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Editorial EOS

Structural and semiotic diversity in the didactic transition from the natural-number domain to the integers: a study of understanding in the field of additive relativity

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Abstract

Introduction. In the present-day didactic process of transition from natural numbers to integers, a certain kind of measure is involved which is neither easy nor appropriate to integrate into the familiar numerical systems. We refer to measures and comparisons for which we need a third numerical system – what we will call "relative natural" numbers; this numerical system presents five structural differences from the sytems of natural numbers and of integers. In this paper we present part of a study whose purpose was to make evident that such differences are also cognitive.

Method. Qualitative meta-analysis and didactic analysis are used for theoretical study and for analysis of responses, for the descriptive study and correspondence analysis of responses to structured questionnaires.

Results. The algebraic structures and the epistemology and phenomenology of relative natural numbers and of integers are different. Subjects distinguish between the two types of numbers by responding differently to situations which pertain to one or the other type of numerical structure.

Discussion. Results confirm our conjectures, and call for a new organization of the additive numerical field, in which additive relativity and relative natural numbers would be considered, as is the case with proportionality and fractions in the conceptual field of multiplication.

Key words: Integers; Directed numbers; Additive conceptual field; Numerical thinking; Understanding arithmetic.

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Introduction

For over three millennia of the History of Mathematics (Gericke, 1996), in what has been called *"a surprising historical process"* (Glaeser, 1981), different types of numerical concepts in the additive field (Schubring, 1986) have coexisted informally (Clawson, 1994). Of these, what we know today as positive and negative integers (Euler, 1840) are the result of a formalization process culminating in the application of the principle of permanence of formal laws (Pycior, 1981). This process, partially justified from the logical point of view (Russell, 1973)[1], and well established from the mathematical point of view (Feferman, 1989), nonetheless does not resolve numerous problems that appear in Mathematics Education (González et al., 1990). The fundamental problem stems from a type of situation, which we will call *additive relative situations*[2]—in the field of application phenomena of additive and ordinal structures of natural numbers and integers—which involve certain measures whose nature and functioning are compatible with an unusual numerical structure which we have called the system of *relative natural numbers*. We refer to notions typical of the additive conceptual field which have been informally present throughout the historical development of negative numbers (Glaeser, 1981; González et al., 1990) and which have been eclipsed by the formal construction of integers (González, 1995, pp. 195 - 232)[3].

The problem we are addressing is based on a verifiable fact (Maz, 2000): the numerical structure which some mathematicians prior to the 19th century (and even later) have used to begin the introduction of the set of integers is not the totally ordered additive group. The situations that are chosen introduce measures and relationships that do not satisfy the conditions required for exemplifying such numbers. Consider an example:

> *"This sign + is called positive; and this one – negative. From this we gather that whenever a quantity has the + before it, or it does not have the – sign before, it is called a positive quantity; but whenever a quantity has the – sign before it, it is*

[1]The author, based on a logical analysis of the order, attributes the difference of direction and the difference of sign to the generic relationship which exists between an asymmetric relationship and its reciprocal, by identifying negative and positive integers with the successive powers of both relationships. We say that the justification is partial because we believe that Russell's arguments have to do with the existence of relative natural numbers and not of integers.

[2]Those which appear in arithmetic problems that involve elemental, discrete measurements or quantities that operate between themselves, under one of the three types of additive relationships - combination, change or comparison – where at least one of these is a relative natural quantity/measure. Relative natural quantities are natural quantities which are given dichotomous adjectives (positive / negative, more / less) (González, 1995, pp. 261-263).

called a negative quantity, and when this negative quantity is not preceded by another positive quantity, it is called a false quantity, or less than nothing. It is easy to form ideas about these quantities which are less than nothing. Suppose that there are three leagues from C to B and from C to A there are two.

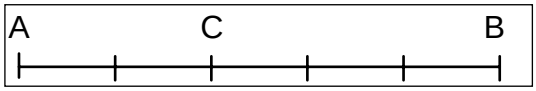


Figure 0. **Illustration from Ulloa's text**

If a traveler is at C with the intention of getting to B, and in effect leaves C and arrives at B, it is truthful to say that he progressed and that his progress is greater than nothing and that the measure of this progress is three leagues. If, despite his intent, he finds himself detained at C, he has made no progress, or, his progress is equal to nothing. If instead of walking from C to B, he arrives at A, in common language we would say he went backward, and in order to express that he did the opposite of what he intended to do, it may be said that he progressed less than nothing, and that his progress is -2 leagues, since here two leagues are less than nothing." (Ulloa, 1706; pp. 20-22).

The example chosen does not fulfill the formal conditions of the structure of the totally ordered additive group of integers. First, the model implies the existence of two totally ordered sets, each with its first element: the one which corresponds to quantities/measurements taken as "positive progress", which begins at "no progress", and one for quantities/measurements taken as retreats or "negative progress", which the author characterizes as "progressing less than nothing". The latter must begin at "no retreat"; therefore, there arises a logical need to consider two zeroes instead of only one.

On the other hand, and even though the author does not mention it, common use indicates that retreating or going backward 7 leagues is *more than* going backward only 3 leagues, fitting into an order that is inversely related to the usual order for negative integers. This indicates that for each of the two subsets ("progress" and "retreat"), the same natural order is maintained. Similarly, comparing a specific amount of progress with a specific amount of

[3]Doctoral thesis read at the University of Granada, Department of Mathematics Didactics.

retreat seems forced and arbitrary, since (1) we are dealing with two different types of actions and of quantities/measures[4], and (2) either of the two directions that we might adopt for the order is equally valid, something which points toward independence between the two subsets, unlike what we see when considering the whole structure. Therefore, the structure being exemplified is not that of a total order with no first or last element, but that of a partial order of two unconnected subsets (the "positive" quantities/measures and the "negative" ones[5]), each of which is totally ordered and has its first element ("no progress" and "no retreat", respectively) (González, 1995; pp. 224). Figure 1 represents two diagrams which illustrate the existing order and the situation and characteristics of the zeroes in the three numeric structures.

