Mathematical problem solving through sequential process analysis

Codina, A.[1,2], Cañadas, M. C.[2], Castro, E. [2]

[1] Dpto. de Educación, Universidad de Almería, Almería
[2] Dpto. de Didáctica de la Matemática, Universidad de Granada, Granada

Spain

Correspondence: Antonio Codina Sánchez, Dpto. de Educación, Edificio A, Universidad de Almería, C.P. 04120, Almería, Spain. E-mail: acodina@ual.es

Abstract

Introduction. The macroscopic perspective is one of the frameworks for research on problem solving in mathematics education. Coming from this perspective, our study addresses the stages of thought in mathematical problem solving, offering an innovative approach because we apply sequential process analysis and the polar coordinates technique in order to study the sequential relations and global interrelations between the different stages of mathematical problem solving.

Method. This investigation is based on observational methodology, taking for our unit of analysis the set of observable processes in a pair of students solving a mathematical problem. Quality of information is assured (intra- and inter-observer reliability and the Chi-square in-dependence test), thus allowing sequential analysis as well as the polar coordinates technique to be applied.

Results. We present two levels of specificity, the individual subject level and the pair level. We analyze the set of basic statistics; periods of collaborative and parallel work; transition probabilities, significant sequences or chains, transferences of execution and the set of global relations maps between the different stages. From the results, we may describe and analyze the behavior of the subjects and the pair during the problem-solving process, as well as the collaborative work that took place.

Discussion. The study reflects a new approach to investigating interrelationships between the stages of problem solving and collaborative work macroscopically, opening a new path for research in mathematics education. The two levels of specificity provide results that describe individual influences within the joint problem-solving process, and thus clarify in greater depth the interrelations between the subjects and the collaborative work that took place. The study reveals the potential of this type of analysis for studying learning difficulties in mathe-matical problem solving.

Key words Stages in mathematical problem solving, transference of execution, sequential analysis, polar coordinates.

La resolución de problemas matemáticos a través del análisis secuencial de procesos

Resumen

Introducción. La perspectiva macroscópica es uno de los marcos desde los que se aborda la investigación sobre resolución de problemas en la enseñanza de las matemáticas. Nuestro estudio, dentro de esta perspectiva, se enmarca dentro de los estadios del pensamiento en la resolución de problemas matemáticos, ofreciendo un enfoque innovador porque aplicamos el análisis secuencial de procesos y la técnica de coordenadas polares para estudiar las relaciones secuenciales e interrelaciones globales entre los distintos estadios en la resolución de problemas matemáticos.

Método. Esta investigación se basa en la metodología observacional, adoptando como unidad de análisis el conjunto de procesos observables de una pareja de estudiantes resolviendo un problema matemático. La calidad de la información (fiabilidad intra e interobservador y el test de independencia de la Chí-cuadrado) está garantizada permitiendo aplicar el análisis secuencial así como la técnica de coordenadas polares.

Resultados. Presentamos dos niveles de concreción, uno para cada sujeto y otro para la pareja. Analizamos el conjunto de estadísticas básicas; periodos de trabajo colaborativo y paralelo; probabilidades de transición, secuencias o cadenas significativas, traslaciones de ejecución y el conjunto de mapas de relaciones globales entre los diferentes estadios. Los resultados permiten describir y analizar el comportamiento de los sujetos y la pareja durante el proceso de resolución así como el trabajo colaborativo puesto en juego.

Discusión. El estudio refleja una nueva aproximación para investigar las interrelaciones entre las etapas de resolución de problemas y el trabajo colaborativo macroscópicamente, abriendo un camino de investigación en educación matemática. Los dos niveles de concreción permiten obtener resultados que describen las influencias individuales en el proceso de resolución conjunto, concretando con mayor profundidad en las interrelaciones surgidas entre los sujetos y el trabajo colaborativo puesto en juego. El estudio muestra el potencial de este análisis para el estudio de las dificultades en el aprendizaje de la resolución de problemas matemáticos.

Palabras Clave. Estadios en resolución de problemas matemáticos, traslación de ejecución, análisis secuencial, coordenadas polares.

Introduction

Mathematical problem solving is a traditional research topic in Mathematics Education, and has been developed especially over the last three decades. An extensive, heterogeneous body of knowledge has been generated, with methodological orientation and diverse analyses (Castro, 2008; Kilpatrick, 1992; Törner, Reiss, & Schoenfeld, 2007). Today, the possibilities offered by digital technology are opening new lines of research on how human-machine interaction, interactive system design or the design of learning-objects relate to mathematical problem solving (Codina, Cañadas & Castro, 2010; Dix, Finlay, Abowd, & Beale, 2004; Mgombelo & Buteau, 2010).

While it is true that research on mathematical problem solving has different orientations, these constitute basically two approaches with different views: problem solving as a activity typical to mathematics or as an educational task that is important for teaching and learning processes. Research may be grouped into two large blocks: (a) research focusing on how to teach problem solving, and (b) research focused on the study of how we think when we solve problems (Castro, 2008). Within the latter, we find studies on the different stages or phases of thought followed by a single problem solver or problem-solving group when solving mathematical problems. Our study belongs to this category. Prior research on these stages in found in the work of Artzt and Armour-Thomas (1990, 1992), Erbas and Okur (2012), Goos, Galbraith, and Renshaw (2002), Lee and Hollebrands (2006), Lester (1985), Poincaré (1908), Pólya (1945), Schoenfeld (1985), Villegas, Castro and Gutiérrez (2009), Yerushalmy (2000) and Yimer and Ellerton (2006, 2010). Before proceeding, since the notion of stage or phase can take on different meanings, we would clarify that we have chosen the term stage, in the sense of episode as described by Schoenfeld (1985): "a period of time during which an individual or a problem-solving group is engaged in one large task … or a closely related body of tasks in the service of the same goal" (p. 292).

The studies mentioned above, which form the background for this research, characterize the stages of mathematical problem solving, establishing connections between them and between their components. The present study seeks to offer a different, novel orientation, consisting of describing the problem-solving process and how its stages interrelate. In order to do this, we begin with observational methodology, obtaining sequential data of the occurrence and passage from one stage to another when a pair of problem solvers solve a mathematical

problem. Next we apply sequential process analysis and the polar coordinates technique to the data (Anguera & Losada, 1999; Bakeman & Gottman, 1989; Bakeman & Quera, 1996, 2011b; Gorospe & Anguera, 2000, Sackett, 1980, Sharpe & Kiperwas, 2003; Yoder & Symons, 2010), for the purpose of obtaining information about concurrence between stages, and any patterns or sequences of stages, frequencies or probabilities of transition, as well as the overall interrelationships between the stages.

Stages in mathematical problem solving

Research on the stages of scientific and mathematical problem solving may be said to have begun with the work of Poincaré (1908) and Dewey (1916). These authors, from a mathematical-psychological viewpoint, and taking into consideration ideal problem solvers, characterize the different stages of thought when mathematical problems are being solved. Decades later, Pólya (1945) takes more of a teaching and educational orientation, considering that the problem-solving process implies that the problem solver must overcome several obstacles on the way toward solving the problem, and that in order to do so, he or she must put different cognitve processes to work. Although Pólya only considers metacognitive processes implicitly, he highlighted the importance of heuristic thought and of mathematical reasoning in the processes of mathematical problem solving. The Pólya model consists of four stages which the problem solver must pass through successively: (a) understanding the problem, (b) conceiving a plan, (c) executing the plan, and (d) examining the solution. The influence of Pólya's work is evident in multiple studies. Among these are the framework for assessing problem solving in PISA-2012 (Organisation for Economic Co-operation and Development [OECD], 2010, 2014), the curriculum documents from different countries (Ministerio de Educación y Ciencia [MEC], 2006, 2007a, 2007b; National Council of Teachers of Mathematics [NCTM], 2000), and diverse investigations (Artz & Armour-Thomas, 1990, 1992; Lee & Hollebrands, 2006; Schoenfeld, 1985, Yerushalmy, 2000; Yimer & Ellerton, 2010).

Schoenfeld (1985), under the influence of Pólya's work, analyzes how decision making (cognition) and measurements of control and self-control (metacognition) are put to work during mathematical problem solving. From a macroscopic perspective, he presents a model of the following stages in mathematical problem solving: (a) reading, (b) analysis, (c) exploration, (d) planning, (e) implementation, and (f) verification. Schoenfeld, taking the problem-solving process of students working individually as his unit of analysis, observes that the pas-

sage from one stage to another and the relationships between stages are not linear, mainly due to metacognitive processes that come into play along with the cognitive processes. This defines a clear difference with respect to the ideas presented by Pólya.

Later, Artz and Armour-Thomas (1990, 1992), taking up Schoenfeld's work again, focus on cognitive and metacognitive processes exercised at each stage. The model proposed by Artz and Armour-Thomas distinguishes cognitive and metacognitive processes at each stage, and unlike Schoenfeld, considers the behavior of pairs of students during problem solving as their unit of analysis. This unit of analysis prompts the appearance of a stage typical of pair work: the stage of observation and listening. They also decide to classify the actions aimed at understanding the problem as a stage. Thus, the model by Artz and Armour-Thomas is made up of the following stages: (a) reading, (b) understanding, (c) analysis, (d) exploration, (e) planning, (f) implementation, (g) verifying and (h) observation and listening. Moreover, the work by Artz and Armour-Thomas suggests that students who are successful in problem solving demonstrate a continuous back and forth between cognitive and metacognitive processes, and between the different stages, whether in the processes that are internal to each stage or in inter-stage processes, this back-and-forth movement being more frequent in processes that successfully solve the problem.

Yerushalmy (2000), based on the Schoenfeld model, focuses her attention especially on the implementation stage and how the use of resources during the problem-solving process creates new ways of interacting between the different stages. Like Artz and Armour-Thomas (1992), she adopts as her unit of analysis pairs of students solving algebra problems, from a functional approach, and using digital technology. Yerushalmy detects back-and-forth movement between small periods in the planning stage, with a strong presence of metacognitive processes, many of which are motivated by the inclusion of digital technology, and between long periods in the implementation stage. The author suggests that these back-and-forth movements need to be studied in greater depth, in order to better understand how strategies and planning are developed, or evolve, in mathematical problem solving. Motivated by this suggestion, Lee and Hollebrands (2006) study the problem solving processes presented in the format of a small computer program that contains a specially designed java applet. For this purpose, they assign problems to primary students and adjust the indicators of each stage, adding a new one: organization. Lee and Hollebrands consider a six-stage model: (a) analysis, (b) planning, (c) implementation, (d) assessment, (e) verification and (f) organization. These

authors observe that in students who more frequently manage to solve the problems, the stage sequence "implementation->assessment->implementation" is also more frequent. However, the less successful students presented fewer control processes after periods of time in the implementation stage. Moreover, they note that use of the applets generally takes place in the implementation stage and is practically not used in the stages of analysis and planning.

