Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 366 (2023) 292–319

Journal of Differential Equations

www.elsevier.com/locate/jde

A priori estimates for non-coercive Dirichlet problems with subquadratic gradient terms

José Carmona [a,*], Salvador López-Martínez [b], Pedro J. Martínez-Aparicio [a]

[a] *Departamento de Matemáticas, Universidad de Almería, Ctra. Sacramento s/n, La Cañada de San Urbano, 04120, Almería, Spain*

[b] *Departamento de Matemáticas, Universidad Autónoma de Madrid, Facultad de Ciencias, Ciudad Universitaria de Cantoblanco, 28049, Madrid, Spain*

Received 19 July 2022; revised 28 March 2023; accepted 7 April 2023

Abstract

We deal with some quasilinear elliptic problems posed in a bounded smooth convex domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$, namely

$$\begin{cases} -\Delta u = \lambda u + \mu(x)|\nabla u|^q + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where the data satisfy

$$\mu \in L^\infty(\Omega),\ \mu \gneq 0; \quad f \in L^p(\Omega),\ p > N,\ f \gneq 0 \text{ and } 1 < q \leq 2.$$

We provide sufficient conditions on f, μ (allowing μ to vanish on $\partial\Omega$) that yield the sharp estimate $\lambda\|u\|_{L^\infty(\Omega)} \leq C$ for any bounded solution u with $\lambda \in (0, \lambda_1)$, which is the non-coercive regime. The estimate leads to remarkable consequences such as a multiplicity result and a precise asymptotic behavior of the bounded but *blowing up* solutions as $\lambda \to 0^+$.

* Corresponding author.

E-mail addresses: jcarmona@ual.es (J. Carmona), salvador.lopez@uam.es (S. López-Martínez), pedroj.ma@ual.es (P.J. Martínez-Aparicio).

https://doi.org/10.1016/j.jde.2023.04.012

MSC: 35J25; 35J62; 35F21; 35B30

Keywords: Subquadratic gradient terms; A priori estimates; Multiplicity; Bernstein method; Asymptotic behavior

1. Introduction

For any $\lambda \in \mathbb{R}$, let us consider the following elliptic problem

$$\begin{cases} -\Delta u = \lambda u + \mu(x)|\nabla u|^q + f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $\mathcal{C}^2$, and the most basic condition on the data that we assume is

$$\mu \in L^\infty(\Omega),\ \mu \gneq 0; \quad f \in L^p(\Omega),\ p > N,\ f \gneq 0 \text{ and } 1 < q \leq 2. \tag{H_0}$$

Problem (P_λ) can be seen as a viscous Hamilton-Jacobi equation with a linear zero-order term and under homogeneous Dirichlet boundary conditions. This is by now a classical model. We may establish its popularization in the '80s with the celebrated works by Boccardo, Murat and Puel, see [6,7] and references therein, which have motivated a vast amount of research through the last decades. Some fundamental problems regarding (P_λ) are still open in the present, for instance those related to the *maximal regularity conjecture* of Lions [27], which in turn is linked to the existence of solution to systems of PDEs arising in *mean field games*. Recent advances in this direction can be found in [10,21].

The aim of the present paper is to analyze the structure of the set of solutions to (P_λ) for $\lambda > 0$. In this range, the problem is not coercive, which means an important obstacle for proving a priori estimates. A further difficulty in our case is that, for $\lambda > 0$, problem (P_λ) does not satisfy a comparison principle; we will give more details about this point later.

To be more precise, in our work we will be concerned with the set of *bounded weak solutions*, i.e. solutions belonging to the space $H_0^1(\Omega) \cap L^\infty(\Omega)$. Actually, as we describe later, the boundedness of the solution and the just regularity of the data imply that the solution belongs to $\mathcal{C}_0^1(\overline{\Omega})$. Although there are examples of unbounded solutions, even for very regular data [1], the literature shows that the set of solutions in the space $H_0^1(\Omega) \cap L^\infty(\Omega)$ enjoys an interesting and rich structure. Let us review some works that, in our opinion, help understand the global behavior of the bounded weak solutions to (P_λ) (from now on, we will call them simply *solutions* unless confusion may arise).

A first observation, as was done in [4], is that every solution to (P_λ) with $\lambda < \lambda_1$ is positive in Ω, where λ_1 denotes the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Moreover, there exists no positive solution to (P_λ) for $\lambda \geq \lambda_1$. See Section 2 below for further details.

As far as the existence of solution is concerned, it was first proved in [7] for $\lambda < 0$, which is called the *coercive case*. The keystone was to prove a global L^∞-estimate on the solutions. The proof of the estimate is based on the choice of a suitable exponential test function that somehow gets rid of the gradient dependence in the equation.

We clarify that, actually, the needed summability for f in the result of [6] is only $p > N/2$. Furthermore, if $q < 2$, it is shown in [22] that unbounded solutions (but enjoying certain regularity) exist and are a priori bounded provided one assumes even less summability on f, specifically

$p = N(q-1)/q$. In our analysis, it is convenient to assume that $p > N$ for several reasons, e.g. for assuring that every bounded solution belongs to $\mathcal{C}_0^1(\overline{\Omega})$, as in [14].

We turn to the *limit coercive case* corresponding to $\lambda = 0$. In contrast with the coercive case, existence for (P_0) does not hold if f and μ are large in some sense [2], while existence holds provided the norms of f and μ are small enough [17] (see also [16,22]). In fact, under conditions on the data similar to (H_0), one can find necessary and sufficient conditions for the existence of solution to (P_0) in [11,12,29]. In any case, if there exists a solution to (P_λ) for $\lambda \leq 0$, it is unique. The uniqueness is a consequence of a comparison principle that holds for $\lambda \leq 0$ [3,5].

Let us discuss the situation in which nonexistence for (P_0) occurs. In this case, the author proved in [29] (for $\mu \equiv 1$ and $f \in L^\infty(\Omega)$ but possibly nonpositive or sign-changing) that the (unique) solutions u_λ for $\lambda < 0$ diverge locally uniformly in Ω as $\lambda \to 0^-$. In particular, zero is a bifurcation point from infinity to the left. Moreover, the author also shows a precise asymptotic behavior of the solutions as $\lambda \to 0^-$, namely

$$\lim_{\lambda \to 0^-} \lambda \|u_\lambda\|_{L^\infty(\Omega)} = c_0, \qquad u_\lambda = v + \|u_\lambda\|_{L^\infty(\Omega)} + \epsilon_\lambda,$$

where $\epsilon_\lambda \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$, $\epsilon_\lambda \to 0$ in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$, and $(v, c_0) \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \times \mathbb{R}$ is the unique pair that satisfies that $\max_{x \in \Omega} v(x) = 0$ and

$$\begin{cases} -\Delta v = c_0 + |\nabla v|^q + f(x), & x \in \Omega, \\ v \to -\infty, & x \to \partial\Omega. \end{cases} \tag{1.1}$$

Problem (1.1) is the so-called *ergodic problem* associated to (P_λ). We emphasize that c_0 is an unknown in (1.1). The unique constant for which the ergodic problem admits a solution is referred to as the *ergodic constant*. We stress also that the solution to (1.1) is unique up to adding constants to v. See [24,25] for further details.