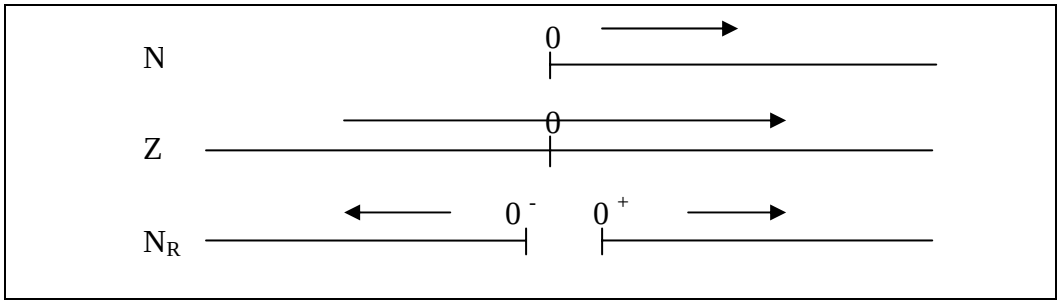


Figure 1. **A representation of the three types of numbers**

On the other hand, the operational structure of the numbers presented in the example is not that of the additive group, either. A detailed analysis yields a structure identifiable as a *non-commutative additive semi-group, without a neutral element and with two null elements which are not permutable between themselves,* with the usual operations of adding and subtracting between natural numbers for values of the same "sign" and a law based on certain rules of "annulment-compensation"[6] for values with opposite signs. Consequently, we can affirm that the set being exemplified (or studied) is that of relative natural numbers and not that of integers. The confusion becomes greater when, in addition, elements from both structures are mixed together, and especially when, lacking a specific symbolization for relative

[4]Known as "directed" or "adjectified" (Bell, 1982).
[5]The quotation marks here indicate that these terms are not being used with the same meanings as in the case of numbers with a sign, but rather to describe the duality inherent to the relative natural structure.
[6]Additive operation composed of a comparison, by which the order of compared values is determined, and a subtraction between natural numbers (whichever of the two is possible in N). Two parts, therefore, are presented: natural subtraction and its direction.

natural numbers, the integer notation is used for the two types of numbers, leading to their being identified with the resulting dysfunctionalities.

The problem at hand can be addressed from different approaches (epistemological, historical or mathematical, for example), but it is not our intent to address considerations other than what is exclusively didactic. Our interest, therefore, focuses on phenomena which take place when one tries to transmit, teach, explain or "make understood" formal mathematical notions using the classical didactic recourse of concrete, intuitive and familiar examples; a necessary and suitable recourse in many cases, but problematic in others, where the examples themselves produce misalignments, even of a structural nature, which must be identified and properly handled. Along these lines, then, we have developed a study which aims to identify, analyze and organize—through epistemological, cognitive, phenomenological and didactic considerations—the part of the additive conceptual field (Vergnaud, 1993, pp. 97-) which we have called the "conceptual field of the relative natural numbers".

The present article presents part of an empirical study carried out in order to reveal that differences between the different metric and numeric notions involved are not only structural or phenomenological, but also cognitive, that is, that subjects handle and assign different meanings to situations which we have usually taken as being the same as integers, and given the same treatment. This differentiation gives rise to the existence of and the didactic need for the new conceptual field, being justified by the following arguments:

a) *Relative natural numbers are a different numerical concept than natural numbers and integers*. The differences with natural numbers are obvious, while with integers, differences are seen: in the logical-formal sphere (inversion between comparison and equivalence in construction processes (González, 1995, pp. 214-219)); five differences in the ordinal and algebraic structures (op. cit., pp. 235-236)); in the phenomenological sphere (different phenomena (op. cit., pp. 260-277)) and in the cognitive sphere (as we will see, subjects establish a clear differentiation in what pertains to their ordinal structures).

b) *Relative natural numbers open new perspectives in didactic research*. They provide a new distribution of the additive conceptual field (Op. cit., pp. 224-226) which modifies and improves classification of additive problems stated verbally (Op. cit., pp. 259-283), facilitates the review of earlier research, offering new explanations for the results obtained (Coquin-

Viennot, 1985; Vergnaud, 1982, pp. 48-49) and becoming an instrument for analyzing this issue from different approaches.

c) *Relative natural numbers are didactically necessary and useful.* Taking them into account can resolve questions that come up in educational practice, encouraging a didactic treatment which is well-adapted to the characteristics of "natural" cognitive development, encouraging the transition from natural numbers to integers and from arithmetic to algebra, avoiding student errors and difficulties (Bell, 1982).

Method

Participants

An intentional sample of 77 subjects was chosen according to the following criteria: 1) Subjects with the minimal level of mathematical knowledge needed to answer the questionnaires; 2) Subjects with a high level of school achievement and general formation; 3) Subjects with different socio-cultural backgrounds, age and type of activity. Sample composition is as follows: 54 students from the following levels: 2nd year Secondary Education (13-14 year-olds) (4); 3rd year Secondary (14-15 year-olds) (8); Vocational training (2); First, second and third year of university training for pre-service primary teachers (40); 23 non-students: University professors (10), University staff (5) and people who perform activities outside the University (8); Gender: male (14), female (63); home: rural (25), urban (52); ages from 14 to 47 years.

Approaches

Complete analysis of the problem was covered from three different approaches: formal, phenomenological and cognitive, and following two lines of methodology: one theoretical and one empirical.

In the *theory phase,* qualitative meta-analysis techniques were used (Fernandez Cano, 1995; Ogawa & Malen, 1991), following a process which we call didactic analysis (González, 1995, pp. 58-62; González, 1998), and the intuitive-constructive theory of scientific conceptual forms was taken as the theoretical framework of reference (Stegmüller, 1979). Under these assumptions, a detailed analysis of the domain of the elementary application of additive

and ordinal structures of natural numbers and integers was carried out (González, 1998, pp.185-189), revealing three types of relationships (comparisons, tranformations and combinations), three types of measures (natural, integers and what we have called “relative natural measurements”) and two types of numbers (natural and integers) which, as we have seen, do not fulfill the formal conditions needed to be isomorphous with the set of relative natural measurements or to act as a support for direct resolution of the corresponding arithmetic problems (González, 1998, pp.194-200).

The theoretical study, therefore, reveals the didactic need to consider a third numeric system that provides adequate coverage for a third identified metric structure (relative natural measures), forcing us to characterize and call the new domain an entity, separating it from what is applied to the integer structure. For this purpose the concept of relative natural number is defined, and we produce a semi-formalized construction of the non-commutative, additive semi-group, without a neutral element, with two null values, and partially ordered (op. cit., pp. 189-200).