Yimer and Ellerton (2006, 2010) develop a five-stage model: (a) engagement, (b) transformation-formulation, (c) implementation, (d) evaluation and (e) internalization, in order to describe the variety of transitions during problem solving processes. The authors confirm Schoenfeld's observations, observing how the transitions between the five stages are not linear or unidirectional. Yimer and Ellerton add that these transitions usually involve a period of re-reading that acts as "a catalyst for metacognitive decisions to take place either in the form of choosing a path or choosing other metacognitive actions within the specific cognitive phase in which problem solvers are engaged." (p. 251) Kursat and Okur (2012) reach similar conclusions, adding that, in order for problem solvers to be successful, they must know when and how to use strategies in each episode, knowing the metacognitive skills and transitions that must be put to use.

Prior studies use different models of problem solving stages and different units of observation (individual subjects or pairs). One element common to all is use of the think-aloud technique (Ericsson & Simon, 1980, 1993) for recording the problem-solving process. This method has allowed researchers to analyze the verbalizations and actions that problem solvers explicitly carry out, as well as those that can be inferred from their talk, thereby greatly increasing the number of processes that may be observed, compared to actions produced in silence (Villegas et al., 2009). The think-aloud technique is not unique to research on problem solving. For example, Armengol (2007) uses it to describe stages in the process of learning to write. It has also been used in studies of self-control and self-regulation in pairs of problem solvers (Goos et al., 2002) and representation pattern analysis in solving optimization problems (Villegas et al., 2009). It seems reasonable then to consider the use of this technique for recording the problem-solving process.

Sequential analysis and polar coordinates

The research presented above confirms the effort and progress of researchers in Mathematics Education in identifying and characterizing the different stages of problem solving, the passage from one to another, their intra- and inter-relationships and analysis of their cognitive and metacognitive components. Although the research mentioned above shows progress in our understanding of problem-solving processes, the authors concur in pointing to the need for further inquiry into the stages and process of mathematical problem solving. Taking this need on ourselves, we proposed the use of observational methodology (Anguera, 1990, 2003; Riba, 1993) in order to obtain probabilities of causality, concurrence, patterns and inter-relationships between the processes under observation – in our case, the stages of mathematical problem solving. One advantage of using this methodological approach is that it offers the possibility to complement qualitative and quantitative treatments (Anguera, 2010; Riba, 1993), representing a distinguishing element with regard to former studies. We submit the problem-solving process to observation, using the think-aloud technique (Ericsson & Simon, 1980, 1993), and to a certain method of recording, with the support of a codification system or system of categories comprising the different stages in problem-solving, described further on. This system allows sequential data to be obtained, to which observational procedures may be applied for a quantitative treatment of the observation. Some of the most notable quantitative observational procedures are those related to sequential process analysis (Bakeman & Gottman, 1989; Bakeman & Quera, 1996, 2011b; Sharpe & Koperwas, 2003; Yoder & Symons, 2010). This type of analysis allows us to move away from the descriptive and exploratory models that have been used traditionally in research on mathematical problem solving, and to consider: (a) the study of concurrences and sequential patterns/relations between the different stages and at different *lags* [lag 0 indicates concurrence, lag 1 indicates following, lag -1 indicates preceding, etc.] through conditioned probabilities, transition frequencies and z scores (adjusted normalized residuals) with a significance level of $\alpha < .05$; (b) exploration of the global interrelationships between stages using the polar coordinates technique (Sackett, 1980) in its genuine retroactivity variant (Anguera & Losada, 1999; Gorospe & Anguera, 2000). According to these authors, polar coordinates make it possible to objectively measure how much each category (in our case, each stage of the problem-solving process) encourages or hinders the other categories (stages). Application of this technique is an innovative element in the study of interrelationships between the stages of mathematical problem solving.

In the scientific literature, sequential process analysis has been used in psychological studies. For example, Symons, Hoch, Dahl and McComas (2003) use it to study the relationship between self-destructive behavior and communicative behavior in persons. García (1993) uses sequential process analysis to study the relationship between cognitive and communicative development in autistic children, in both cases the unit of analysis are the individual subjects. However, other studies use sequential process analysis when focusing on social interaction. Thus, Gimeno, Anguera, Berzosa and Ramírez (2006) use it to study communication style relationships in families with teenage children. The use of sequential analysis made it possible to obtain sequence patterns between self-destructive behavior and the communicative behavior that encouraged it, the existence of symmetrical interactions between the family and the adolescent child, and that the communication style is a function of the adolescent's gender.

While sequential analysis, and especially the polar coordinates technique, are relatively recent, and have emerged from psychology, its potential for obtaining patterns and relationships between observables has not gone unnoticed by other scientific disciplines. It is especially interesting to apply this when inquiring into social interaction processes, as is seen in studies from the sphere of Physical Education and Sports. For example, Castellano (2000) and Perea (2008) analyze relationships between the behavior of football players on a team and game actions; Gorospoe and Anguera (2000), Gorospe, Hernández, Anguera, and Martínez (2005) focus on tennis; Cayero (2008) on volleyball. Recently, studies from educational research have begun to use sequential process analysis. Kapur (2011) uses it to obtain patterns of interaction within a group of students that work collaboratively on problem solving in physics, over chat; and Liao, Chen, Cheng and Chan (2012) use it for studying learning behaviors in interaction, as exercised by groups of students in game-based learning environments. The potential of sequential analysis goes beyond studies that involve subjects. For example, Jiménez and Perales (2000) use it to discover underlying didactic patterns or scripts in written texts and illustrations from physics and chemistry text books in secondary education.

As we have seen above, application of the polar coordinates technique, in its genuine retrospectivity variant (Anguera & Losada, 1999; Gorospe & Anguera, 2000), is an innovative element in studies on the interrelationships between stages in mathematical problem solving. This technique allows us to analyze global interrelationships that either stimulate or inhibit observation categories, in this case, the different stages of problem solving.

Suppose that we have an exhaustive, mutually exclusive system of stages (categories), made up of stages {A, B, C, D, E}. This technique would consist of fixing a Stage E, and considering: (a) the forward-looking perspective, examining *z* scores in the number of positive lags, taking E as criterion stage, or given stage, and the rest (A B C D) as target stages; and (b) the retrospective perspective, analogous, but now considering negative lags (Figure 1).

Z scores are obtained by using a standardized residual, proposed by Haberman (1979, cited in Bakeman & Quera, 2011a):

$$Z_{ij} = \frac{(X_{ij} - M_{ij})}{\sqrt{M_{ij} \times (1 - \frac{X_{ij}}{X_{+j}}) \times (1 - \frac{X_{ij}}{X_{i+}})}}$$

Where

- X_{ij} = observed frequency for the cell *ij,* which indicates the number of transition occurrences between stage *i* and stage *j* in the lag being considered. For example, given the sequence ABCAB, for lag+1 (following); $X_{11}=0$; $X_{12}=2$ or $X_{13}=0$ taking rows and columns (A, B, C).

- X_{+j} and X_{j+} observed frequencies by columns and rows.

- X_{++} total observed frequency.

- $M_{ij} = \frac{X_{i+} \times X_{+j}}{X_{++}}$ is the expected frequency for cell *ij.*
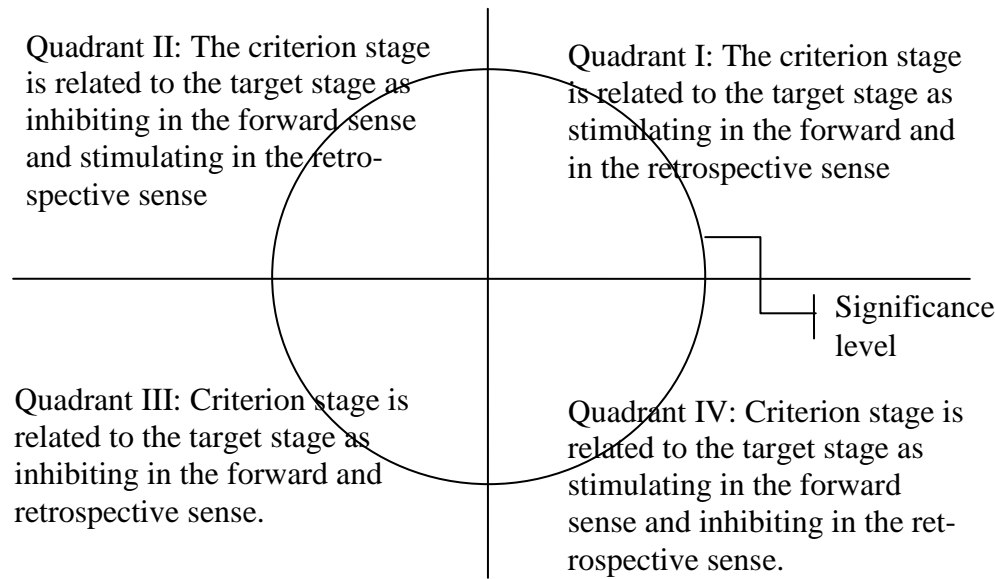
Retrospective perspective   Forward perspective

Target stages: A, B, C, D
Criterion stage: E
z-scores matrix: A matrix of r
rows (nº of negative lags) and k
columns (as many as stages)

Target stages: A, B, C, D
Criterion stage: E
z-scores matrix: A matrix of r
rows (nº of positive lags) and k
columns (as many as stages)

Figure 1. **Adaptation of the polar coordinates model**
(Gorospe & Anguera, 2000, p. 279)

Beginning, then, with *z* scores on the respective lags, we calculate the values of the statistic $Z_{sum} = \frac{\sum z}{\sqrt{n}}$, where *n* is the number lags considered (Sackett, 1980). According to this

author, the Z_{sum} statistic is a powerful data reducer able to provide much information on the relationships between the stages. The genuine retrospectivity variant introduces calculation of the statistic for positive lags (from +1 to +5) in the forward-looking perspective [Z_{sumX}] and negative lags (from -1 to -5) in the retrospective perspective [Z_{sumY}]. Anguera and Losada (1999) and Gorospe and Anguera (2000) consider the pair (Z_{sumX} , Z_{sumY}) to be coordinates that allow vectors to be defined originating from the coordinate center (0,0). The representation of such vectors in a coordinate system provides a map of inter-stage relations, either inhibiting or stimulating. Given that *z* scores are used, the relationships will be significant when the vector modules are greater than 1.96, while the angle with respect to the origin gives us the nature of the relationship (stimulating or inhibiting). We summarize the interpretation of the quadrants of polar coordinates in Figure 2.