The ergodic problem has a stochastic interpretation and motivation, as explained for instance in [29]. From another point of view, (1.1) can be seen, via a change of unknown, as a singular weighted eigenvalue problem with homogeneous Dirichlet boundary condition, where the principal eigenvalue coincides with the ergodic constant and the uniqueness up to additions turns into uniqueness up to multiplications [9].

An essential tool in the proofs in [29] is the mentioned comparison principle for $\lambda \leq 0$. Indeed, it is straightforward to check by comparison that

$$-\lambda \|u\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \tag{1.2}$$

provided that $\lambda < 0$ and $f \in L^\infty(\Omega)$. This is one of the key ingredients to prove the mentioned asymptotic behavior.

It remains to describe the case in which (P_0) admits a solution. If this is the situation, then the branch of solutions for $\lambda \leq 0$ can be continued, using topological degree, to the range $\lambda > 0$, leading to existence of solution in the *non-coercive case*. In this regime, several recent results show that zero is a bifurcation point from infinity to the right and, as a consequence, there exist at least two solutions to (P_λ) for every $\lambda > 0$ small enough (notice that this is consistent with the validity of the comparison principle for $\lambda \leq 0$). However, the lack of coercivity forces several restrictions involving q, μ and N, in contrast to the previous coercive and limit coercive cases. Let us clarify this point:

1. In the pioneering works [4,23], the multiplicity result is achieved under the *natural growth* condition $q = 2$. This allows the authors to perform the well-known Cole-Hopf transformation in order to obtain a semilinear equation in [23] and two semilinear inequalities in [4]. Even more, in [4] it is proved that the multiplicity result is a consequence of the fact that $\lambda = 0$ is a bifurcation point from infinity to the right. More information about the blowing up solutions as $\lambda \to 0^+$ is obtained in [14].
2. In the literature it is frequently assumed also that $\mu \geq \mu_0$ in Ω for some constant $\mu_0 > 0$. In some cases it has been shown that this condition can be relaxed to hold only in a subset of Ω at the expense of localizing somehow the zero order term, see [13]. As far as we know, the first (and only) work in which μ is allowed to vanish on $\partial\Omega$ is [31]. The counterpart, apart from the assumption $q = 2$, is that the proof of the multiplicity result is valid only for low dimension N.
3. Finally, a multiplicity result for $1 < q < 2$ is proved in [28]. However, and similarly as in [31], a restriction depending on the dimension appears. Specifically, the result is valid only for $q \leq Q_N$, where $Q_N \in (1, 2]$ and $Q_N \to 1$ as $N \to \infty$.

It is worth to remark that the original idea in [4] was to prove the multiplicity result as a direct consequence of an L^∞-estimate of the type

$$\|u\|_{L^\infty(\Omega)} \leq C(\lambda_0). \tag{1.3}$$

Under the conditions $q = 2$ and $\mu \geq \mu_0 > 0$, the authors of [4] prove that (1.3) holds for every solution u to (P_λ) with $\lambda \in (\lambda_0, \lambda_1)$, for any fixed $\lambda_0 \in (0, \lambda_1)$ and for some constant $C(\lambda_0) > 0$ independent of λ.

In the present paper, we are able to prove a multiplicity result in the non-coercive regime for any $q \in (1, 2]$ and any dimension $N \geq 3$, and allowing μ to vanish on $\partial\Omega$. In this sense, the result means a novelty with respect to the previous works. However, the techniques we employ require the following condition on μ

$$\mu \in W^{1,\infty}_{\text{loc}}(\Omega), \quad \forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 :\ \mu \geq c_\omega,\ x \in \omega. \tag{H$_1$}$$

We will also need the convexity of the domain Ω, as well as the following regularity and monotonicity assumptions on μ, f, precisely

$$\begin{cases} \text{There exists } \delta_0 > 0 \text{ such that } f, \mu \in \mathcal{C}(\overline{\Omega_{\delta_0}}) \text{ and,} \\ \text{for every } x_0 \in \partial\Omega \text{ and } \eta \in [\eta_0 - \delta_0/2, \eta_0),\ \eta_0 = x_0 \cdot \nu(x_0), \\ \text{one has that } \Sigma_{\nu(x_0),\eta} \cup \Sigma'_{\nu(x_0),\eta} \subset \Omega_{\delta_0} \text{ and} \\ \mu(x) \leq \mu(x_{\nu(x_0),\eta}) \text{ and } f(x) \leq f(x_{\nu(x_0),\eta}),\ x \in \Sigma_{\nu(x_0),\eta}. \end{cases} \tag{H$_2$}$$

In (H$_2$), roughly speaking, Ω_{δ_0} represents a neighborhood of the boundary at distance δ_0, $\nu(x_0)$ is the normal unit vector at x_0 pointing outwards, $\Sigma_{\nu(x_0),\eta}$ is the cap in Ω at the exterior side of the hyperplane $x \cdot \nu(x_0) = \eta$, $x_{\nu(x_0),\eta}$ is the reflected point of x with respect to the mentioned hyperplane and $\Sigma'_{\nu(x_0),\eta}$ denotes the set of all the reflected points. This is a usual notation when dealing with the *moving plane* technique (see Notation below). Observe that since Ω is convex, then (H$_2$) is an assumption on the behavior of μ and f near $\partial\Omega$. More specifically, both functions are continuous and essentially nonincreasing, along normal directions, near the boundary. Apart from constant functions, examples of nontrivial functions μ and f satisfying (H_2) are the distance function to the boundary, the first eigenfunction φ_1 and, in general, any increasing function of them.

In spite of the convexity of Ω and (H_2), our conclusions suggest that, if there is a solution to (P_0), a multiplicity result for $\lambda > 0$ small should hold true assuming only (H_0) and, possibly, a suitable control on μ from below, but for general bounded smooth domains Ω and for any dimension $N \geq 3$. Moreover, we confirm that the condition $q \leq Q_N$ in [28], as well as several conditions regarding μ and N in Theorem 3 and Theorem 4 of [31], are technical, at least under the mentioned assumptions, providing thus partial answers to some questions posed by the authors of both papers.

In the spirit of [4], the keystone of the proof of our multiplicity result is an L^∞-estimate. Specifically, we prove the following result.

Theorem 1.1. *Assume that Ω is convex and (H_0), (H_1), (H_2) are satisfied. Then, there exists $C > 0$ such that*

$$\lambda \|u\|_{L^\infty(\Omega)} \leq C \tag{1.4}$$

holds for every solution u to (P_λ) with $\lambda \in (0, \lambda_1)$. In particular, the only possible bifurcation point from infinity to the right for bounded solutions to (P_λ) is $\lambda = 0$.