The process described and its results can be summed up in the following points, which constitute the hypotheses which were critically evaluated in the complete theoretical study:

I. In the usual concrete application domain of the additive and ordinal structure of natural numbers and integers, there is a subdomain characterized by the intervention of a type of discrete measures related to the comparison of natural measures, and which we will call “relative natural measures”, for which a partially-ordered structure and a law with a specific additive internal composition can be established.

II. There is a set of numbers which we will call relative natural numbers, which, with proper addition and ordering, is isomorphous with the set of relative natural measures.

III. The set of relative natural numbers, with its defined addition and order, present the following five basic structural differences with respect to the additive, ordered group of the integers:

1) Partial order with inversion in the negative region, instead of a total order;
2) Having a first element for each of its regions, instead of lacking any first element;

3) disconnection between the positive and negative regions;
4) two null values, instead of a single zero;
5) additive annulment-compensation (relative natural addition), instead of integer addition.

IV. The relative natural numbers open a new channel for extending the natural numbers to the integers, in their additive and ordinal aspects, when they take their place alongside these in a theoretical model which relates all the internal elements of the domain, regulating the corresponding additive arithmetic structures.

V. The new model makes it possible to establish a semantic-logical classificaion of the problems and situations of the domain, which expand and further clarify other classifications already proposed for verbally stated additive problems.

In the empirical phase, attention was directed to the first of these differences established in Section III, that is: total order for the integers, partial order for the relative natural numbers, although broken down into the following four aspects:

a) attribution of meanings, signs and dichotomous adjectives to the regions;
b) global comparison-assessment of regions;
c) comparison of measures with negative numeric values;
d) comparison of measures with numeric values of a different sign or region

A questionnaire was constructed for each of these, and applied to an intentional sample selected according to the characteristics of the problem. A cross-sectional descriptive study was performed with the data (Bisquerra, 1989, pp. 217-) with a present focus and correspondence analysis (Cornejo, 1988). The results favor the following hypothesis: individuals with an education at or beyond the level of 14-year-olds give a differentiated semantic treatment to relative natural numbers and to integers, on the basis of their ordinal differences, when they appear in basic situations involving comparison of discrete measures.

The present article addresses the study performed using the first of the four questionnaires, that is, the study on attribution of meanings, signs and dichotomous adjectives to the regions. Even though choosing this first part of the research is due to space considerations,

essential information about the whole study is not deleted, since the three remaining questionnaires have the same structure, orientation and format as the first, the same types of situations, the results are similar and in line with the hypothesis, and all of them fulfill the characteristics of studies on semantial differential. Consequently, we refer the reader to the original report (González, 1998) for detailed information about the complete study.

Instruments

The intent is to observe whether subjects give a different overall semantic assessment, through pairs of opposing terms or short phrases, to situations expressed in the form of generic statements without numeric specification and in concrete, familiar contexts. These situations propose a comparison between a "positive" quantity/measure and a "negative" one, both of which correspond to the same magnitude and to one of two different cases: i) when the two quantities/measures are relative natural numbers (which we will call "relative situations") and ii) when the two quantities/measures are integers (which we will call "integer situations"). Our conjecture holds that if subjects respond differently to these two types of situations, where the properties and operations of integers are usually applied indiscriminately, we can affirm that cognitive differences exist, thus empirically reinforcing the adequacy and didactic utility of the conceptual field of the relative natural numbers.

As will be justified below, we take it that subjects are responding differently to these situations when (1) for comparisons between integer measures, they choose fixed, objective terms (greater-less, before-after, etc.), although we also allow choices of dichotomous subjective terms (better-worse, etc.), and (2) for comparisons between relative natural measures, they choose subjective terms, or terms which imply the absence of a fixed, objective assessment (it depends, etc.). The purpose of the study is, therefore, to establish a semantic differential between the integer measures and the relative natural measures with respect to the order relationship, but making our inquiry into the cognitive structure and not into attitudinal aspects, as has usually been the case in this type of study (Osgood, Suci & Tannerbaum, 1957; Summers, 1982).

The following process was used to prepare the instrument:

1) A pilot study, where we drafted an initial, exploratory questionnaire, applied it to an intentional sample and analyzed results;

2) Drafting of the final questionnaire as it appears in Appendix 1. The definitive study was carried out by applying the final questionnaire to a new intentional sample.

Criteria for preparing the instrument

The total order of integers means that any two integer measures are comparable/assessable, and in particular those measures which belong to different regions. This comparison/assessment is usually expressed through pairs of opposites which belong to the following categories:

- ordinal or comparative terms, establishing a direct relationship between the elements: greater-less, higher-lower, before-after, better-worse, etc; these may be objective (greater-less, higher-lower, before-after) or subjective (better-worse).
- terms of classification, establishing an indirect relationship: positive-negative; more-less; good-bad; etc., as long as there is some agreement that connects the two categories through something intermediate which establishes the order; these may be objective (more-less; positive-negative; greater than zero - less than zero), or subjective (good-bad; easy-difficult; high-low).

By contrast, in the partial order structure of the relative natural numbers, only measures with the same sign are comparable (Ex.: a gain of 3 is greater than a gain of two 2, but it is not greater than a loss of 2; if anything, it would be better). Here, objective comparison/assessment of measures with a different sign is inappropriate; the only options allowed are the following: a) a fixed but subjective comparison/assessment (Examples: "better/worse"; "easy/difficult", etc.), b) a variable comparison/assessment, depending on the circumstances and the context (Example: "it depends on the situation"), c) total independence in any situation (Example: "they are independent"); these three can be summarized in a single quality: the absence of fixed, objective assessment, as opposed to the case of the total order structure of integers.

Questionnaire content and structure

The questionnaire is built around two crossed categorizations:

a) Columns: assessment using a verbal expression, in one of two categories:

1) a fixed, objective assessment using pairs of opposite, objective terms. The following options were used for this category: "positive - negative (+ sign / - sign)"; "greater than zero – less than zero"; "higher - lower" and "before - after";

2) the absence of a fixed, objective assessment, including the remainder of possible assessments: subjective, fixed assessment, with three pairs of opposite terms ("good-bad (positive/negative)"; "easy-difficult"; "high-low"), variable or dependent assessment and the absence of a fixed assessment ("it depends on the situation" and "they are independent, there is no way to compare").

b) Rows: a type of situation for the participants to qualify with adjectives, using the term pairs or short phrases given above, the two dual regions in which the universe of values is divided. It is a qualitative variable present in two modalities: situations with opposing integer measures (integer situations) and situations with opposing relative natural measures (relative situations). All situations have the same structure: a pair of generic, opposite quantities stated verbally and separated by a slash (example: "a stock market index indicating a rise / a fall").