Quadrant II: The criterion stage is related to the target stage as inhibiting in the forward sense and stimulating in the retrospective sense

Quadrant I: The criterion stage is related to the target stage as stimulating in the forward and in the retrospective sense

Significance level

Quadrant III: Criterion stage is related to the target stage as inhibiting in the forward and retrospective sense.

Quadrant IV: Criterion stage is related to the target stage as stimulating in the forward sense and inhibiting in the retrospective sense.

Figure 2. **Interpretation of the polar coordinate quadrants**

Objective and hypothesis

In this study we pose two hypotheses in relation to solving mathematical problems in pairs. On one hand, we hypothesize the existence of patterns and causality relations between the different stages in mathematical problem solving, in other words, the existence of sequential relations between stages. On the other hand, it will be possible to specify how each stage either encourages or inhibits the other stages in the mathematical problem solving process, in other words, the existence of global interrelationships between stages. Therefore, our objective is:

The study of sequential relationships and global interrelationships between the different stages in mathematical problem solving.

Method

Participants

Given the innovative, descriptive nature of this study, we analyze the problem-solving process of a pair of students who are solving a mathematical problem. In this case, we use an

optimization problem of distance calculation (Camacho & González, 1998). The pair of students were university students from the Mathematics degree program, both female, ages 22 and 23. The choice of subjects (A1 and A2) is intentional, from the students who volunteered for the experience. The main reasons for choosing these students is that they were accustomed to working in pairs and to collaborative work, to using digital technology, to solving mathematical problems and accustomed to reasoning, explaining and discussing their problem-solving process aloud.

Procedure

The physical space where the subjects worked was a seminar room with elements and technical equipment distributed as shown in Figure 3, i.e., two videocameras, an audio recorder, a 32-inch television, a laptop computer showing the problem to be solved, and a projector. One of the cameras was filming the subjects and was connected to the television. The other camera targeted the images that were projected from the laptop computer via the projector. The television made it possible for an observer to view the subjects' actions close up, while at the same time being hidden from the subjects' field of vision. The recorder was placed close to the pair, capturing even faint verbalizations, in order to minimize information loss.
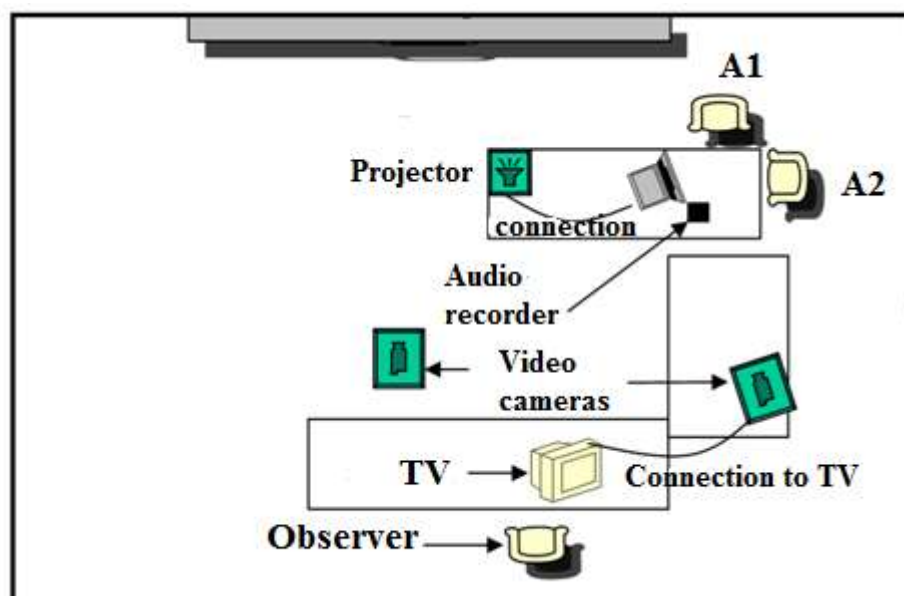

Figure 3. **Work space and technical equipment**

The subjects were allowed 30 minutes to solve the problem. They could use the computer and sheets of paper to solve it, and we invited them to use the think-aloud technique. The infrastructure was designed so as to obtain two video recordings, one audio recording and two written logs *in situ*, one from the observer and another from the pair.

The video recordings were synchronized and superposed using the CromaKey technique and Studio 9.0 video editing software (Avid Technology, 2005). We then obtained two new audiovisual recordings, one with the subjects as the main image and another with the actions-reactions carried out on the laptop as a secondary image. We transcribed the audio using Subtitle Workshop 2.51 (free software, Gluskin, 2004). Occasionally, for intervals with poor audio quality, we used the audio recording. Finally, we synchronized the transcription to the videos by using VirtualDub 1.9.11 (free software, Lee, 2010) and the subtitler.vdf filter (Figure 4). The process we used minimizes a bias from information loss, whether visual or auditory, and provides two registers that capture the subjects' performance while solving the problem.
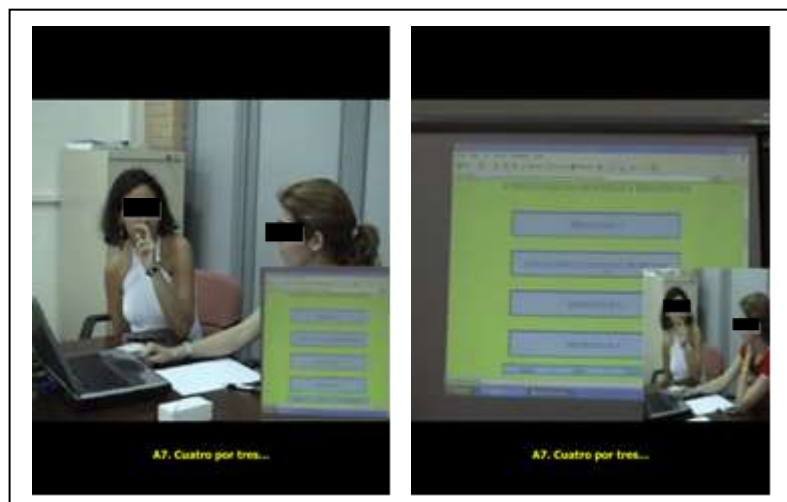

**Figure 4. Superposed videos with audio
transcription as subtitles**

Mathematical problem assigned

The mathematical problem assigned was an optimization problem [Figure 5] integrated into a web design as an i-Activity: "activities in a web format whose objective is to facilitate development of the activity itself and subsequent learning, through computer-student interactivity" (Codina, Cañadas & Castro, 2011, p. 159). The i-Activity consists of a set of 6 pages hyperlinked to instructions, warnings, suggestions and small applets with which the pair may interact. These are designed to guide and facilitate passage from one stage to another and through the problem-solving process in general. The choice of this type of problem, as presented in Figure 5, was motivated by the following: (a) these problems are naturally present in human activity in a large variety of applications and daily situations, thus facilitating activation of previous experience and intuition in the problem solvers (Malaspina, 2007); (b) different possible representations and modeling can be made, opening up a variety of methods, strategies and connections (Villegas et al., 2009); (c) technological advances, especially the development of educational software, facilitate their treatment and solution in the classroom (Camacho & González, 1998; Forbes, 2001; Scher, 1999; Verderber, 1992) and; (d) they are studied in school mathematics (González & Sierra, 2004; Malaspina, 2011; MEC, 2006, 2007a, 2007b; NCTM, 2000).
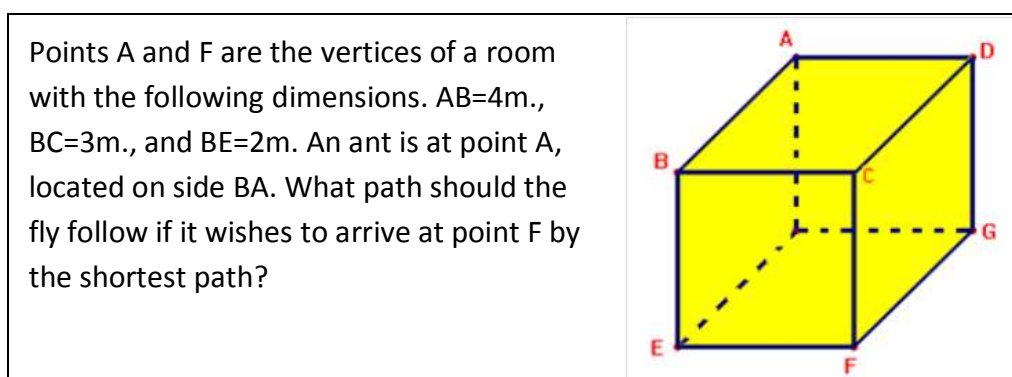

Points A and F are the vertices of a room with the following dimensions. AB=4m., BC=3m., and BE=2m. An ant is at point A, located on side BA. What path should the fly follow if it wishes to arrive at point F by the shortest path?

Figure 5. Problem statement of the i-Activity

Instrument

In order to collect contextualized, comprehensive and objective data from the processes related to the stages of mathematical problem solving, we elaborated an *ad hoc* observation

instrument using a mixed system that combines the field format and category systems (Bakeman & Gottman, 1989). Beginning with a germinal system based on the work by Pólya (1945), Schoenfeld (1985) and Artzt and Armour-Thomas (1990, 1992), we elaborated an initial observation instrument, which was subjected to readjustment and modification based on unsystematic observations from three prior pilot experiences. These readjustments took into account details related to the influence of interactivity with the i-Activity and other details related to pair work. Finally, the observation instrument contained eight comprehensive, mutually exclusive stages (only one stage can be assigned to each observed process); these constitute the categories used for codifying the information:

(a) Reading (Rdg): The subject reads the problem statement and internalizes the conditions and objective of the problem.

(b) Analysis (Ana): In Analysis, the subject makes an attempt to "fully understand a problem, to select an appropriate perspective and reformulate the problem in those terms, and to introduce for consideration whatever principles or mechanisms might be appropriate" (Schoenfeld, 1985, p. 298). The subject "considers domain-specific knowledge that is relevant to the problem" (Artz & Armour-Thomas, 1992, p.172).

(c) Exploration (Exp): The subject uses strategies, understood as procedures or rules that allow him or her to address questions, making use of relationships and concepts from a conceptual structure (Rico, 1997). Ideally, the subject does not have a structured action procedure, and needs to exercise greater control over his or her progress through local and global assessment. In a sense, this is a review of the problem structure in search of relevant information that may be incorporated in an analysis-plan-execution sequence.