Observe that (1.4) contains more information than (1.3) as we show an explicit dependence on λ. Actually, (1.4) is more reminiscent of (1.2). However, in contrast to the result in [29], we develop a proof of (1.4) in the absence of any comparison principle. Instead, we combine the following ingredients:

1. We employ the integral Bernstein method in order to get local estimates on ∇u in Lebesgue spaces. This is a classical method that dates back to [26] (there are recent improvements, see [10,21] and references therein). Nevertheless, gradient estimates in the non-coercive case and for x-dependent μ seem not to be available in the literature. We verify that, indeed, they can be obtained assuming (H_1), and that they do not depend on λ.
2. Local estimates on λu in Lebesgue spaces are proved thanks to regularity theory on weighted Lebesgue spaces developed in [18].
3. An estimate near the boundary is achieved by using the *moving plane* method [19,30], in the spirit of [15]. This is the point where the convexity of Ω and condition (H_2) come into play.

Once the estimate (1.4) is at hand, a multiplicity result can be proved by following the ideas in [4]. More precisely, we obtain the following result.

Theorem 1.2. *Assume that Ω is convex and (H_0), (H_1), (H_2) are satisfied. Assume in addition that (P_0) admits a solution $v_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then there exists an unbounded and connected set Σ with*

$$(0, v_0) \in \Sigma \subset \{(\lambda, u) \in [0, \lambda_1) \times L^\infty(\Omega) : u \text{ solves } (P_\lambda)\},$$

and Σ bifurcates from infinity to the right of the axis $\lambda = 0$. In particular, there exists $\lambda_0 \in (0, \lambda_1)$ such that, for every $\lambda \in (0, \lambda_0)$, ($P_\lambda$) admits at least two solutions $u_\lambda, v_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ that satisfy

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{L^\infty(\Omega)} = \infty \quad \text{and} \quad \lim_{\lambda \to 0^+} \|v_\lambda - v_0\|_{L^\infty(\Omega)} = 0.$$

Another remarkable by-product of (1.4), combined with the complementary local and boundary estimates mentioned above, is a precise asymptotic behavior of the *blowing up* solutions u_λ as $\lambda \to 0$. Specifically, we obtain in the non-coercive case the same behavior (actually slightly better thanks to (1.5) below) that is shown in [29] in the coercive case.

The precise statement reads as follows.

Theorem 1.3. *Assume that Ω is convex and (H_0), (H_1), (H_2) are satisfied. For any sequences $\{\lambda_n\} \in (0, \lambda_1)$ and $\{u_n\} \subset H_0^1(\Omega) \cap L^\infty(\Omega)$ such that u_n solves (P_{λ_n}) for all n with*

$$\|u_n\|_{L^\infty(\Omega)} \to \infty,$$

there exists $\{k_n\} \subset (0, \infty)$ such that $k_n \to \infty$ as $n \to \infty$ and

$$u_n \geq k_n \varphi_1 \quad \text{in } \Omega. \tag{1.5}$$

Moreover,

$$\lim_{n \to \infty} \lambda_n \|u_n\|_{L^\infty(\Omega)} = c_0, \qquad u_n = v + \|u_n\|_{L^\infty(\Omega)} + \epsilon_n,$$

where $\epsilon_n \in W^{1,\infty}_{loc}(\Omega)$, $\epsilon_n \to 0$ in $W^{1,\infty}_{loc}(\Omega)$, and $(v, c_0) \in W^{1,\infty}_{loc}(\Omega) \times \mathbb{R}$ satisfies $\max_{x \in \Omega} v(x) = 0$ and

$$\begin{cases} -\Delta v = c_0 + \mu(x)|\nabla v|^q + f(x), & x \in \Omega, \\ v \to -\infty, & x \to \partial\Omega. \end{cases} \tag{1.6}$$

In addition, if we assume that $f \in L^\infty(\Omega)$ and there exist $\sigma, b > 0$ such that

$$\lim_{d(x) \to 0} \mu(x) d(x)^{-\sigma} = b, \tag{1.7}$$

then there exists at most one constant $c_0 \in \mathbb{R}$ for which (1.6) admits a solution in $W^{1,\infty}_{loc}(\Omega)$ and the solution is unique up to adding constants.

The last part of Theorem 1.3 allows to completely characterize the singular behavior of u_λ. Let us remark that, unlike the classical case (1.1), i.e. for μ constant, neither the uniqueness of c_0 nor of v (up to adding constants) is known for problem (1.6) with general $\mu \in L^\infty(\Omega)$ vanishing on $\partial\Omega$, to the best of our knowledge. Here we provide the proof of the uniqueness in the particular case of μ satisfying (1.7).

The plan of the paper is the following: in the second section we state some preliminary and essentially well-known results that will be needed in our proofs. The third section is devoted to proving the local and boundary estimates commented above. Finally, in the last section we collect the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3.

Notation.

We will denote the distance function to the boundary of Ω in the following way:

$$d(x) = \operatorname{dist}(x, \partial\Omega).$$

A neighborhood of the boundary at distance $\delta > 0$ will be denoted by

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}.$$

We will write λ_1 and φ_1 to denote, respectively, the first eigenvalue and the first eigenfunction (normalized as $\|\varphi_1\|_{L^\infty(\Omega)} = 1$) of $-\Delta$ with homogeneous Dirichlet boundary conditions, i.e.,

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1, & x \in \Omega, \\ \varphi_1 > 0, & x \in \Omega, \\ \varphi_1 = 0, & x \in \partial\Omega. \end{cases}$$

For each $0 \neq \nu \in \mathbb{R}^N$ and $\eta \in \mathbb{R}$ we will use the notation $\Sigma_{\nu,\eta} = \{x \in \Omega : x \cdot \nu > \eta\}$ and $x_{\nu,\eta}$ will denote the symmetric point of $x \in \mathbb{R}^N$ with respect to the hyperplane $\{x \in \mathbb{R}^N : x \cdot \nu = \eta\}$.

2. Preliminary results

Let us start by clarifying some basic aspects of the solutions to (P_λ). First of all, under (H_0), it is clear that $-\Delta u \geq \lambda u$ in Ω, so the weak maximum principle implies that every solution $u \in H_0^1(\Omega)$ to (P_λ) with $\lambda < \lambda_1$ is non-negative. Moreover, since $f \not\equiv 0$, the strong maximum principle implies that $u > 0$ in Ω. On the other hand, multiplying the equation in (P_λ) by φ_1 and integrating by parts, it is straightforward to check that there exists no positive $H_0^1(\Omega)$ solution to (P_λ) for any $\lambda \geq \lambda_1$. This was already observed in [4].

Let us also state the following well-known existence result for the non-coercive regime. See [4] or [28] for more details.

Theorem 2.1. *Assume that* (H_0) *is satisfied and that there exists a solution* $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *to* (P_0). *Then, there exists an unbounded and connected subset of* $\{(\lambda, u) \in [0, \lambda_1) \times L^\infty(\Omega) :$ *u solves* (P_λ)$\}$ *that contains* $(0, u_0)$. *In particular, for any* $M > 0$, *there exist* $\lambda \in (0, \lambda_1)$ *and* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *solution to* (P_λ) *satisfying that* $\|u\|_{L^\infty(\Omega)} > M$.