A total of 24 situations are included in the questionnaire (11 integer and 13 relative). The situations in positions 2, 3, 5, 8, 10, 11, 13, 16, 17, 18, 19, 21 and 24 were selected as cases with relative measures. The situations in positions 1, 4, 6, 7, 9, 12, 14, 15, 20, 22 and 23 were selected as cases with integer measures. It must be noted that this distinction is not always simple, since there are cases where the two characterizations can be combined.

Responses

The connection between rows and columns is carried out through a single sentence which begins in a row-situation, and ends, via a linking word or short frase, in a grammatical hiatus where one or several of the possible values from the columns is placed (example: "The following opposing situations … are usually described using this pair of terms"). The composition: pair of opposing terms (row) – linking phrase – pair of words or short phrase (column) produces a text which may or may not make sense. The participant is asked to mark, for each

row, all the column modalities that make sense. The dependent variable is the type of connection, with two modalities: selected or not selected, coded as 1 or 0 respectively.

Procedure

The questionnaire is applied to five groups of students, two of them being complete class groups (8^{th} and 9^{th} graders), during their class hours and without any type of clarification. Teachers and non-students were given the test individually, asking them to complete it as soon as possible. Everyone was asked to attempt a response for all questions.

123 completed questionnaires were collected, out of which fifteen were eliminated, having left five or more answers blank, or showing a systematic pattern of responses. Of the remaining 108, a new intentional sampling was made through a second, more restrictive application of the criteria used in choosing the initial sample; this new selection process resulted in a sample of 77 for this study.

Results were displayed in frequency tables and diagrams. Data analysis was performed using Excel 2.2 spreadsheets, the SPSS package and the SAS program on the VAX/VMS computer, version v5.5-2, of the Data Processing Center at the University of Málaga.

Statistical analyses

Descriptive analysis

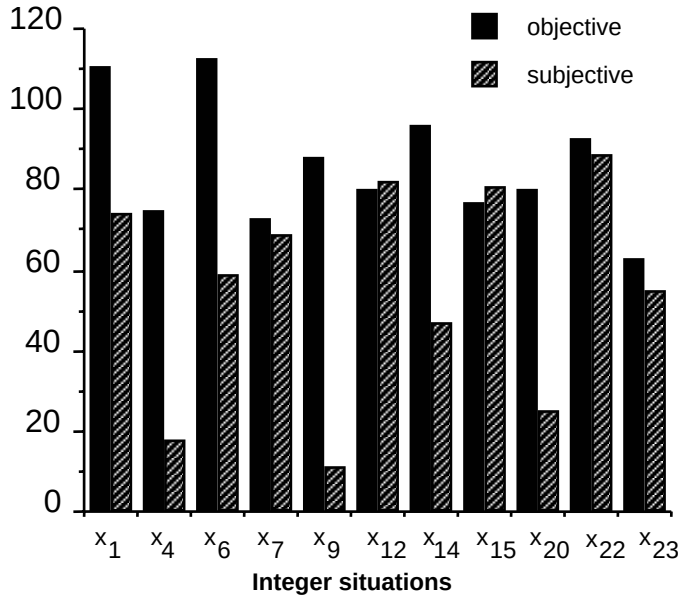
Table 1 shows the absolute frequencies of responses to the questionnaire; shaded regions represent where the greatest number of responses for that row should appear, for each of the 24 situations, according to our theoretical predictions: in Block I, the integer situations (objective responses, or Type 1) and in Block II, the relative natural situations (subjective responses, or Type 2) and other responses (no assessment, it depends, Type 3).

Table 1. Absolute frequencies, and expected response areas

	Block I					Block II						
	Type 1 Responses					Type 2 Responses			Type 3 responses			
	positive negative	greater less	above below	before after	SUM	good bad	easy difficult	high low	it depends	independent	SUM	TOTAL SUM
x1	68	32	11	0	111	53	8	12	1	0	74	185
x2	23	11	4	1	39	48	19	0	12	1	80	119
x3	47	8	9	0	64	58	16	6	2	0	82	146
x4	2	1	4	68	75	3	4	0	11	0	18	93
x5	44	7	18	2	71	33	9	14	9	3	68	139
x6	23	60	26	4	113	16	1	32	10	0	59	172
x7	9	11	49	4	73	7	0	56	6	0	69	142
x8	3	0	26	6	35	3	3	35	17	11	69	104
x9	7	9	2	70	88	5	1	1	3	1	11	99
x10	7	1	16	3	27	12	37	24	12	3	88	115
x11	14	2	14	1	31	41	32	8	27	1	109	140
x12	58	12	7	3	80	54	17	5	6	0	82	162
x13	12	6	14	7	39	13	3	7	29	12	64	103
x14	9	16	52	19	96	7	0	35	0	5	47	143
x15	41	10	25	1	77	36	25	16	3	1	81	158
x16	15	8	11	5	39	21	6	5	24	11	67	106
x17	21	32	24	3	80	29	3	26	19	2	79	159
x18	15	2	5	5	27	35	24	0	11	9	79	106
x19	18	2	19	7	46	33	9	5	22	6	75	121
x20	13	6	2	60	80	15	2	0	6	2	25	105
x21	11	4	26	1	42	10	7	47	18	5	87	129
x22	57	9	24	3	93	44	16	25	4	0	89	182
x23	20	4	5	34	63	22	7	2	20	4	55	118
x24	32	4	21	10	67	58	9	10	3	4	84	151

The graphs from Figure 2 represent absolute frequencies of the objective and subjective answers to the relative natural situations and the integer situations, through double bars (in black or shaded, for the two response types). This gives us a visual appreciation of the type of response which was predominant for each situation.

Questionnaire a.1.- Absolute frequencies of objective and subjective response types to integer situations.



Questionnaire a.1.- Absolute frequencies of objective and subjective response types to relative situations.