(b) Planning (Pla): The subject selects the steps and strategies that may potentially lead to a solution of the problem.

(e) Implementation (Imp): The subject executes actions that were previously structured in Planning.

(f) Verification (Ver): The subject carries out an assessment or local control during the problem-solving process, as well as when performing a global assessment of the solution. This stage, therefore, may cut across the former stages.

(g) Observation and listening (Obs): The subject seems to be listening to and observing his or her partner's work. This stage may occur more often in students who work in groups.

(h) Conversation (Con): The subject holds conversation with his/her partner, apparently unrelated to the problem-solving process, or perhaps makes some comment to the teacher-observer.

Since we consider a stage as "a period of time during which an individual or a problem-solving group is engaged in one large task … or a closely related body of tasks in the service of the same goal" (Schoenfeld, 1985, p. 292), the observation instrument allows us to record the occurrence or non-occurrence of the different stages in a single flow of sequential behavior. Using Atlas-ti 6.0 software (Muhr, 2010), we obtained a set of sequential data (codes) with times, transformed later into a specific software format for GSEQ 5.1.07 sequential analysis (Bakeman & Quera, 2011a). This transformed flow describes the problem-solving process as a function of the occurrence, duration and transitions between the different stages, through codes, in such a way that we may implement statistical analyses available in the GSEQ software.

Reliability and validity

The resulting data flow is continuous, showing intra-session constancy (no exceptional circumstances interrupted the activity during the session). There is no bias from non-observable data, the only bias is technological, and does not exceed 5% of the total observation time, as recommended (Anguera, 1990; Hernández, 1996). The stages are activated by transitions and are only logged at the instant when a transition between occurrences takes place (Bakeman & Gottman, 1989).

Given that session recordings are available, we used the consensus technique in order to ensure intra-observer reliability, reaching an agreement between observers before recording

the data (Anguera, 1990; 2003). As for inter-observer reliability, we compared the codification of 15% of the total session, selected at random and coded by consensus among three observers, with another observer's codification of the same period. Following the recommendation from Bakeman and Quera (2011b), we obtained Kappa indices per time unit with a tolerance of ±2 seconds and per event alignment with a tolerance of 5 and overlap of 80%, the maximum Kappa and the percentage of agreement. Table 1 shows the values obtained, which, according to Landis and Koch (1977, taken from Bakeman & Quera, 2011b), are excellent.

Table 1. **Inter-observer reliability**

Tolerance ±2sec			Tolerance 5, 80% overlap		
	Maximum		Per	Maximum	
Per time unit	Kappa	Agreement	Alignment	Kappa	Agreement
.84-.86	.89-.89	88%-89%	.76	.88	82%

In order to confirm whether there were associations of some kind between the stages, at the different lags, we apply the Chi-square test at a significance level of p<.01. The results show the existence of associations, except perhaps for lag 5 & -5 (Table 2).

Table 2. **Chi-square per lag**

Lags	χ^2	df
lag -5	47.27	49%
lag -4	82.63	49*
lag -3	81.56	49*
lag -2	138.43	49*
lag -1	88.83	41*
lag 1	88.83	41*
lag 2	138.43	49*
lag 3	81.56	49*
lag 4	82.63	49*
lag 5	47.27	49%

* *p<.01*

Design and data analyses

We proposed a data analysis design with two levels of specificity: (a) for each subject (A1, A2), and (b) for the pair (P). For each level, we obtained:

1) a set of basic descriptive statistics for each stage: frequency, duration, simple probability, and average duration (Table 3).

Table 3. **Basic statistics, students A1, A2 and pair**

| Stage	Frequency			Duration			Probability			Mean Duration		
	A1	A2	P	A1	A2	P	A1	A2	P	A1	A2	P
Rdg	19	13	32	204	178	382	.2	.2	.2	11.0	14.0	12.0
Ana	13	17	30	96	157	253	.1	.1	.1	7.4	9.2	8.4
Exp	18	7	25	215	58	273	.2	.1	.1	12.0	8.3	11.0
Pla	1	0	1	13	0	13	.0	.0	.0	13.0	0.0	13.0
Imp	7	10	17	189	82	271	.2	.1	.1	27.0	8.2	16.0
Ver	24	21	45	177	184	361	.2	.2	.2	7.4	8.8	8.0
Con	18	7	25	46	26	72	.0	.0	.0	2.6	3.7	2.9
Obs	52	55	107	207	462	669	.2	.4	.3	4.0	8.4	6.3
Total episodes	152	130	282									
Total seconds				1147	1147	2294						

2) Periods of parallel work, that is, time intervals (t_n, t_m) with $m > n$, where the two students' action is coded in different stages of the problem solving criterion; and collaborative work, or time intervals (t_n, t_m) with $m > n$, where students' action is coded in the same stage of problem solving (Table 4). In other words, during collaborative work, both subjects were focusing their attention on problem-solving aspects that fall into the same stage; while in parallel work, their attention was focused on aspects that are not necessarily complementary, in different stages of the problem-solving process. This study offers information about the periods of parallel and collaborative work that enables a description of how the pair jointly approached solving the problem.

Table 4. **Co-occurrences between stages in A1 and A2**

	A2Rdg		A2Ana		A2Exp		A2Pla		A2Imp		A2Ver		A2Con		A2Obs		Total	
	fr	T	fr	T	fr	T	fr	T	fr	T	fr	T	fr	T	fr	T	fr	T
A1Rdg	*11*	*149*	1	6	1	6	0	0	0	0	2	12	0	0	8	31	23	204
A1Ana	1	1	*7*	*51*	0	0	0	0	1	2	2	27	0	0	6	15	17	96
A1Exp	5	15	3	7	*7*	*22*	0	0	0	0	2	4	2	11	20	156	39	215
A1Pla	0	0	1	4	0	0	*0*	*0*	0	0	0	0	0	0	0	9	1	13
A1Imp	1	2	0	0	0	0	0	0	*9*	*75*	6	48	0	0	14	64	30	189
A1Ver	1	2	7	42	2	8	0	0	0	0	*13*	*66*	0	0	18	59	41	177
A1Con	1	1	0	0	0	0	0	0	0	0	0	0	*4*	*8*	13	37	18	46
A1Obs	3	8	9	47	4	22	0	0	3	5	9	27	4	7	*28*	*91*	60	207
Total	23	178	28	157	14	58	0	0	13	82	34	184	10	26	107	462	229	1147

Note: fr=frequency; T=time measured in seconds. In italics, frequency and time in collaborative work.

3) Graphs of probabilities of between-stage transitions, which report the likelihood of one stage being followed successively by another (we only show probabilities greater than .2), and the set of significant diadic and triadic chains at a level of α < .05. Given that there are insufficient data for using conditioned probabilities, we use tables of p-values obtained with the PSEQ permutations analysis software (Bakeman, Robinson & Quera, 1996). The chains, at the level of α <.05, are formed by considering significant relationships in the same direction (stimulating or inhibiting) at each lag, and taking into account that: (a) the transitive property is not fulfilled, and (b) they end when there is a double, triple, etc. bifurcation, or there are two lags with no significant relationships (max-lag) (Figure 6).
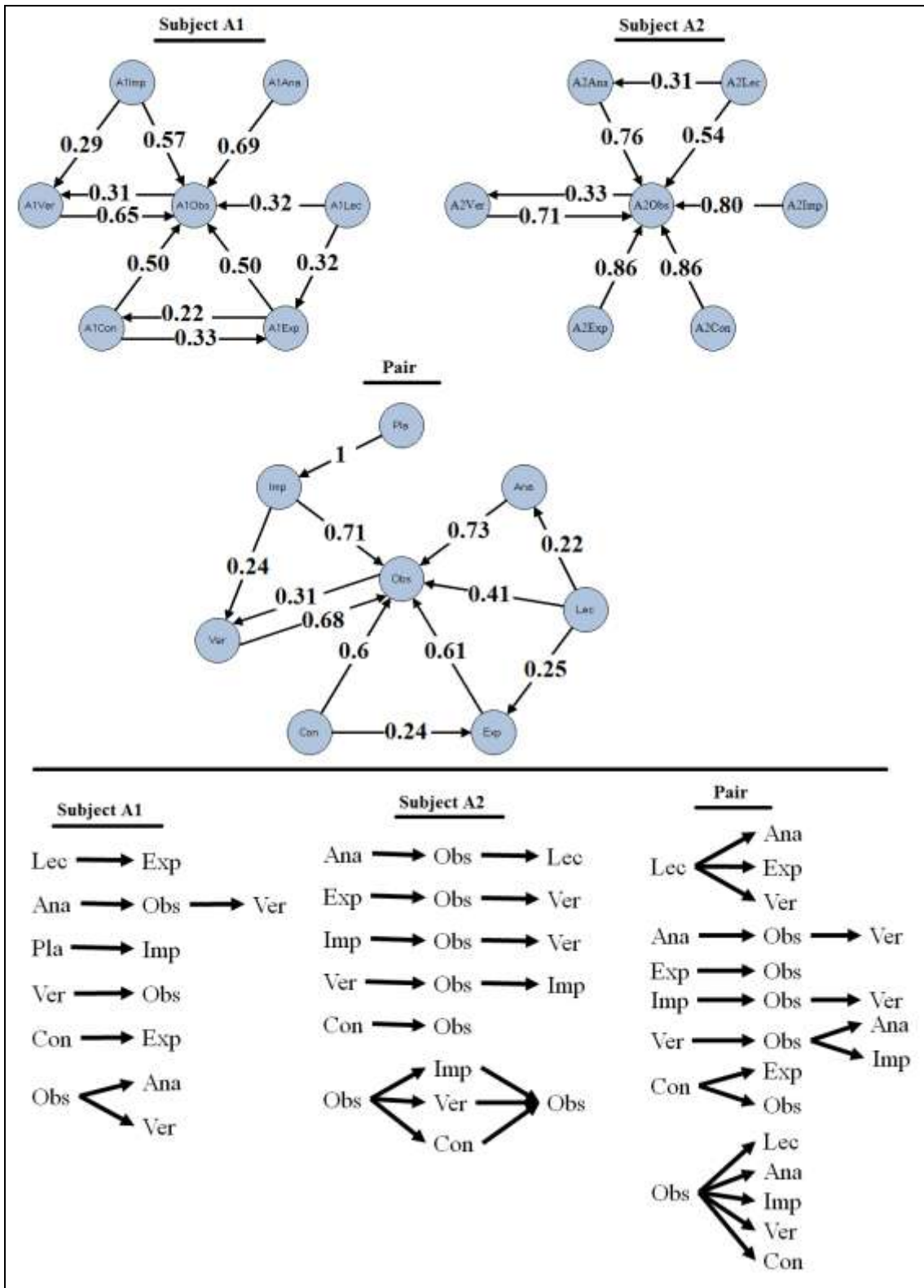
(T.N.: Lec corresponds to Rdg)

