deg(I - T_k(.), B_R(0) \setminus (\overline{U}_\varepsilon(K_{k-1}) \bigcup \overline{B}_\epsilon(v_k)), 0)$. For this purpose we know that by Proposition 3.2, Lemma 4.1 and (17)

$$\begin{aligned} deg(\phi_k'(.), B_R(0), 0) &= 1, \\ deg(\phi_{k-1}'(.), U_\varepsilon(K_{k-1}), 0) &= 1, \\ deg(\phi_k'(.), B_\varepsilon(v_k), 0) &= 1. \end{aligned}$$

Then due to the additivity property of the degree given in Theorem (1.1)

$$\begin{aligned} deg(I - T_k, B_R(0) \setminus (\overline{U}_\varepsilon(K_{k-1}) \bigcup \overline{B}_\varepsilon(v_k)), 0) &= deg(I - T_k, B_R(0), 0) \\ &- deg(I - T_{k-1}, U_\varepsilon(K_{k-1}), 0) - deg(I - T_k, B_\varepsilon(v_k), 0) \\ &= -1. \end{aligned}$$

This means that there exists $\hat{v}_k \neq v_k$ critical point of ϕ_k' in $B_R(0) \setminus (\overline{U}_\varepsilon(K_{k-1}) \bigcup \overline{B}_\varepsilon(v_k))$, so $\hat{v}_k \in K_k \setminus K_{k-1}$.

Then, we have at least two points in $K_k \setminus K_{k-1}$, which thanks to Lemma 3.1 they satisfy that $a_{k-1} < \|\hat{v}_k\|_\infty, \|v_k\|_\infty < a_k$. □

Now we can finish the proof of the main result.

Proof of Theorem 4.1 completed. Let consider $\lambda > \overline{\lambda}$ with $\overline{\lambda} = \max\{\lambda_k : 2 \leq k \leq m\}$ and λ_k given by Proposition 4.1. Then the existence of $\hat{u}_1, \hat{u}_2, \dots, \hat{u}_m$ with $0 < \|\hat{u}_1\|_\infty < a_1$ and $a_{k-1} < \|\hat{u}_k\|_\infty < a_k$ for $k = 2, \dots, m$ is deduced from Theorem 3.2. Moreover, the other $m-1$ solutions $u_2, \dots, u_m$ with $a_{k-1} < \|u_k\|_\infty < a_k$ for $k = 2, \dots, m$ is deduced from Proposition 4.1.

Let us now prove that $\hat{u}_1 < \hat{u}_2 < \dots < \hat{u}_m$ with the natural ordering in $C(\overline{\Omega})$. Observe that:

a) $\underline{u} = 0$ is a subsolution to problem (3). Indeed, it satisfies that $\underline{u} \in H_0^s(\Omega) \bigcap L^\infty(\Omega)$ and

$$\begin{cases} (-\Delta)^s \underline{u} = 0 < \lambda f(0) = \lambda f(\underline{u}) & \text{in } \Omega, \\ \underline{u} = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

b) $\overline{u}_k = a_k$ with $k = 1, \dots, m$ is a supersolution to problem (3) since $\overline{u}_k \in L^2_{loc}(\mathbb{R}^N)$, $\dfrac{\overline{u}_k(x) - \overline{u}_k(y)}{|x-y|^{N/2+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$ and

$$\begin{cases} (-\Delta)^s \overline{u}_k = 0 = \lambda f(a_k) & \text{in } \Omega, \\ \overline{u}_k > 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then, Theorem 1.9 (see also Remark 1.6) assures the existence of a maximal solution $u_{k,M}$ to (3) between 0 and a_k. Moreover, since a solution in $(0, a_k)$ is also a solution in $(0, a_{k+1})$ we inmediately have that $u_{k,M} \leq u_{k+1,M}$. Even more, Remark 1.5 also gives us that if $u_{k,M} \not\equiv u_{k+1,M}$ then $u_{k,M} < u_{k+1,M}$.

On the other hand, due to Lemma 3.1, $u_{k,M} \in K_k$ and $\hat{u}_k \leq u_{k,M}$ since $u_{k,M}$ is maximal. In addition, for $\lambda > \overline{\lambda}$, using Theorem 3.2 we have $a_{k-1} < \|\hat{u}_k\|_\infty \leq \|u_{k,M}\|_\infty < a_k$. Hence we can choose $\hat{u}_k$ to be the maximal solution of the problem in $[0, a_k]$ and it is clear that $\hat{u}_1 < \ldots < \hat{u}_m$ for large λ. □

REMARK 4.1. *Since we can choose $\hat{u}_k$ to be the maximal solution of the problem in $[0, a_k]$ then we can assure also that $u_k < \hat{u}_k$ but we cannot assure that $\hat{u}_{k-1} \leq u_k$ but only that $\|\hat{u}_{k-1}\|_\infty < a_{k-1} < \|u_k\|_\infty$.*

CHAPTER 5

Conclusions

Thanks to this essay, I have been able to test my knowledge about the things that I have learned during the degree in Mathematics and in the Master's degree in Mathematics, particularly about the branch of Mathematical Analysis. In addition, it has been useful to expand that knowledge that I knew about weak convergences, which I also studied in the subject called Advanced Functional Analysis in the Master, and especially about the Fractional elliptic equations, and the fractional Laplacian Operator which I began to study in my TFG, [10].

Another important thing that have helped me this work with, is the introduction to the world of the research in mathematics by studying and understanding, with the great help of my tutor, the work of some important researchers in this fields of the Mathematics.

This research that I have done, in order to do this manuscript, I would like to follow it in the future to study some other properties of the solutions that I haven't got the time to do, like the behaviour of them when they are closed to the the boundary of the space or when λ goes to ∞, or the study of how many solutions are between every two roots when $k \geq 2$.

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