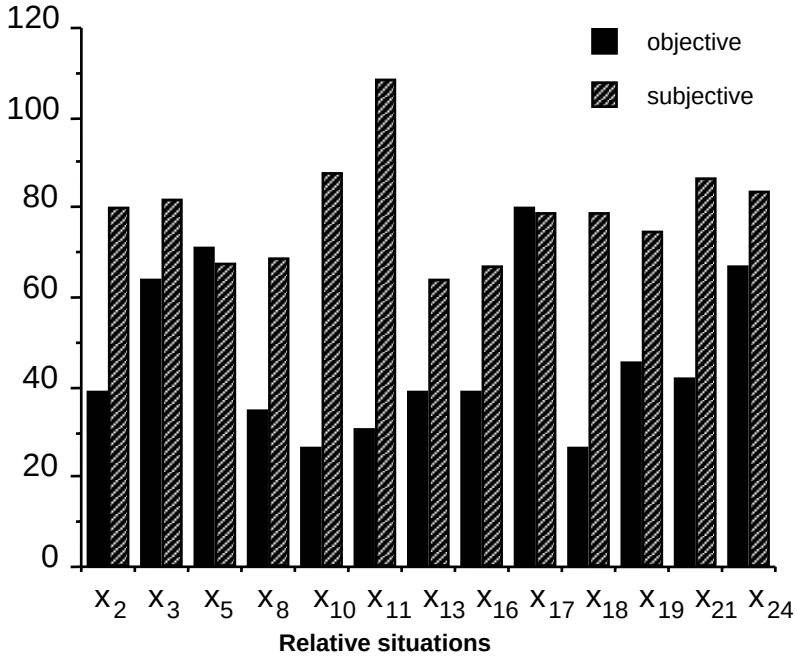


Figure 2. **Absolute frequencies of responses to the two situation types**

One can conclude from Table 1, and visualize in Figure 2, the tendency in 20 of the 24 situations is to group integer situations around objective assessments and relative natural situations around the absence of fixed, objective assessment, representing 83.33% of the total. In the integer situations x12 and x15, there is a predominance of Type 2 answers (subjective), which we allowed for in our conjecture. In the two remaining situations, x5 and x17, the grouping tendency is the inverse of what was expected, although the differences are small; these situations account for 8.33% of the total.

This first generic description seems to point in a direction that sustains our conjecture. However, the differences between a fixed, objective assessment and the absence of fixed, objective assessment are not always as pronounced as one would wish; some results require interpretation, leading us to continue here with a more detailed description.

Concrete comparative analysis

Effect size was calculated for each situation, comparing objective assessment (Block I) to subjective assessment or absence of assessment (Block II). It can be observed in Table 2 that the strongest concordances are those predicted by the hypothesis: 15 of them are not due to chance, with an error type $p< 0.01$, and five are weak (those which correspond to items: x3, x7, x22, x23 and x24). Discordances are seen only for four items (x5, x12, x15 and x17), with trivial effect sizes and no statistical significance.

Overall comparative analysis

The arithmetic mean and standard deviation were calculated for the four variables that result from combining the two answer types (objective assessment and subjective assessment or lack of assessment) with the two situation types. Intervals for the four variables are represented in Figure 3, where appreciable differences can be found between the objective and subjective answers, as well as between objective answers to the two types of situations.

Table 2. ***Effect size and inferential contrast***

Items	(effect size)	Concordance	χ^2	P
1	0.41	yes	7.40	0.01*
2	-0.73	yes	14.13	0.00*
3	-0.25	yes	2.22	0.14
4	1.55	yes	39.94	0.00*
5	0.04	no	0.06	0.81
6	0.66	yes	16.95	0.00*
7	0.06	yes	0.11	0.74
8	-0.69	yes	11.12	0.00*
9	2.47	yes	59.89	0.00*
10	-1.25	yes	32.36	0.00*
11	-1.34	yes	43.46	0.00*
12	-0.02	no	0.02	0.89
13	-0.50	yes	6.07	0.01*
14	0.73	yes	16.79	0.00*
15	-0.05	no	0.10	0.75
16	-0.55	yes	7.40	0.01*
17	0.01	no	0.01	0.92
18	-1.13	yes	25.51	0.00*
19	-0.49	yes	6.95	0.01*
20	1.23	yes	28.81	0.00*
21	-0.74	yes	15.70	0.00*
22	0.04	yes	0.09	0.76
23	0.14	yes	0.54	0.41
24	-0.23	yes	1.91	0.17

With regard to variability, one notes the homogeneity of the subjective answers or absence of assessment in relative situations, in contrast to the dispersion of these values in the case of integer situations. This disperson can be explained by considering that the questionnaire includes both classic integer situations (temperatures, chronology, stock market indices, etc.) and less familiar integer situations (golf or family income and expenses). We must recall

that within our hypothesis we do not exclude the treatment of integer situations using subjective terms, but we do exclude the absence of assessment for integer situations as well as treating relative situations with objective terms, as is evident in the graph shown in Figure 3.

Questionnaire a.1.- Intervals of the means of the two answer types for each of the two situation types ± standard deviation.

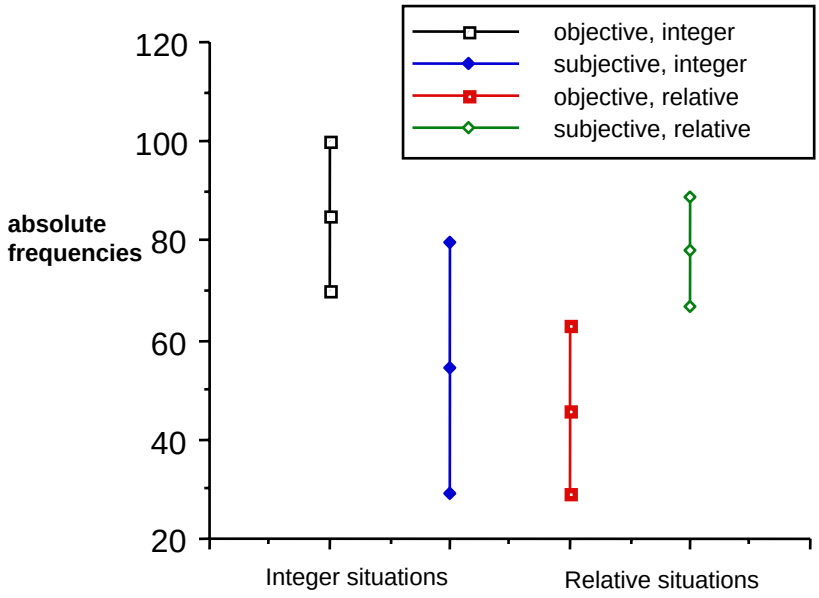


Figure 3. **Intervals for the types of questions**.