Figure 6. **Graphs of transitions and significant chains between stages**

4- Maps of global interrelationships between stages. We applied the polar coordinates technique on *z* scores for lags -5 to 5 and α<.05 (Figure 7a, 7b, 7c).
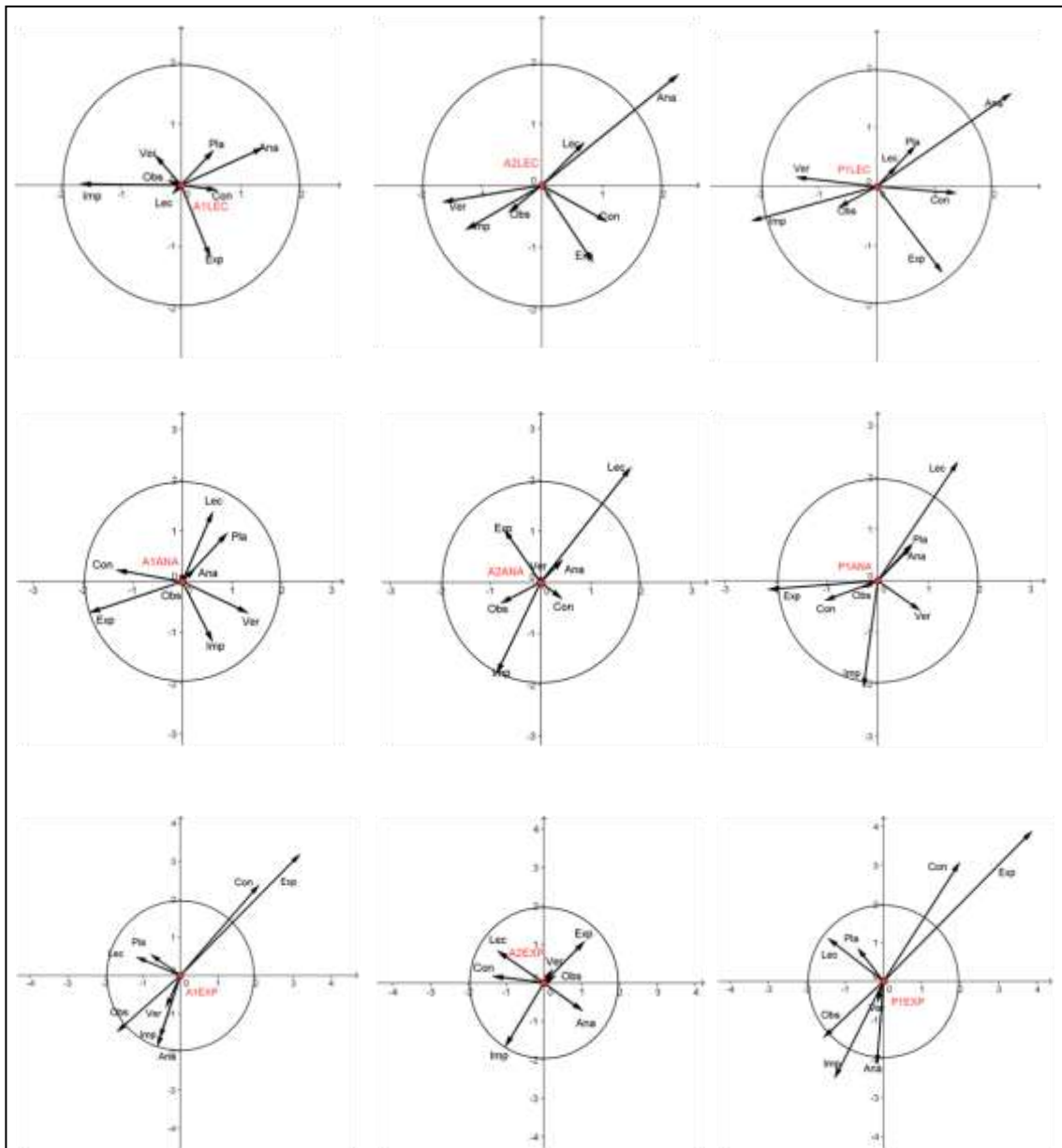
(T.N.: Lec corresponds to Rdg)

***Figure (7a).* Graph of polar coordinates, Reading, Analysis and Exploration stages**

As an example, let us look at how to interpret one of these maps. The relationship map with exploration as criterion behavior for the pair (Figure 7a) informs us that there is a stimulating relationship between the exploration and conversation stages, and self-stimulation of the exploration category in both directions, forward-looking and retrospective. This means

that, beyond chance occurrence, there are periods in the pair where exploration is a close fore-runner of conversation, and vice versa. It also means that exploration periods precede later occurrences of exploration. Similar reasoning follows for the three inhibiting relationships obtained in both directions, namely, between exploration and observation and listening; between exploration and implementation; and between exploration and analysis.
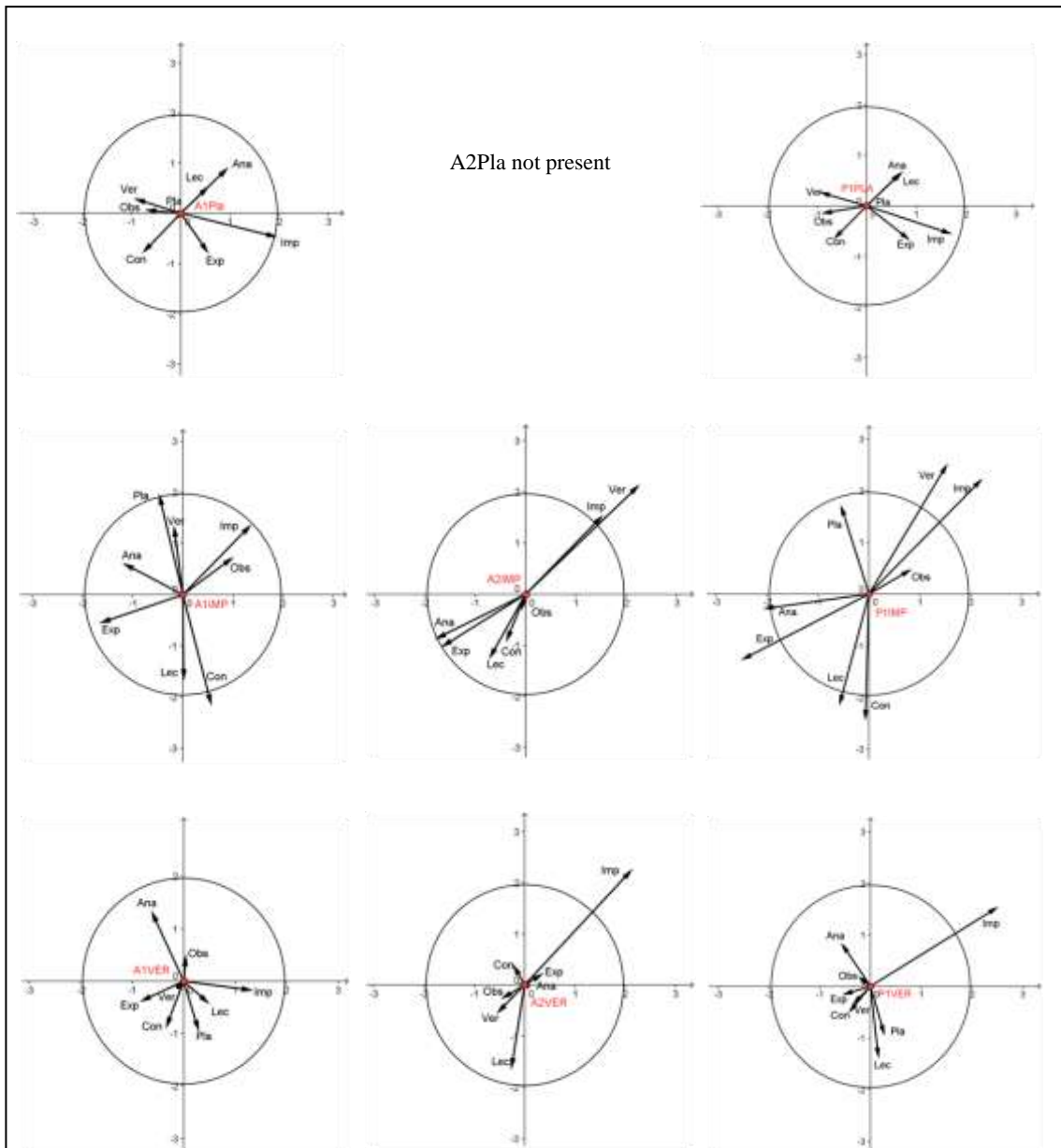

Figure 7b. **Graph of polar coordinates, Planning, Implementation and Verification stages**
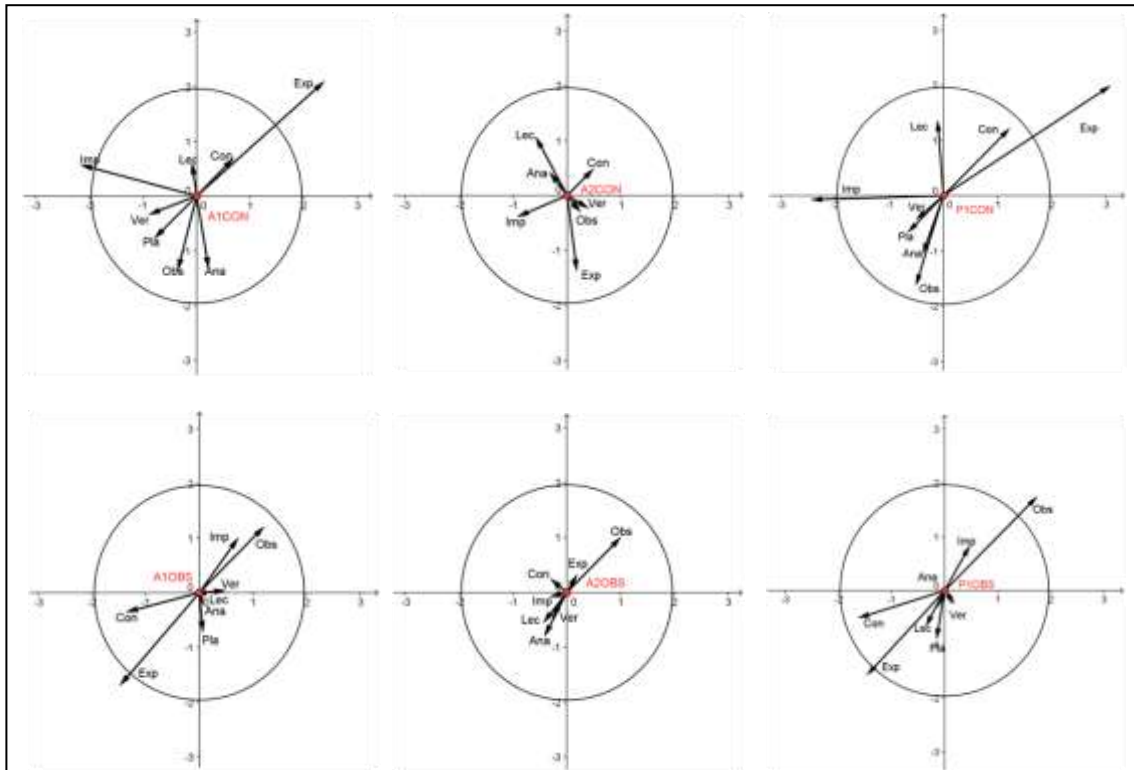
***Figure 7c.* Graph of polar coordinates, Conversation and Observation and Listening stages**