Let us emphasize that the main motivation of the present paper is to prove that, in fact, there exist at least two solutions to (P_λ) for every $\lambda \in (0, \lambda_0)$, for some $\lambda_0 > 0$.

Throughout the paper, we will need to use a suitable comparison principle for subsolutions and supersolutions to the equation

$$-\Delta u = \mu(x)|\nabla u|^q + f(x), \quad x \in \Omega.$$

Since the functions we work with are not smooth and may even blow-up near $\partial\Omega$, we state here for convenience of the reader a valid comparison result that is essentially contained in [3].

Theorem 2.2. *Assume that* (H_0) *holds and let* $u, v \in W^{1,N}_{loc}(\Omega) \cap \mathcal{C}(\Omega)$ *be such that*

$$
\begin{aligned}
&-\Delta u \le \mu(x)|\nabla u|^q + f(x), \quad x \in \Omega,\\
&-\Delta v \ge \mu(x)|\nabla v|^q + f(x), \quad x \in \Omega,\\
&\limsup_{d(x)\to 0}(u(x) - v(x)) \le 0.
\end{aligned}
\tag{2.1}
$$

Then $u \le v$ *in* Ω.

Regarding the previous statement, we only clarify that in the original result (Lemma 2.2 in [3]) it is required that $u, v \in \mathcal{C}(\overline{\Omega})$ and $u \le 0 \le v$ on $\partial\Omega$. However, checking carefully the proof, it is only needed that, for any $k > 0$, there exists $\delta_k > 0$ such that $u - v - k \le 0$ in Ω_{δ_k}. This condition clearly holds as a consequence of the boundary condition (2.1).

Finally, we prove the following regularity result that is essentially contained in [14].

Proposition 2.3. *Assume that* (H_0) *is satisfied. Then every* $H^1_0(\Omega) \cap L^\infty(\Omega)$ *solution to* (P_λ) *belongs to* $\mathcal{C}^1_0(\overline{\Omega})$.

Proof. Let $\lambda \in \mathbb{R}$ and let $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ be a solution to (P_λ). Let us consider the following problem

$$
\begin{cases}
v - \Delta v = \mu(x)|\nabla v|^q + f(x) + (\lambda+1)u, & x \in \Omega,\\
v = 0, & x \in \partial\Omega.
\end{cases}
\tag{2.2}
$$

Obviously, u is a solution to (2.2). We aim to prove that there exists a $\mathcal{C}^1_0(\overline{\Omega})$ solution to (2.2), which by uniqueness (see Theorem 1.1 in [3]) coincides with u. In order to do so, let $\overline{u} \in \mathcal{C}^1_0(\overline{\Omega})$ be the unique solution, given by Theorem 2.2 in [14] and Theorem 1.1 in [3], to

$$
\begin{cases}
v - \Delta v = \mu(x)|\nabla v|^2 + \overline{g}(x), & x \in \Omega,\\
v = 0, & x \in \partial\Omega,
\end{cases}
$$

where $\overline{g} = |f| + |\lambda+1||u| + \mu \in L^p(\Omega)$. One readily shows that $\overline{u}$ is a supersolution to (2.2). Moreover, the maximum principle implies that $\overline{u} \gneq 0$.

On the other hand, let $\underline{u} \in \mathcal{C}^1_0(\overline{\Omega})$ be the unique solution to the linear problem

$$
\begin{cases}
v - \Delta v = -\underline{g}(x), & x \in \Omega,\\
v = 0, & x \in \partial\Omega,
\end{cases}
$$

where $\underline{g} = |f| + |\lambda+1||u| \in L^p(\Omega)$. It is easy to see that $\underline{u}$ is a subsolution to (P_λ) and that $\underline{u} \lneq 0$ by virtue of the maximum principle. In sum, Theorem 2.1 in [14] implies that there exists a solution $v \in \mathcal{C}^1_0(\overline{\Omega})$ to (2.2). $\square$

3. Local and boundary estimates

We dedicate this section to proving several results that will lead to the L^∞-estimate (1.4). We state such results in three subsections.

3.1. Local estimates on ∇u: the Bernstein method

We use the integral Bernstein method in order to demonstrate local estimates on ∇u in Lebesgue spaces. In the first result we assume a stronger hypothesis than (H_1) on the growth of μ, i.e.

$$\begin{cases} \mu \in W^{1,\infty}_{\text{loc}}(\Omega) \text{ and there exist } \sigma, \tau \geq 0, b_1, b_2 > 0 \text{ such that} \\ \mu \geq b_1 \varphi_1^\sigma, \quad |\nabla \mu| \leq b_2 \varphi_1^{-\tau}, \quad \text{in } \Omega. \end{cases} \tag{3.1}$$

Then, using that (H_1) implies (3.1) in compactly embedded subsets of Ω, we will deduce as a corollary the local estimates assuming only (H_1).

Theorem 3.1. *Let* $1 < q \leq 2$, $f \in L^N(\Omega)$ *and* $\mu \in L^\infty(\Omega)$ *satisfying* (3.1). *Then, for every*

$$\alpha > \frac{2\max\{\sigma + 1, 2\sigma + \tau\}}{q - 1} \tag{3.2}$$

and every $r > 0$, *there exists* $C > 0$ *such that*

$$\int_\Omega |\nabla u|^r \varphi_1^{\frac{r\alpha}{2}} \leq C, \tag{3.3}$$

for every non-negative $u \in W^{2,p}(\Omega)$ *(*$p > N$*) satisfying*

$$-\Delta u = \lambda u + \mu(x)|\nabla u|^q + f(x), \quad x \in \Omega$$

with $\lambda \in (0, \lambda_1)$.

Remark 3.2. We observe explicitly that the convexity of the domain Ω is not needed in the proof of this theorem.

Proof. In the proof, ε will denote a positive constant that can be taken as small as necessary, its value may vary from line to line. Furthermore, C will denote a positive (possibly large) constant whose value may vary from line to line too.

We divide the proof in two steps. In the first one, we assume further regularity on f and u in order to be able to differentiate the equation satisfied by $|\nabla u|^2$. In the second one, we clarify that the result is actually true for less regular f and u by performing a regularizing argument. This is a usual procedure when dealing with the Bernstein method.