Correspondence analysis

A contingency table was constructed from the original data, using the 24 situations for rows, and three answer groups for columns (group 1: objective assessments; group 2: subjective assessments; group 3: absence of fixed assessment). With this data, a correspondence analysis was performed, yielding two factors that explain 100% of the inertia, namely, 59.5% and 40.5%, respectively.

Analysis of frequencies observed is summarized in Table 3, where we see that the quality of all items is 1, indicating that they are well represented in this bidimensional space. Answer frequencies fall in the interval [0.029, 0.058], showing great uniformity. The largest contribution to total inertia corresponds to x4, x9, x10, x11, x13, x16, and x20; while x5, x7, x12, x17 and x24 show a null or minimal contribution.

Table 3. ***Basic statistics for correspondence analysis***

Item	Frequency over sample size	Quality of inclusion in the plane	Contribution of the row to total inertia	Correlation of the item with axis 1	Correlation of the item with axis 2
x_1	0.058	1.00	0.007	0.900	0.100
x_2	0.037	1.00	0.004	0.356	0.644
x_3	0.046	1.00	0.007	0.089	0.911
x_4	0.029	1.00	0.014	0.344	0.656
x_5	0.043	1.00	0.000	0.857	0.143
x_6	0.054	1.00	0.006	0.848	0.152
x_7	0.044	1.00	0.002	0.526	0.474
x_8	0.033	1.00	0.009	0.852	0.148
x_9	0.031	1.00	0.020	0.654	0.346
x_{10}	0.036	1.00	0.010	0.490	0.510
x_{11}	0.044	1.00	0.013	0.873	0.127
x_{12}	0.051	1.00	0.003	0.336	0.664
x_{13}	0.032	1.00	0.027	0.513	0.487
x_{14}	0.045	1.00	0.007	0.944	0.056
x_{15}	0.049	1.00	0.004	0.286	0.714
x_{16}	0.033	1.00	0.016	0.622	0.378
x_{17}	0.050	1.00	0.000	0.032	0.968
x_{18}	0.033	1.00	0.007	0.872	0.128
x_{19}	0.038	1.00	0.006	0.823	0.177
x_{20}	0.033	1.00	0.010	0.575	0.425
x_{21}	0.040	1.00	0.005	0.945	0.055
x_{22}	0.057	1.00	0.005	0.444	0.556
x_{23}	0.037	1.00	0.005	0.084	0.916
x_{24}	0.047	1.00	0.003	0.052	0.948

Table 4 indicates the statistics for each assessment type, while a figure in Appendix 2 represents the distribution of the row points and column points over the coordinate plane.

Table 4. ***Statistics related to the assessment types***

Assessment type	Frequency over sample size	Axis 1 coordinate	Correlation with Factor 1	Relative inertia of the row to Axis 1	Axis 2 coordinate	Correlation with Factor 2	Relative inertia of the row to Axis 2
objective	0.487	-0.293	0.789	0.368	-0.152	0.211	0.145
subjective	0.402	0.146	0.176	0.076	0.317	0.824	0.522
absence	0.111	0.753	0.711	0.556	-0.480	0.289	0.333

We draw the following conclusions from the correspondence analysis:

a) Dimension 1 is the result of a greater contribution from column points 1 and 3, while dimension 2 has a greater contribution from column point 2, referring to subjective type assessments. The objective assessment terms contribute to the inertia of dimension 1, the subjective assessment terms to dimension 2, and those that indicate absence of assessment contribute to the inertia in both directions, with the advantage going to dimension 1.

b) There is opposition between point 1 and points 2 and 3, separated by the dimension 2 axis; the opposition is produced between fixed objective assessments and other types of assessments falling under the category “lack of fixed assessment”.

c) The vertical axis forms a line of separation between objective assessment situations and situations that lack fixed assessment, with the exception of statements x3, x5, x23 and x24. The former take negative values for the abscissa, while the latter are placed with positive abscissas, to the right of the vertical axis.

d) Considering only the first dimension, the following associations are confirmed:

For Type 1 assessments:

- Situations x1, x4, x6, x9, x14 and x20, all of them integer situations, present markedly objective assessments (group 1).

- Situations x5, x7, x17, x22 and x23, of which x5 and x17 are relative, are found in an intermediate zone between point 1 and point 2, presenting a balance of assessments between group 1 (objective assessments) and group 2 (fixed, subjective assessments), with a slight advantage in favor of Type 1.

For Type 2 assessments:

- Situations x3, x12, x15 and x24, of which x12 and x15 are integer situations, are found in an intermediate zone between point 1 and point 2, presenting a balance of assessments between group 1 and group 2, with a slight advantage in favor of Type 2.

- Situations x2, x10, x11, x18 and x21, all of them relative, are clearly associated with point 2, corresponding to fixed, subjective type assessments.

For Type 3 assessments:

- Relative situations x8 and x19 are found in an intermediate zone between points 2 and 3, such that their assessment lies between fixed subjective and the absence of fixed assessment.

- Relative situations x13 and x16 present the greatest tendency toward dependent assessment or the lack of a concrete assessment.

e) There are four situations, some integer and others relative, that are not well represented in factor or dimension 1, given their low correlation ($| r | < 0.25$); however, these are very well represented on axis 2. These situations, namely x3, x17, x23 and x24, should be reconsidered, since they are the ones that reduce the confirmatory possibilities of the proposed model.

Results

From the analyses and examination of the data, we highlight the following results:

A) In the set of 11 situations of comparison between integer measures, we find that:

- an objective assessment greater than the subjective is produced in 81.8% of the situations, where subjects respond using terms typical of a total order relationship;

- frequencies of type 3 responses (absence of assessment) are very low in all cases, except in situation x23; frequencies of objective assessments are greater than the other two types, except in two cases, x12 and x15, where there is no concordance with what was predicted; in the remainder of the cases there is concordance, and in six of these it is above chance;

- correlations of the 11 integer situations with respect to axis 1 are higher than 0.30, except for situation x23 (minutes passed or remaining at an appointment), which is also the exception to the separation established by the vertical axis between the situations with fixed objective assessment and other situations.

- there are three types of integer situations as a function of the level of objective assessment:

- situations with a strongly objective assessment: balance in a bank account (x1); time with respect to a moment or date (x4, x9 and x20); temperatures (x6) and position of floors in a building, with respect to the ground floor (x14). All of these have a high correlation with axis 1, except x4.
- situations with objective assessment greater than subjective: sea level (x7) and stock market index (x22); the correlation of these two with axis 1 is acceptable.
- situations with a subjective assessment slightly higher than the objective: family financial balance (x12) and golf game (x15); their correlation with axis 1 is low.