5) Transferences of execution: those situations where at least one partner's immediately preceding stage is seen and recorded in the other partner's stage immediately following, at a given period. The existence of such relationships stems from self-stimulations and self-inhibitions that appear in the polar coordinates analysis. This type of study is an innovative element of inquiry in the field of mathematical problem solving in pairs (Table 5).

Three types of execution transferences have been defined for the two subjects (A and B):

a) Continuity transference in A's execution: if in a period of time (t_n, t_m) subject B is in stage X, then in the period immediately following, subject A is in stage X. At this point, B may either remain in stage X or not, possibly continuing to a different stage Y.

b) Continuity transference in B's execution: the same as above, exchanging B for A and A for B.

c) Mutual transference of execution: in a period of time (t_n, t_m), subject A is in stage X and B is in stage Y, then in the period immediately following, A is in stage Y and B is in stage X (stages are switched).

Table 5. **Transferences of Execution between A1 and A2**

Type		A1	A2	NextA1[a)]	NextA2[b)]	Freq	Total
Mutual		Ana	Obs	Obs	Ana	1	
		Obs	Exp	Exp	Obs	1	
		Obs	Ver	Ver	Obs	1	
		Obs	Con	Con	Obs	1	4
A1->A2	A2 Maintains	Exp	Obs	Obs	Obs	6	
		Con	Obs	Obs	Obs	6	
		Obs	Ver	Ver	Ver	5	
		Ver	Obs	Obs	Obs	3	
		Rdg	Obs	Obs	Obs	2	
		Imp	Obs	Obs	Obs	2	
		Obs	Ana	Ana	Ana	2	
		Obs	Imp	Imp	Imp	2	
		Rdg	Ana	Ana	Ana	1	
		Exp	Con	Con	Con	1	
		Ana	Obs	Obs	Obs	1	
		Ana	Rdg	Rdg	Rdg	1	
		Exp	Ana	Ana	Ana	1	
		Obs	Exp	Exp	Exp	1	
		Obs	Rdg	Rdg	Rdg	1	
		Obs	Con	Con	Con	1	36
A2 Does not maintain		Exp	Con	Con	Obs	1	
		Imp	Obs	Obs	Con	1	
		Ver	Obs	Obs	Con	1	
		Ana	Obs	Obs	Ver	1	
		Imp	Obs	Obs	Ver	1	
		Ver	Obs	Obs	Imp	1	
		Obs	Imp	Imp	Ver	1	7

A2->A1	A1 Maintains	Imp	Obs	Imp	Imp	7	
		Ver	Obs	Ver	Ver	5	
		Exp	Obs	Exp	Exp	4	
		Exp	Rdg	Exp	Exp	2	
		Obs	Ver	Obs	Obs	2	
		Obs	Ana	Obs	Obs	1	
		Ver	Ana	Ver	Ver	1	
		Ana	Obs	Ana	Ana	1	
		Obs	Con	Obs	Obs	1	24
	A1 Does not maintain	Obs	Ana	Ver	Obs	3	
		Obs	Con	Ana	Obs	1	
		Ver	Obs	Imp	Ver	1	5

[a] NextA1= next stage for A1

[b] NextA2= next stage for A2

Results

The pair made up of subjects A1 and A2 is able to solve the problem. The total time invested is 1147 seconds (19 minutes and 17 seconds). A total of 282 time periods or events are recorded for the pair. Of these, 152 correspond to A1 and 130 to A2. The stage of Observation and Listening is strongly present during the problem-solving process, with a frequency of 107 events, total duration of 669s, mean duration of 6.03s and probability of .3. Of these 107 events, 52 correspond to A1 and 55 to A2, while the total duration of Observation and Listening for A2 is twice the total duration for A1 (462 x and 207 x respectively). This suggests that A1 took the leading role in solving the problem. This inference is reinforced by the fact that: (a) for the exploration stage, A1 has double the event frequency and total duration of A2 (18 vs. 7 events; 215s vs. 58s respectively); (b) A1 has more reading processes than A2 (19 vs. 13 events; 204s vs. 178s respectively); (c) A1, though she performs fewer processes of implementation than A2 (7 vs. 10 events), her total time duration in this stage is double that of A2 (182s vs. 82s); (d) only A1 presents an event in the category of Planning, with a duration of 13 s, and (e) A2 does more analysis processes than A1 (17 vs. 13 events; 157s vs. 96s respectively). Finally, while logic suggests to us *a priori* that the verification stage ought to

have been more frequent and with greater duration in A2, the fact that it remains roughly equivalent in the two partners reveals the pair's intent to create and maintain a context of joint problem solving (Roschelle & Teasley, 1995), where both students continually monitor the problem-solving process.

As for concurrences between stages (Table 4), the results show that the students worked collaboratively about 40% of the total time (462s out of 1147s), representing 35% of the total recorded events (79 of 229 events). Of the 462s, 149s correspond to the Reading stage, which is to be expected, especially at the beginning of the problem-solving process; both students invest time reading and understanding the problem statement. Similar reasoning explains the 91s that belong to the Observation and Listening stage, given that when there is any interaction with an i-Activity applet, both students usually observe the computer screen. As for the remaining stages, we would note that during 51s, A1 and A2 are simultaneously in the analysis stage, corresponding respectively to 53% and 32% of the total time in this stage (96s and 157s respectively). During this period, the students try to analyze and choose a perspective that would allow them to successfully go about solving the problem. Likewise, during 75s both students simultaneously perform their own processes within the Implementation stage, corresponding to 40% for A1 and 92% for A2 of the total time in the stage (189s and 82s respectively). Notice that A2 carries on implementation tasks independently only 8% of the time. As for the exploration stage, the students had difficulty in choosing common viewpoints from which to explore the problem, or, each one simply carried out independent explorations. Proof of this is seen in the 22s when A1 and A2 are simultaneously in the exploration stage, representing only 10% of A1's total exploration time (215s) while representing 40% for A2 (58s). Finally, the two partners simultaneously carry out verification tasks during 37% of the total time in this stage (66s out of 177s and 184s).

We now turn our attention to the passage between the different stages, for which we created transition graphs, and graphs of significant diadic and triadic stage transitions (Figure 6). With the transition graphs, whether for subjects or for the pair, the Observation and Listening stage is strongly related to the other stages, with probabilities even approaching .8. Figure 6 also shows how the only probabilities that exceed .2 in either direction are produced between the stages of Verification and Observation and Listening. This suggests that the students, once they have performed some local assessment, observe their partner's reaction, and vice versa, observation produces the appearance of local assessments. In fact, the diadic tran-

sition Obs→Ver is significant at both levels (by subject and in the pair). On the other hand, the probabilities of transition between the categories Conversation and Exploration, along with the significant diadic changes Con→Exp for A1, and Con→Obs for subject A2, again suggest that during the Exploration stage, the pair did not coordinate, or they did not carry out collaborative work.

Elsewhere, probabilities greater than .2 between the Implementation and Verification stages at both levels, and the triadic chain Imp→Obs→Ver, significant for the pair and for A2, indicate that the pair tried to control and minimize possible calculation errors during the Implementation stage, especially so in subject A2. Finally, the fact that the diadic chain Rdg→Exp is significant for A1 and for the pair, and that in all of A2's chains we find the category Observation and Listening, reinforces the reality that A1 is the one who took the lead in the problem-solving process, while A2 exercised the role of monitoring.

The analysis of results to this point has taken into account a maximum of two lags. Use of the polar coordinates technique allows us to go further and report on stimulation or inhibition between stages, whether forward-looking or retrospective and regardless of the lag, in other words, from a global view of the process. The relationship maps that are pictured in Figures 7a, 7b and 7c go beyond the study of next events, and therefore, they allow us to relate the subjects' behaviors further out than the instant or immediately preceding or following. As much as possible, we will try to describe the stimulation relationships on one hand, and the inhibition relations on the other.

Let us begin with stimulation relationships. The Reading stage stimulates the stage of Analysis in A2, both forward and back, while for A1 this relationship does not appear. However, for the pair, Reading and Analysis mutually stimulate each other, both forward and back. In other words, as a pair, after a Reading process, they proceed to analyze the information, and at the same time, when this analysis generates doubts, they return to reading. As a pair, they try to reach full comprehension of the problem, especially driven by A2.

The Exploration and Conversation stages mutually stimulate each other for A1, both forward and back, while for A2 this relationship does not appear. However, again at the pair level, these stages mutually stimulate each other in both directions. In other words, while ex-

ploring the problem, the pair takes little "breaks" that may help build awareness of the problem-solving process. These breaks are promoted by A1 who, as we have seen in previous analyses, is the one leading the problem-solving process. On the other hand, in the pair, exploration is self-stimulating, implying that the two students monitor their progress and look for information that they can incorporate into the problem-solving effort. This fact, in relation to the results shown above, reveals how independent actions involve a coordinated view of actions at the pair level, to a greater or lesser degree, contrary to what might be expected.

The Planning stage stimulates Implementation when looking forward, and inhibits it in the retrospective perspective, only for A1. This is logical, since only A1 establishes a problem-solving plan that is afterward carried out; no stimulation is found in the retrospective sense, as A1 does not show any doubts about her plan.