Step 1: The case of further regularity

In addition to the hypotheses in the statement, let us assume that $u \in \mathcal{C}^3(\Omega)$ and also the following condition on f.

$$f \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \text{ and there exists } \gamma > 0 \text{ such that } \varphi_1^{\gamma}|\nabla f| \in L^{\infty}(\Omega). \tag{3.4}$$

Let us fix α satisfying (3.2). Let us denote $w = |\nabla u|^2$ and $\phi = \varphi_1^{\alpha}$. Notice that

$$|\nabla\phi|^2 \le C\phi^{\frac{2(\alpha-1)}{\alpha}}. \tag{3.5}$$

It is straightforward to check that

$$\begin{aligned} 2|D^2u|^2\phi - \Delta(w\phi) =& (2\lambda + \alpha\lambda_1)w\phi + \left(1+\frac{1}{\alpha}\right)\frac{|\nabla\phi|^2}{\phi}w \\ & - 2\frac{\nabla\phi}{\phi}\nabla(w\phi) + q\mu w^{\frac{q-2}{2}}\nabla u\nabla(w\phi) \\ & + 2w^{\frac{q}{2}}\phi\nabla\mu\nabla u - q\mu w^{\frac{q}{2}}\nabla u\nabla\phi + 2\phi\nabla u\nabla f, \end{aligned} \tag{3.6}$$

almost everywhere in Ω. Observe that (3.6) makes sense thanks to the facts that $u \in \mathcal{C}^3(\Omega)$ and $\mu, f \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

On the other hand, by Cauchy-Schwarz inequality, using the inequality $(a-b)^2 \ge \frac{a^2}{2} - 2b^2$ with $a = \lambda u + \mu(x)|\nabla u|^q$ and $b = -f$, and taking into account that $\lambda u \ge 0$, we get

$$|D^2u|^2 \ge \frac{1}{N}(-\Delta u)^2 \ge \frac{\mu^2(1-\varepsilon N)}{2N}|\nabla u|^{2q} - \frac{2(1-\varepsilon N)}{N}f^2 + \varepsilon(-\Delta u)^2. \tag{3.7}$$

Combining (3.6) and (3.7) leads to

$$\begin{aligned} \frac{\mu^2(1-\varepsilon N)}{N}w^q\phi + 2\varepsilon(-\Delta u)^2\phi - \Delta(w\phi) \le& (2\lambda + \alpha\lambda_1)w\phi \\ & + \left(1+\frac{1}{\alpha}\right)\frac{|\nabla\phi|^2}{\phi}w \\ & - 2\frac{\nabla\phi}{\phi}\nabla(w\phi) + q\mu w^{\frac{q-2}{2}}\nabla u\nabla(w\phi) + 2w^{\frac{q}{2}}\phi\nabla\mu\nabla u \\ & - q\mu w^{\frac{q}{2}}\nabla u\nabla\phi + 2\phi\nabla u\nabla f + \frac{4(1-\varepsilon N)}{N}f^2\phi. \end{aligned} \tag{3.8}$$

Now we multiply (3.8) by $(w\phi)^{\beta}$ for some $\beta > 1$ to be chosen later, and we integrate over Ω, so we obtain

$$\begin{aligned} & \frac{(1-\varepsilon N)}{N}\int_{\Omega}\mu^2 w^{\beta+q}\phi^{\beta+1} + 2\varepsilon\int_{\Omega}(-\Delta u)^2 w^{\beta}\phi^{\beta+1} + \beta\int_{\Omega}|\nabla(w\phi)|^2(w\phi)^{\beta-1} \\ & \le (2\lambda + \alpha\lambda_1)\int_{\Omega}(w\phi)^{\beta+1} + \left(1+\frac{1}{\alpha}\right)\int_{\Omega}w^{\beta+1}|\nabla\phi|^2\phi^{\beta-1} \\ & - 2\int_{\Omega}\nabla\phi\nabla(w\phi)w^{\beta}\phi^{\beta-1} - q\int_{\Omega}\mu w^{\beta+\frac{q}{2}}\phi^{\beta}\nabla u\nabla\phi \end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&+ q \int_\Omega \mu w^{\beta+\frac{q-2}{2}} \phi^\beta \nabla u \nabla (w\phi) + 2 \int_\Omega w^{\beta+\frac{q}{2}} \phi^{\beta+1} \nabla \mu \nabla u \\
&+ 2 \int_\Omega w^\beta \phi^{\beta+1} \nabla u \nabla f + \frac{4(1-\varepsilon N)}{N} \int_\Omega w^\beta \phi^{\beta+1} f^2.
\end{aligned}
$$

Notice that, by virtue of (3.4) and (3.1), the terms involving ∇f and $\nabla \mu$ are finite if β is taken large enough.

Next, we estimate the right-hand side of (3.9) term by term using Young inequality. We clarify that the terms that will appear depending on ε will be absorbed by one of the three positive terms in the left-hand side of (3.9), being ε a positive constant that can be chosen as small as needed. The remaining terms will have the form $C \int_\Omega w^\beta \phi^{\beta+1} f^2 + C$, where $C > 0$ is a possibly large constant that does not depend on f, λ nor u. We clarify that C may be taken independent on Ω as well, but depending on two fixed bounded domains Ω', Ω'' such that $\Omega' \subset\subset \Omega \subset\subset \Omega''$ (this will be used in Step 2).

Regarding the first term in (3.9), bearing in mind (3.1), we derive

$$
\begin{aligned}
(2\lambda + \alpha\lambda_1) \int_\Omega (w\phi)^{\beta+1} &\le \varepsilon \int_\Omega \mu^2 w^{\beta+q} \phi^{\beta+1} + C \int_\Omega \varphi_1^{\left(\alpha - \frac{2\sigma}{q-1}\right)(\beta+1)} \\
&= \varepsilon \int_\Omega \mu^2 w^{\beta+q} \phi^{\beta+1} + C.
\end{aligned}
$$

About the second one, taking (3.5) into account, we obtain

$$
\begin{aligned}
\left(1 + \frac{1}{\alpha}\right) \int_\Omega w^{\beta+1} |\nabla \phi|^2 \phi^{\beta-1} &\le C \int_\Omega \phi^{\frac{2(\alpha-1)}{\alpha}-2} w^{\beta+1} \phi^{\beta+1} \\
&\le \varepsilon \int_\Omega \mu^2 w^{\beta+q} \phi^{\beta+1} + C \int_\Omega \varphi_1^{(\beta+1)\left(\alpha - \frac{2(\sigma+1)}{q-1}\right)-2}.
\end{aligned}
\tag{3.10}
$$

Hence, we take β large enough in order to have

$$
\int_\Omega \varphi_1^{(\beta+1)\left(\alpha - \frac{2(\sigma+1)}{q-1}\right)-2} < \infty.
$$

We estimate the third term in the following way:

$$
-2 \int_\Omega \nabla \phi \nabla (w\phi) w^\beta \phi^{\beta-1} \le \varepsilon \int_\Omega |\nabla (w\phi)|^2 (w\phi)^{\beta-1} + C \int_\Omega w^{\beta+1} |\nabla \phi|^2 \phi^{\beta-1},
$$

and the second part of the right-hand side term of the previous inequality can be controlled as in (3.10).

Similarly, we treat the fourth term as follows:

$$
\begin{aligned}
-q \int_\Omega \mu w^{\beta+\frac{q}{2}} \phi^\beta \nabla u \nabla \phi &\le q \int_\Omega \mu w^{\beta+\frac{q+1}{2}} |\nabla \phi| \phi^\beta \\
&\le \varepsilon \int_\Omega \mu^2 w^{\beta+q} \phi^{\beta+1} + C \int_\Omega w^{\beta+1} |\nabla \phi|^2 \phi^{\beta-1},
\end{aligned}
$$

so we are reduced again to (3.10).