- discrepancies with what was expected are acceptable, such that, except in the case of situation x23, the hypothesis is confirmed for comparison of integers.

In general, subjective terms are widely used and are often more common and familiar than objective terms. This may mean that the difference between the two response types is less in the case of integer situations than in the relative situations, and in some integer situations the subjective answers may even exceed the objective ones.

Although the differences are very small, assessments of integer situations x12 and x15 (representing 18.2%) are not markedly objective. This lack of concordance consistency may be due to the situation. For x12, family financial balance, the Spanish term “balance” is not often used and involves two types of variables: balance (integer variable) and income and expenses (relative natural variables). For situation x15 (golf), confusion arises from unfamiliarity with the game and the peculiarities of its scoring system, which sometimes contradict the norm. In any case, it seems reasonable that for these two situations, when in doubt, subjects prefer to use valid terms which are less risky and less rigorous than those classified as objective. Finally, situation x23 ("minutes that have passed/minutes left") is presented in an unclear

fashion; opposing verbs are used which can be considered from a relative point of view, which may have brought about the high frequency of type 3 assessments.

B) In the 13 comparisons of relative natural measures, we find that:

- the absence of objective assessment surpasses objective assessments in 84.6% of the situations; answers predominantly express a subjective assessment or the absence of assessment. In the remaining 15.4%, there is a balance between objective and non-objective assessments.

- Type 3 assessments (absence of assessment) show frequencies greater than 10% of the total, except for situations x3 and x24; objective assessments total less than the sum of the other two, except in cases x5 and x17, where they come out equal and no concordance is produced; concordance is seen in the rest of the cases and in 9 of these it is above chance.

- correlations of the 13 situations with respect to axis 1 are above 0.30, except for situations x3, x17 and x24. The first and third have a low percentage of type 3 assessments, and the second has a higher percentage of objective assessments than the rest.

- we find four types of relative situations as a function of the different types of answers:

*strongly subjective assessment: not enough/more than enough, rise/descend (steps, a balloon), increase/decrease and find/lose (x2, x10, x11, x18 y x21). All of these have an acceptable correlation with axis 1.

*subjective assessment higher than objective, with low absence of assessment: gain/loss (x3) and discount/surcharge on a price (x24); correlation with factor I is very low, and even though there is concordance, error is greater than 0.01.

*predominance of absence of fixed, objective assessment (subjective assessment similar to objective and to the absence of assessment) (frequencies for the three answer types is practically the same): rise/fall and exit/entry (x8, x13, x16 and x19); correlations with axis 1 are high, and concordance is above chance in every case.

*absence of objective assessment equal to objective assessment (total frequencies for subjective assessment and absence of assessment are equal to frequencies for objective assessment): situations x5 and x17 (deposits-withdrawals and temperature variations); these are

usually presented as integers and it is more unusual to treat them as relative. These are the only two cases where there is no concordance and no perceptible correlation with axis 1.

- We can affirm that the starting hypothesis holds true except in the case of situations x5 and x17, although these discrepancies from the expected allow for interpretation, namely:

Situation x5 refers to the assessment of deposits-withdrawals in a bank account, where the most frequent answer was “positive (+ sign), negative (- signo)”. We understand this to be so because bank deposits and withdrawals can be considered in two different, equally valid manners: as opposite actions that are commonly referred to as putting in and taking out money (relative natural structure), the sense considered when we included this situation in the questionnarie; and as a line item which the bank records as a number with a sign, indicating an amount to be added or subtracted from the prior amount (interpretation related to the bank balance, and as such, framed in an integer structure). The latter interpretation is what was predominant in this case.

We believe that the result from situation x17 is explained by priority use of the temperature scale. A variation in temperature, besides being seldom referred to, is not meaningful until it is associated with the base temperature from which the variation is produced. Thus it is common to associate a rise in temperature with a “greater” or “higher” temperature; it is an easy step from this to “above zero” or “positive (+ sign)”, substituting the partial order for the corresponding total order, which in this case is more familiar. (*T.N. Spanish subjects refer to the Celsius scale, where weather temperatures above and below zero are not unusual.*)

Discussion

Results from the questionnaire analyzed above, as well as from the rest of the empirical study (González, 1995, pp. 341-390), offer evidence in favor of the stated hypotheses, and point in the direction of the goodness of the theory, that is, differences between the conceptual fields being compared are not only logical-formal and phenomenological, but they are also manifest from the cognitive point of view.

It is evident that this research would not be of interest if there were homogeneity between the different phenomena, situations and problems in this domain, from the cognitive point of view—that is, if subjects considered the integer numeric structure as the only suitable structure to be applied in a "natural" fashion, without difficulty and evenly across all situations appropriate to the additive field. But this is not so, the study makes evident a heterogeneity of meanings, terms and structures associated with two types of situations and phenomena under consideration: those that are modeled using the integer structure and those modeled using the relative natural structure. This differentiation takes shape in the preferential use of (1) objective terms, closely tied to a total order structure (positive-negative, greater than zero –less than zero, etc.), when making comparisons between integer measures, although subjective terms can possibly be used; or (2) terms which indicate the absence of fixed objective assessment and which can refer to a dichotomous structure with an imprecise or undetermined connection between the parts (typical of a partial order with inversion in the "negative" region), for comparisons between relative natural measures.

It has been shown that participants do not have the same mastery of all the specific topics and contexts used in the questionnaire; familiar situations with largely uniform responses coexist with other situations whose characteristics and functioning are not evenly understood. However, in summary, it can be affirmed that individuals in the sample assess the two types of situations differently, for the most part defining the field of relative natural numbers and the field of integers. For this reason we can make two affirmations: (1) the criteria used in the questionnaire were correct except for some small, isolated modifications, that the questionnaire constitutes an instrument which is well-suited to reflect the differentiated treatment given to the two situation types, and (2) results uphold the plausibility of the empirical hypothesis set out for examination, that is: Individuals with an education level corresponding to 14 years of age or higher give differentiated semantic treatment to the relative natural numbers and to the integers, based on their ordinal differences, when they appear in basic situations of comparison between discrete measures.