The Implementation and Verification stages mutually stimulate each other for A2, both forward and back, while for A1 this relationship does not appear. But at the pair level, the Implementation and Verification stages mutually stimulate each other in both directions. In other words, at the pair level, execution of the problem-solving plan is monitored, especially by subject A2. Moreover, the fact that the Implementation stage stimulates Conversation in the retrospective sense for A1, informs us that A1 takes small conversation breaks during execution of the problem-solving process. This in turn explains the self-stimulation relationship of Implementation, for both A2 and for the pair.

Finally, the fact that the Observation and Listening stage self-stimulates in both directions at the pair level is consistent with the functioning of a collaborative, pair-based problem-solving process, because it indicates that both subjects constantly observe their partner's actions.

As for inhibition relations, the fact that the Reading and Implementation stages are mutually inhibiting at both levels is consistent with a problem-solving process where there is a well-established plan to solve the problem; if there were no such inhibition, it would indicate that there was only a superficial reading of the problem, and that actions were being executed without a pre-established, coherent plan.

The stages of Analysis and Exploration inhibit each other in both directions for A1; this suggests that this subject has reached an understanding of the problem statement that will later allow her to design a problem-solving plan that she feels confident about. For her part, A2 adopts this plan, since at the pair level we also find this inhibition relationship. On the other hand, the existence of mutual inhibition between the Analysis and Implementation stages for A2 and for the pair is consistent with a problem-solving process where there is a structured problem-solving plan prior to implementation. All of this in turn explains the mutual inhibition relations in both directions between the stages Exploration and Observation and Listening for A1 and for the pair, and between the stages Exploration and Implementation for A2 and for the pair. Finally, the Implementation stage inhibits the Planning and Conversation stages in the forward-looking direction in A1, showing that this subject feels secure with her plan to solve the problem.

Finally, with regard to transferences of execution (Table 5), results show that all of them involve the Observation and Listening stage, indicating the existence of two-way continuity in the stages involved. In addition, of the 43 transferences of execution for subject A1, A2 maintains her stage in 36 of them. This informs us that there are 36 time periods of collaborative work that are motivated by the action of A2, in which A1 tries to follow or maintain the proposals-suggestions of A2. Worthy of note are 5 occasions that involve the Verification stage (A1 also validates the observations made by A2) and 4 occasions of the Analysis stage (A1 continues the actions begun by A2). With regard to the transferences of A1 toward A2, where A2 does not maintain her category, most of them involve the Observation category.

Inversely, 29 transferences of execution are identified for subject A2. Of these, A1 maintains her stage during 24 periods, that is, 24 more periods of collaborative work are produced. Unlike the inverse transferences described above, in this case, the transferences of A2 toward A1 do not mainly involve the Observation and Listening stage, but most frequently involve the stages of Implementation, Verification or Exploration. This means that A2 collaborates in implementing the problem-solving plan proposed by A1, that A2 needs to explore the problem, unlike A1, and that A2 acts as overseer of the problem-solving process.

Discussion and Conclusions

Research on the stages of mathematical problem solving has traditionally focused on describing the passage between stages or analyzing the components of the different stages. The present study reveals that the use of sequential analysis and the polar coordinates technique makes it possible to describe the passage from one stage to the next, to obtain transition patterns and causality relations between stages, as well as to obtain significant, global interrelationships of stimulation or inhibition between stages, occurring during the process of solving a mathematical problem. Our study confirms our hypotheses, and is original in its method and technique, opening new possibilities for research not only in the stages of problem solving, but in the field of Mathematics Education itself and by extension, in any scholastic activity where there may be sequential causality relationships.

In order to be able to explore the possibilities of sequential analysis, we elaborated an observation instrument that incorporates the Conversation stage as a new element to consider in the study of mathematical problem solving in pairs. As we have stated, this stage enabled us to establish the influence of these small periods of conversation in the problem-solving process of pairs, especially in the stages of exploration and implementation. The conversation stage, for the pair of solvers, acts as a bridge between problem exploration and implementation processes.

Next, we carried out an analysis with different levels of specificity. On one hand, we took the set of actions manifested by both subjects (pair level) as our unit of analysis; on the other, we considered the actions of each subject individually (subject level). These levels of specificity allowed us to obtain results that describe the individuals' influences in the actions of the pair; for example, the existence of the triadic transition Imp→Obs→Ver, found in the pair and in A2, shows that A2 is the one who does the most monitoring and who minimizes the implementation process in the pair. The possibility of establishing different units of analysis implies that, if there are a sufficient number of subjects of observation (pairs), one could consider the generic macro unit of the set of actions manifested by all the subjects. This macro unit would then make possible global results that may be considered to a certain extent subject-independent, opening a new path of interpretation in mathematical problem solving.

By obtaining significant interrelationships of stimulation and inhibition along with diadic and triadic transitions between stages, we described subject and pair behavior during the problem-solving process. The diadic and triadic transitions informed us about certain regularities in the subjects' manner of proceeding, while the interrelationships of stimulation and inhibition informed us of the nature and intensity of the relationships during the problem-solving process.

In this way, if this analysis procedure were reiterated for a diversity of situations, contexts and mathematical problems, for a subject (or groups of subjects), the teacher or researcher would have the possibility of identifying those aspects or stages of problem solving where certain difficulties appear, and thus have new information available for proposing specific measures for learning mathematical problem solving. Finally, sequential analysis techniques have proven to be powerful tools for describing the collaborative work exercised by the subjects during problem solving. In particular, the study of the subjects' concurrences between stages enabled us to identify the transferences of execution, and with these, a more in-depth identification of the interrelationships that appear between the subjects during collaborative work.

References

Anguera, M. T. (1990). Metodología observacional [Observational methodology]. In J. Arnau, M. T. Anguera & J. Gómez (Eds.), *Metodología de la investigación en ciencias del comportamiento* (pp. 125-236). Murcia, Spain: Universidad de Murcia.

Anguera, M. T. (2003). La observación [Observation]. In C. Moreno (Ed.), *Evaluación psicológica. Concepto, proceso y aplicación en las áreas del desarrollo y de la inteligencia* (pp. 271-308). Madrid, Spain: Sánz y Tórres.

Anguera, M. T. (2010). Posibilidades y relevancia de la observación sistemática por el profesional de la psicología [Possibilities and relevance of systematic observation by the psychology professional]. *Papeles del psicólogo, 31*(1), 122-130.

Anguera, M. T. & Losada, J. L. (1999). Reducción de datos en marcos de conducta mediante la técnica de coordenadas polares [Data reduction in behavioral frameworks using the polar coordinates technique]. In M. T. Anguera (Ed.), *Observación de conducta interactiva en contextos naturales: aplicaciones* (pp. 163-188). Barcelona, Spain: Universidad de Barcelona.

Armengol, L. (2007). Los protocolos de pensamiento en voz alta como instrumento para analizar el proceso de escritura [Think-aloud protocols as instrument to analyze the writing process]. *Revista Española de Lingüística Aplicada, 20,* 27-36.

Artzt, A. F., & Armour-Thomas, E. (1990, Abril). *Protocol analysis of group problem solving in mathematics: a cognitive-metacognitive framework for assessment*. Paper presented at the Annual Meeting of the American Educational Research Association, Boston.

Artzt, A. F., & Armour-Thomas, E. (1992). Development of a cognitive-metacognitive framework for protocol analysis of mathematical problem solving in small groups. *Cognition and Instruction, 9*(2), 137-175. doi:10.1207/s1532690xci0902

Avid Technology (2005). Pinnacle Studio (Version 9.0) [Software].

Bakeman, R., & Gottman, J. M. (1989). *Observación de la interacción: Introducción al análisis secuencial.* [Observing interaction: An introduction to sequential analysis]. Madrid, Spain: Morata.

Bakeman, R., & Quera, V. (1996). *Análisis de la interacción* [Interaction analysis]. Madrid, Spain: Ra-Ma.

Bakeman, R., Robinson, B. F., & Quera, V. (1996). Testing sequential association: estimating exact p values using sampled permutations. *Psychological methods*, *1*(1), 4-15.

Bakeman, R., & Quera, V. (2011a). GSEQ (Version 5.1) [Software]. Barcelona. Retrieved from http://www2.gsu.edu/~psyrab/gseq/index.html

Bakeman, R., & Quera, V. (2011b). *Sequential analysis and observational methods for the behavioral sciences*. New York: Cambridge University Press.

Camacho, M., & González, A. (1998). Una aproximación a los problemas de optimización en libros de texto de bachillerato y su resolución con la TI-92 [An approach to optimization problems in high school textbooks and their resolution using TI-92]. *Aula, 10*, 137-152.

Castellano, J. (2000). *Observación y análisis de la acción de juego en el Fútbol* [Observation and analysis of game action in Football]. (Unpublished doctoral dissertation). Vitoria-Gasteiz: Universidad del País Vasco.

Castro, E. (2008). Resolución de problemas: Ideas, tendencias e influencias en España [Problem solving: Ideas, tendencies and influences in Spain]. In G. Luengo (ed.), *Investigación en Educación Matemática XII, actas del XII simposio de la Sociedad Española de Investigación en Educación Matemática* (pp. ¶). Badajoz, Spain: Sociedad Española de Investigación en Educación Matemática.

Cayero, R. (2008). Observación de la acción del juego del voleibol: análisis secuencial y variabilidad [Observation of game action in volleyball: sequential analysis and variability]. (Unpublished doctoral dissertation). Málaga: Universidad de Málaga.

Codina, A., Cañadas, M. C., & Castro, E. (2010). Diseño de una e-actividad orientada a la resolución de problemas de matemáticas [An e-activity design for mathematical problem solving]. In F. Albuquerque, G. Lobato, J. F. De Matos, I. Chagas & E. Cruz (eds.), *I Encontro internacional tic e educaÇão. InovaÇão curricular com TIC* (pp. 1-7). Lisbon, Portugal: Instituto de EducaÇao da Universidade de Lisboa.

Codina, A., Castro, E., & Cañadas, M. C. (2011). Un sistema de categorías para el análisis de la interactividad en una i-actividad de resolución de problemas [An category system to analyze the interactivity in a problem solving i-activity]. In J. L. Lupiáñez, M. C. Cañadas, M. Molina, M. Palarea, M. & A. Maz (eds.), *Investigaciones en pensamiento numérico y algebraico e historia de la matemática en educación matemática* (pp. ¶). Granada, Spain: Departamento de Didáctica de la Matemática. Universidad de Granada.

Dewey, J. (1916). *Essays in experimental logic*. Chicago: University of Chicago Press

Dix, A., Finlay, J. E., Abowd, G. D., & Beale, R. (2004). *Human-computer interaction*. Harlow: Pearson Education Limited.