Let us deal with the fifth term:

$$
\begin{aligned}
q \int_\Omega \mu w^{\beta+\frac{q-2}{2}} \phi^\beta \nabla u \nabla (w\phi) &\le (\beta - 3\varepsilon) \int_\Omega |\nabla (w\phi)|^2 (w\phi)^{\beta-1} \\
&\quad + \frac{q^2}{4(\beta-3\varepsilon)} \int_\Omega \mu^2 w^{\beta+q} \phi^{\beta+1}.
\end{aligned}
$$

Hence, taking β even larger if necessary, the fifth term is absorbed by the left-hand side of (3.9).

We consider now the sixth term:

$$
\begin{aligned}
2 \int_\Omega w^{\beta+\frac{q}{2}} \phi^{\beta+1} \nabla \mu \nabla u &\le 2 \int_\Omega |\nabla \mu| w^{\frac{2\beta+q+1}{2}} \phi^{\beta+1} \\
&\le \varepsilon \int_\Omega \mu^2 w^{\beta+q} \phi^{\beta+1} + C \int_\Omega |\nabla \mu|^{\frac{2(\beta+q)}{q-1}} \mu^{-\frac{2(2\beta+q+1)}{q-1}} \phi^{\beta+1} \\
&\le \varepsilon \int_\Omega \mu^2 w^{\beta+q} \phi^{\beta+1} + C \int_\Omega \varphi_1^{\left(\alpha - \frac{4\sigma+2\tau}{q-1}\right)(\beta+1) - 2(\sigma+\tau)}.
\end{aligned}
$$

Once more, we choose β large enough so that

$$
\int_\Omega \varphi_1^{\left(\alpha - \frac{4\sigma+2\tau}{q-1}\right)(\beta+1) - 2(\sigma+\tau)} < \infty.
$$

In order to control the seventh term, we integrate by parts, resulting

$$
\begin{aligned}
2 \int_\Omega w^\beta \phi^{\beta+1} \nabla u \nabla f &= -2 \int_\Omega \operatorname{div}(w^\beta \phi^{\beta+1} \nabla u) f \\
&= 2 \int_\Omega \left(w^\beta \phi^{\beta+1} (-\Delta u) - \beta w^{\beta-1} \phi^\beta \nabla (w\phi) \nabla u - (w\phi)^\beta \nabla u \nabla \phi \right) f.
\end{aligned}
$$

We treat these last three terms separately. On the one hand,

$$2\int_\Omega w^\beta\phi^{\beta+1}(-\Delta u)f \le \varepsilon\int_\Omega(-\Delta u)^2 w^\beta\phi^{\beta+1} + C\int_\Omega w^\beta\phi^{\beta+1}f^2.$$

On the other hand,

$$-2\beta\int_\Omega w^{\beta-1}\phi^\beta\nabla(w\phi)\nabla u f \le \varepsilon\int_\Omega|\nabla(w\phi)|^2(w\phi)^{\beta-1} + C\int_\Omega w^\beta\phi^{\beta+1}f^2.$$

And lastly,

$$-2\int_\Omega(w\phi)^\beta\nabla u\nabla\phi f \le C\int_\Omega w^\beta\phi^{\beta+1}f^2 + C\int_\Omega w^{\beta+1}|\nabla\phi|^2\phi^{\beta-1},$$

where we recall that this last integral can be dealt with as in (3.10).

Then, combining all the estimates for the right-hand side terms in (3.9) we obtain,

$$\left(\frac{1}{N} - O(\varepsilon)\right)\int_\Omega \mu^2 w^{\beta+q}\phi^{\beta+1} + \varepsilon\int_\Omega(-\Delta u)^2 w^\beta\phi^{\beta+1} + \\ +\varepsilon\int_\Omega|\nabla(w\phi)|^2(w\phi)^{\beta-1} \le C\int_\Omega w^\beta\phi^{\beta+1}f^2 + C,$$

where dropping the first two positive terms in the left-hand side we arrive to

$$\int_\Omega|\nabla(w\phi)^{\frac{\beta+1}{2}}|^2 \le C\int_\Omega w^\beta\phi^{\beta+1}f^2 + C.$$

Moreover, Sobolev inequality yields

$$\left(\int_\Omega(w\phi)^{\frac{N(\beta+1)}{N-2}}\right)^{\frac{N-2}{N}} \le C\int_\Omega w^\beta\phi^{\beta+1}f^2 + C.$$

Next, we apply Hölder inequality and derive

$$\left(\int_\Omega(w\phi)^{\frac{N(\beta+1)}{N-2}}\right)^{\frac{N-2}{N}} \le C\left(\int_\Omega(w\phi)^{\frac{N(\beta+1)}{N-2}}\right)^{\frac{\beta(N-2)}{(\beta+1)N}}\left(\int_\Omega|f|^{p_\beta}\phi^{\frac{N(\beta+1)}{N+2\beta}}\right)^{\frac{N+2\beta}{N(\beta+1)}} + C,$$

where $p_\beta = \frac{2N(\beta+1)}{N+2\beta}$. Observe that $p_\beta < N$ for every $\beta > 0$.

We now apply Young inequality and arrive at

$$\left(\int_\Omega (w\phi)^{\frac{N(\beta+1)}{N-2}}\right)^{\frac{N-2}{N}} \le \varepsilon \left(\int_\Omega (w\phi)^{\frac{N(\beta+1)}{N-2}}\right)^{\frac{N-2}{N}} + C\left(\int_\Omega f^{p_\beta}\phi^{\frac{N(\beta+1)}{N+2\beta}}\right)^{\frac{N+2\beta}{N}} + C.$$

In conclusion, for every β large enough, the following estimate holds:

$$\left(\int_\Omega (w\phi)^{\frac{N(\beta+1)}{N-2}}\right)^{\frac{N-2}{N}} \le C\left(\int_\Omega |f|^{p_\beta}\phi^{\frac{N(\beta+1)}{N+2\beta}}\right)^{\frac{N+2\beta}{N}} + C. \tag{3.11}$$

Thus, since the right-hand side of the previous inequality is bounded and $w = |\nabla u|^2$ and $\phi = \varphi_1^\alpha$, we obtain (3.3) with $r = \frac{2N(\beta+1)}{N-2}$. The restrictions on β allow to assure that (3.3) is true for $r > \frac{2N(\beta+1)}{N-2}$ and then, by Hölder inequality, for every $r > 0$.

Step 2: The general case.

Now, our objective is to prove the result (in the generality of the statement) by applying Step 1. In order to do that, we consider a sequence $\{\Psi_n\}$ of standard non-negative compactly supported regularizing kernels. The usual example is $\Psi_n(x) = Cn^N\Psi(nx)$, where $\Psi(x) = e^{\frac{1}{|x|^2-1}}$ for $|x| < 1$, $\Psi(x) = 0$ for $|x| \ge 1$ and $C = 1/\|\Psi\|_{L^1(\mathbb{R}^N)}$. Thus, we get $0 \le \Psi_n \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, $\mathrm{supp}(\Psi_n) = \overline{B(0, 1/n)}$, $\|\Psi_n\|_{L^1(\mathbb{R}^N)} = 1$.