This affirmation is at the same time surprising, if we take into account that mathematics instruction received by subjects in our sample has been based on the assumption that the integer structure is the only suitable context for directly handling all the situations put forward in the questionnaire, all of them under the same conditions. Our results contradict this assumption, leaving open the didactic question of how to address the divergences detected be-

tween relative natural numerical thinking and integer numerical thinking, or between arithmetic and algebraic thinking, and how to approach the transition from natural numbers to integers, and from arithmetic to algebra.

Finally, there is an evident need for replication studies which corroborate the evidence found here, delving further into the typologies of responses given on all the questionnaires. Likewise, a specific study is needed in order to further understand the disparities seen in the irregular situations, or situations with unexpected responses, which, although they are a minority, they are found to be irregular on all the questionnaires.

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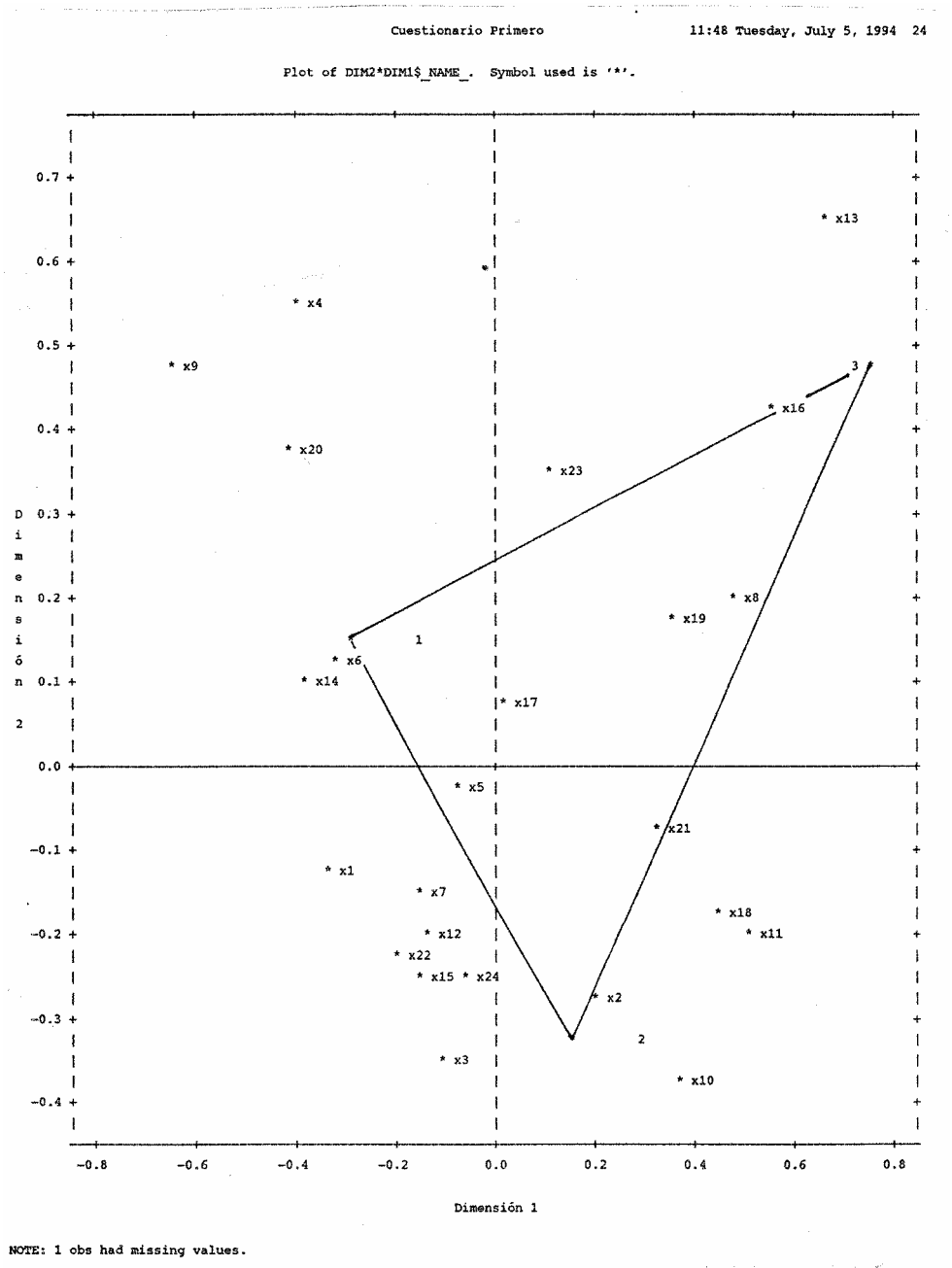
Appendix 1. Final questionnaire used in the study

(Check the boxes that seem best to you) (a.1) (Before answering each question, read the options in the upper portion of the table.) If you want to cross out an answer marked by mistake, draw a circle over it.	THE FOLLOWING OPPOSING SITUATIONS … … ARE USUALLY DESCRIBED USING THIS PAIR OF TERMS:							... DO NOT HAVE ANY FIXED VALUE, THEREFORE:	
	Positive - Negative (+ sign / - sign)	Greater than zero – Less than zero	Above - Below	Before – After	Good - Bad (positive - negative)	Easy - Difficult	High - Low	It depends	They are independent and there is no way to compare
In my bank account, a balance in the black / in the red									
When I go shopping, not enough money / more than enough money									
When I am playing, a win / a loss									
Regarding a certain date, a prior day / a later day									
In my bank account, a deposit / a withdrawal									
A temperature below zero / above zero									
With reference to sea level, a higher altitude / a lower altitude									
Rising / Descending in an elevator									
A year before the birth of Christ / a year after									
Climbing / Descending a number of steps									
For a person, weight gain / weight loss									
On my family balance sheet, more expenses than income / vice-versa									
At a bus stop, a number of people get off / get on									
In a building, a level below the ground floor / above the ground floor									
A golf score “above par” / “below par”									
In a coffee shop, a number of people leave / come in									
In a city, rising / falling temperatures									
Lose a number of buttons / find a number of buttons									
To someone’s age, add / take off years									
With respect to launch time, a second prior / a second after									
For a balloon, rising / falling a number of feet									
A stock market index indicating a rise / a fall									
At an appointment, minutes that have passed / minutes remaining									
A discount on the price of an item / a surcharge on that price									

Appendix 2. Diagram of the correspondence analysis



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