Erbas, A. K., & Okur, S. (2012). Researching students' strategies, episodes and metacognitions in mathematical problem solving. *Quality & Quantity, 46*(1), 89-102. doi:10.1007/s11135-010-9329-5

Ericsson, K. A., & Simon, H. A. (1980). Verbal report as data. *Psychological Review, 87*(1), 215-251.

Ericsson, K. A., & Simon, H. A. (1993). *Protocol analysis. Verbal reports as data.* Cambridge: MIT Press.

Forbes, I. (2001). The circle of understanding. Optimisation with the TI-92. *Micromath, 17*(3), 15-18.

García, J. N. (1993). Aplicación del análisis secuencial en la observación del desarrollo de los niños autistas [Sequential analysis applied to observing the development of autistic children]. In Asociación Española de Profesionales del Autismo (Ed.), *Actas del VII congreso nacional de autismo, "el autismo 50 años después de kanner"* (pp. 163-181). Salamanca, Spain: Amarú.

Gimeno, A., Anguera, M. T., Berzosa, A., Ramírez, L. (2006). Detección de patrones interactivos en la comunicación de familias con hijos adolescentes [Interactive patterns detection in family communication with adolescents]. *Psicothema*, *18*(4), 785-790.

Gluskin, R. (2004). *Subtitle workshop* (Version 2.51) [Software]. *URUSoft*. Retrieved from http://www.urusoft.net/products.php?cat=sw

González, M. T., & Sierra, M. (2004). Metodología de análisis de libros de texto de matemáticas. Los puntos críticos en la enseñanza secundaria en España durante el siglo XX [Methodology for analyzing math textbooks. The critical points in secondary education in Spain during the twentieth century]. *Enseñanza de las Ciencias, 22*(3), 389-408.

Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics, 49*(2), 193-223. doi:10.1023/A:1016209010120

Gorospe, G., & Anguera, M. T. (2000). Modificación de la técnica clásica de coordenadas polares mediante un desarrollo distinto de la retrospectividad: Aplicación al tenis [Retrospectivity in polar coordinates analysis: application to tennis]. *Psicothema, 2*(2), 279-282.

Gorospe, G., Hernández, A., Anguera, M. T., & Martínez, R. (2005). Desarrollo y optimización de una herramienta observacional en el tenis de individuales [Development and optimization of an observational tool for singles tennis]. *Psicothema, 17*(1), 123-127.

Hernández, A. (1996). *Observación y análisis de patrones de juego en deportes sociomotores* [Observation and patterns analysis of play in sociomotor sports]. Santiago de Compostela, Spain: Servicio de publicaciones de la Universidad de Santiago de Compostela.

Jiménez, J. D. & Perales, F. J. (2000). Aplicación del análisis secuencial al estudio de texto escrito e ilustraciones de los libros de física y química en la ESO [Sequential analysis applied to the study of written text and illustrations in physics and chemistry textbooks]. *Enseñanza de las Ciencias, 19*(1), 3-19.

Kilpatrick, J. (1992). A history of research in mathematics education. En D. Grouws (ed.), *Handbook of research on mathematics teaching and learning* (pp. 3-38). New York: MacMillan.

Kapur, M. (2011). Temporality matters: Advancing a method for analyzing problem-solving processes in a computer-supported collaborative environment. *International Journal of Computer-Supported Collaborative Learning*, *6*(1), 39-56. doi: 10.1007/s11412-011-9109-9

Kursat, A., & Okur, S. (2012). Researching students' strategies, episodes, and metacognitions in mathematical problem solving. *Quality & Quantity*, *46*(1), 89-102, doi: 10.1007/s11135-010-9329-5

Lee, A. (2010). *VirtualDub* (Version 1.9.11) [Software]. Retrieved from http://www.virtualdub.org/

Lee, H. S. & Hollebrands, K. F. (2006). Students' use of technological features while solving a mathematics problem. *The Journal of Mathematical Behavior, 25*(3), 252-266. doi: 10.1016/j.jmathb.2006.09.005

Lester, F. K. (1985). Methodological considerations in research on mathematical problem solving instruction. In E. Silver (ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 41-69). Hillsdale, NJ: Lawrence Erlbaum.

Liao, C., Chen, Z. H., Cheng, H., & Chan, T. W. (2012). Unfolding learning behaviors: a sequential analysis approach in a game-based learning environment. *Research and Practice in Technology Enhanced Learning*, *7*(1), 25-44.

Malaspina, U. (2007). Intuición, rigor y resolución de problemas de optimización [Intuition, rigor and solving optimization problems]. *Revista Latinoamericana de Investigación en Educación Matemática, 10*(3), 365-399.

Malaspina, U. (2011). Resolución de problemas y estímulo del pensamiento optimizador en la educación básica [Problem solving and encouragement of optimizing thought in basic education]. In A. Ruiz (Ed.) *XIII Conferencia Interamericana de Educación Matemática* (pp. ¶). Recife, Brazil: Comisión Interamericana de Educação Matemática. Retrieved from
http://www.cimm.ucr.ac.cr/ocs/index.php/xiii_ciaem/xiii_ciaem/paper/viewFile/2818/1142

Ministerio de Educación y Ciencia [MEC] (2006). Real Decreto 1513/2006, de 7 de diciembre, por el que se establecen las enseñanzas mínimas de la Educación primaria. *Boletín Oficial del Estado,* n. 293, 43053-43102.

Ministerio de Educación y Ciencia [MEC] (2007a). Real Decreto 1631/2006, de 29 de diciembre, por el que se establecen las enseñanzas mínimas de la Educación Secundaria Obligatoria. *Boletín Oficial del Estado,* n. 5, 677-773.

Ministerio de Educación y Ciencia [MEC] (2007b). Real Decreto 1467/2007, de 2 de noviembre, por el que se establecen la estructura del bachillerato y se fijan sus enseñanzas mínimas. *Boletín Oficial del Estado,* n. 266, 45381-45477.

Mgombelo, J., & Bateau, C. (2010). Prospective secondary teachers repositioning by designing, implementing and testing mathematics learning objects: A conceptual framework. *International Journal of Mathematical Education in Science and Technology, 8*(40), 1051-1068. doi: 10.1080/00207390903236459

Muhr, T. (2010). In Scientific Software Development GmbH (Ed.), *Atlas-Ti 6.0.* Berlin:

National Council of Teacher of Mathematics [NCTM] (2000). *Principles and Standards for school mathematics*. Reston: NCTM.

Organisation for Economic Co-operation and Development [OECD] (2010). PISA 2012. Field Trial Problem Solving Framework. Retrieved from http://www.oecd.org/pisa/pisaproducts/46962005.pdf

Organisation for Economic Co-operation and Development [OECD] (2014). *PISA 2015. Draft Collaborative Problem Solving Framework.* Retrieved from http://www.oecd.org/pisa/pisaproducts/pisa2015draftframeworks.htm

Perea, A. (2008). Análisis de acciones colectivas en el fútbol de rendimiento [Collective actions analysis in football performance]. (Unpublished doctoral dissertation). Vitoria-Gasteiz: Universidad del País Vasco.

Poincaré, H. (1908). *Science at method* [Science and method]. Paris, France: Flammarion. From http://www.ac-nancy-metz.fr/enseign/philo/textesph/Scienceetmethode.pdf

Pólya, G. (1945). *How to solve it* [Zugazagoitia, J. (1978), trans. *Cómo plantear y resolver problemas. México, D.F.: Trillas*]. Princeton: Princeton University Press.

Riba, C. (1993). Proceso de categorización [Categorization process]. In M. T. Anguera (Ed.), *Metodología observacional en la investigación psicológica. vol.1 fundamentación* (pp. 115-168). Barcelona, Spain: Promociones y publicaciones universitarias.

Rico, L. (1997). Consideraciones sobre el currículo de matemáticas para educación secundaria [Considerations about mathematics curriculum in secondary education]. In L. Rico (Coord.), *La Educación Matemática en la enseñanza secundaria* (pp. 15-59). Barcelona, Spain: Horsori.

Sackett, G. P. (1980). Lag sequential analysis as a data reduction technique in social interaction research. In D. B. Sawin, R. C. Hawkins, L. O. Walker & J. H. Penticuff (Eds.), *Exceptional infant. Psychosocial risks in infant-enviroment transactions* (pp. 300-340). New York: Brunner/Mazel.

Scher, D. (1999). Problem solving and proof in the age of dynamic geometry. *Micromath, 15*(1), 24-29.

Schoenfeld, A. (1985). *Mathematical problem solving*. Orlando: Academic Press.

Sharpe, T., y Koperwas, J. (2003). *Behavior and sequential analyses.* London: Sage Publications.

Symons, F. J., Hoch, J., Dahl, N. A., & McComas, J. J. (2003). Sequential and matching analyses of self-injurious behavior: A case of overmatching in the natural environment. *Journal of Applied Behavior Analysis, 36*(2), 267-270. doi: 10.1901/jaba.2003.36-267

Törner, G., Reiss, K. M., & Schoenfeld, A. (2007). Problem solving around the world: summing up the state of the art. *ZDM The International Journal on Mathematics Education, 39*(5-6), 353-353. doi:10.1007/s11858-007-0053-0

Verderber, N. (1992). Tin cans revisited. *The Mathematics Teacher, 85*(5), 346-349.

Villegas, J. L., Castro, E., & Gutiérrez, J. (2009). Representaciones en resolución de problemas: un estudio de caso con problemas de optimización [Representation in problem solving: a case study with optimization problems]. *Electronic Journal of Research in Educational Psychology, 7*(17), 279-308. Retrieved from http://www.investigacion-psicopedagogica.org/revista/articulos/17/espannol/Art_17_297.pdf

Yerushalmy, M. (2000). Problem solving strategies and mathematical resources: a longitudinal view on problem solving in a function based approach to algebra. *Educational Studies in Mathematics, 43*(2), 125-147, doi: 10.1023/A:1017566031373

Yimer, A., & Ellerton, N. F. (2006). Cognitive and metacognitive aspects of mathematical problem solving: An emerging model. In P. Grootenboer, R. Zevenbergen y M. Chinnappan (eds.), *Identities, cultures, and learning spaces* (pp. 575-582). Adelaida: Mathematics Education Research Group of Australasia.

Yimer, A., & Ellerton, N. F. (2010). A five-phase model for mathematical problem solving: Identifying synergies in pre-service teachers' metacognitive and cognitive actions. *ZDM The International Journal on Mathematics Education*, *2*(42), 245-261, doi: 10.1007/s11858-009-0223-3

Yoder, P., & Symons, F. (2010). *Observational measurement of behavior*. New York: Springer.