Denoting again by u, f, μ their extensions by zero outside Ω, we may consider the functions $u_n(x) = (\Psi_n \star u)(x) = \int_{\mathbb{R}^N} \Psi_n(x-y)u(y)dy$ and $f_n = \Psi_n \star f$. Recall that $u_n, f_n \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ and both have support contained in $\overline{\Omega + B(0, 1/n)}$. Moreover, using that $u \in W^{2,p}(\Omega)$, we have

$$\begin{aligned}\Delta u_n(x) =&(\Delta\Psi_n \star u)(x) = \int_{\mathbb{R}^N} \Delta\Psi_n(x-y)u(y)dy\\ &= \int_\Omega \Delta\Psi_n(x-y)u(y)dy = \int_\Omega \Psi_n(x-y)\Delta u(y)dy\\ &- \int_{\partial\Omega\cap B(x,\frac{1}{n})} \left(u(y)\frac{\partial\Psi_n}{\partial\nu}(x-y) + \frac{\partial u}{\partial\nu}(y)\Psi_n(x-y)\right)d\sigma_y.\end{aligned}$$

In particular, $\Delta u_n(x) = (\Psi_n \star \Delta u)(x)$ for every $x \in \Omega\setminus\Omega_{1/n}$. Thus, given $\omega \subset\subset \Omega$, we have that $\omega \subset \Omega\setminus\Omega_{1/n}$ for n large enough and then, in ω, it is satisfied that

$$\begin{aligned}-\Delta u_n &= \lambda u_n + \Psi_n \star \left(\mu(x)|\nabla u|^q\right) + f_n\\ &= \lambda u_n + \mu(x)|\nabla u_n|^q + g_n,\end{aligned}$$

where $g_n := \left[f_n + \Psi_n \star (\mu(x)|\nabla u|^q) - \mu(x)|\nabla u_n|^q\right]$.

Observe that $g_n \in L^N(\omega)$. Furthermore, hypotheses (3.1) and (3.4) are satisfied in ω for $\varphi_{1,\omega}$ the first eigenfunction in ω. Therefore, we can apply Step 1 to get

$$\left(\int_\omega (w_n \varphi_{1,\omega}^\alpha)^{\frac{N(\beta+1)}{N-2}}\right)^{\frac{N-2}{N}} \le C\left(\int_\omega |g_n|^{p_\beta} \varphi_{1,\omega}^{\alpha\frac{N(\beta+1)}{N+2\beta}}\right)^{\frac{N+2\beta}{N}} + C, \tag{3.12}$$

with $w_n = |\nabla u_n|^2$ and with $C > 0$ independent on n.

Now we use the convergence properties of the convolution, i.e. $\Psi_n \star z$ converges to z in $L^r(\mathbb{R}^N)$ whenever $z \in L^r(\mathbb{R}^N)$, $1 \le r < \infty$, and $\Psi_n \star z$ converges uniformly to z in compact sets of Ω' whenever $z \in \mathcal{C}(\Omega')$, being Ω' any open set in $\mathbb{R}^N$. Indeed, on the one hand, recall that $\nabla u_n = \Psi_n \star \nabla u$ in $\Omega \setminus \Omega_{1/n}$ and, by the Sobolev embeddings, $\nabla u \in \mathcal{C}(\Omega \setminus \overline{\Omega}_{1/n})^N$. Therefore, w_n converges uniformly to $w = |\nabla u|^2$ in ω. Furthermore, we also have that $\mu|\nabla u|^q \in \mathcal{C}(\Omega)$, so $g_n - f_n \to 0$ uniformly in ω too. On the other hand, since $f \in L^N(\mathbb{R}^N)$, it follows that $f_n \to f$ in $L^N(\mathbb{R}^N)$.

Thus, we can pass to the limit in (3.12) and we get

$$\left(\int_\omega (w \varphi_{1,\omega}^\alpha)^{\frac{N(\beta+1)}{N-2}}\right)^{\frac{N-2}{N}} \le C\left(\int_\omega |f|^{p_\beta} \varphi_{1,\omega}^{\alpha\frac{N(\beta+1)}{N+2\beta}}\right)^{\frac{N+2\beta}{N}} + C.$$

Finally we get (3.11) when ω tends to Ω taking into account the convergence of $\varphi_{1,\omega}$ to φ_1 and the fact that C does not depend on ω. In conclusion, Step 1 continues to hold for general solutions $u \in W^{2,p}(\Omega)$ $(p > N)$ and less regular datum f. The result is now proved. □

As a consequence of Theorem 3.1 applied in open subsets compactly embedded in Ω we can improve the local estimates using (H_1) instead of (3.1).

Corollary 3.3. *Assume* (H_0) *and* (H_1). *Then, for every $r > 0$ and every $\omega \subset\subset \Omega$, there exists $C_\omega > 0$ such that every solution $u \in \mathcal{C}_0^1(\overline{\Omega})$ to* (P_λ), $\lambda \in (0, \lambda_1)$ *satisfies*

$$\int_\omega |\nabla u|^r \le C_\omega.$$

Proof. Let $\tilde{\omega} \subset\subset \Omega$ be open such that $\omega \subset\subset \tilde{\omega}$. We can apply Theorem 3.1 in $\tilde{\omega}$. Observe that (H_1) implies that (3.1) is satisfied in $\tilde{\omega}$ for b_1 small and b_2 large enough. □

3.2. Local estimates on λu

This subsection is devoted to establishing some local estimates on λu that we will need in the proof of Theorem 1.1.

The following result is essentially proved in [31] (see also [28]). However, since we need an estimate showing an explicit dependence on λ, we include the proof for the sake of clarity.

Proposition 3.4. *Let $1 < q \le 2$ and $0 \lneq f \in L^1(\Omega)$. Assume that $\mu \in L^\infty(\Omega)$ and $\inf_\omega(\mu) > 0$ for every open set $\omega \subset\subset \Omega$. Then, for every $m \in [1, (N+1)/(N-1))$, there exists $C > 0$ such that*

$$\lambda \left(\int_\Omega u^m \varphi_1 \right)^{\frac{1}{m}} \le C$$

for every $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solution to (P_λ) *with $\lambda \in (0, \lambda_1)$.*

Proof. First of all we claim that, for every open set $\omega \subset\subset \Omega$, there exists $c_\omega > 0$ such that

$$\lambda \int_\omega u \le c_\omega, \tag{3.13}$$

for every solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (P_λ) with $\lambda \in (0, \lambda_1)$. Recall that $u > 0$ in Ω.

Indeed, let $\phi \in \mathcal{C}_c^1(\Omega)$ be such that $\omega \subset\subset \operatorname{supp}(\phi)$, $0 \le \phi \le 1$ in Ω and $\phi = 1$ in ω. Let us denote $K = \operatorname{supp}(\phi)$. Multiplying the equation in (P_λ) by ϕ^β for some $\beta > 1$, integrating by parts and using Young inequality, we deduce

$$\begin{aligned} \int_\Omega (\lambda u + \mu(x)|\nabla u|^q + f(x))\phi^\beta &= \beta \int_\Omega \phi^{\beta-1} \nabla u \nabla \phi \\ &\le \frac{\inf_K(\mu)}{2} \int_\Omega |\nabla u|^q \phi^\beta + C \int_\Omega |\nabla \phi|^{\frac{q}{q-1}} \phi^{\beta - \frac{q}{q-1}}, \end{aligned}$$

for some constant $C > 0$ depending on $\inf_K(\mu)$ and q. We now choose $\beta = q/(q-1)$ so that the last integral in the previous inequality is finite. At this point, (3.13) easily follows.

Next, Lemma 3.2 in [8] (see also Lemma 3.9 in [28]) implies that there exists $C > 0$ such that

$$u \ge C\varphi_1 \int_\Omega (-\Delta u)\varphi_1 dx, \tag{3.14}$$

for every solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (P_λ) with $\lambda \in (0, \lambda_1)$. Multiplying (3.14) by λ, integrating over ω and taking (3.13) into account, we derive

$$\int_\Omega (-\Delta(\lambda u))\varphi_1 dx \le C.$$

Finally, applying Proposition 2.2 in [18], the result follows. □

Combining Proposition 3.4 and Corollary 3.3 in a bootstrap argument, we are able to prove the following result.

Corollary 3.5. *Assume that* (H_0) *and* (H_1) *are satisfied. Then, for every* $\omega \subset\subset \Omega$, *there exists* $C_\omega > 0$ *such that*

$$\lambda \|u\|_{W^{2,p}(\omega)} \leq C_\omega, \tag{3.15}$$

for every $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *solution to* (P_λ) *with* $\lambda \in (0, \lambda_1)$.

Proof. For every n, let $\lambda_n \in (0, \lambda_1)$ and let $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution to (P_{λ_n}). Recall that $u_n > 0$ in Ω and $u_n \in \mathcal{C}_0^1(\overline{\Omega})$ by Proposition 2.3. Consider an open set $\omega \subset\subset \Omega$. By virtue of Corollary 3.3 and Proposition 3.4, we immediately deduce that $\{\lambda_n u_n\}$ is bounded in $W^{1,m}(\omega)$ and $\{-\Delta u_n\}$ is bounded in $L^m(\omega)$ for some $m > 1$ (in particular, $\{-\Delta(\lambda_n u_n)\}$ is bounded in $L^m(\omega)$). Then, elliptic regularity (see for instance Problem 3.3, p. 202 in [32]) implies that $\{\lambda_n u_n\}$ is bounded in $W^{2,m}(\omega_1)$ for any $\omega_1 \subset\subset \omega$. If $m \geq p$, then the proof is finished. Otherwise, from the Sobolev embeddings, it follows that $\{\lambda_n u_n\}$ is bounded in $W^{1,m_1}(\omega_1)$, where $m_1 = m^*$. We may repeat the arguments and conclude that $\{\lambda_n u_n\}$ is bounded in $W^{2,m_2}(\omega_2)$ for any $\omega_2 \subset\subset \omega_1$, where $m_2 = m_1^*$. In a finite number of steps, we arrive at some $m_i \geq p$, so that $\{\lambda_n u_n\}$ is bounded in $W^{2,p}(\omega_i)$ for every $\omega_i \subset\subset \omega_{i-1}$. Since ω was arbitrary, we have proved (3.15). □

3.3. Boundary estimate: the moving plane method

Now, we deduce an L^∞ estimate of u near the boundary of Ω using the moving plane method. We will use the following version of Lemma 2.2 in [19].

Lemma 3.6. *Assume that* Ω *is a convex set and that conditions* (H_0) *and* (H_2) *are satisfied. Let* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *be a solution to* (P_λ) *for some* $\lambda \in (0, \lambda_1)$ *such that, for some* $x_0 \in \partial\Omega$ *and some* $\eta \in [\eta_0 - \delta_0/2, \eta_0)$, *it is satisfied that*

$$\frac{\partial u}{\partial \nu}(x) \leq 0, \text{ and } u(x) \leq u(x_{\nu,\eta}),\ u(x) \not\equiv u(x_{\nu,\eta}),\ x \in \Sigma_{\nu,\eta},$$

where $\nu = \nu(x_0)$ *denotes the exterior normal vector at* x_0. *Then,* $u(x) < u(x_{\nu,\eta})$ *for every* $x \in \Sigma_{\nu,\eta}$ *and* $\frac{\partial u}{\partial \nu}(x) < 0$ *for every* $x \in \Omega \cap \partial\Sigma_{\nu,\eta}$.

Proof. The proof is standard and we only illustrate how we use hypothesis (H_2). We write $\Sigma'_{\nu,\eta}$ to denote the reflection of $\Sigma_{\nu,\eta}$ along the hyperplane $x \cdot \nu(x_0) = \eta$. Observe that $v(x) = u(x_{\nu,\eta})$ satisfies in $\Sigma'_{\nu,\eta}$

$$-\Delta v = \lambda v + \mu(x_{\nu,\eta})|\nabla v|^q + f(x_{\nu,\eta}).$$

In particular, using (H_2) and the convexity of $\xi \to |\xi|^q$ for $q > 1$, we have

$$\begin{aligned} -\Delta(u - v) =& \lambda(u - v) + \mu(x)|\nabla u|^q - \mu(x_{\nu,\eta})|\nabla v|^q + f(x) - f(x_{\nu,\eta}) \\ \geq& \lambda(u - v) + \mu(x)\left(|\nabla u|^q - |\nabla v|^q\right) \\ \geq& \lambda(u - v) - B(x) \cdot \nabla(u - v), \end{aligned}$$

where $B(x) = -q\mu(x)|\nabla v(x)|^{q-2}\nabla v(x) \in L^\infty(\Sigma'_{\nu,\eta})^N$. Thus, the proof follows as in [19]. □

Finally, the local estimates in the previous subsection and Lemma 3.6 allow to deal with the usual moving plane method to deduce that the solutions to (P_λ) attain their maximum at a point at distance to the boundary at least $\delta_0/2$. The proof is standard (see [15]), so we only show a sketch of the proof.

Theorem 3.7. *Assume that Ω is convex and that* (H_0) *and* (H_2) *are satisfied. Let $u \in C_0^1(\overline{\Omega})$ be a solution to* (P_λ) *for any $\lambda \in (0, \lambda_1)$. Then*

$$\max_{\overline{\Omega}} u = \max_{\Omega \setminus \Omega_{\delta_0/2}} u.$$

Sketch of the proof. First we observe that given $x_0 \in \partial\Omega$, for some small $\varepsilon \in (0, \delta_0/2)$ and every $\eta \in [\eta_0 - \varepsilon, \eta_0)$, it is satisfied that

$$\frac{\partial u}{\partial \nu(x_0)}(x) < 0, \text{ and } u(x) < u(x_{\nu,\eta}), \ x \in \Sigma_{\nu(x_0),\eta}.$$

Using Lemma 3.6 we can argue as in the proof of Theorem 2.1 in [19] to deduce that the supremum of the values of $\varepsilon \in (0, \delta_0/2)$ satisfying this property is always $\delta_0/2$ (otherwise Lemma 3.6 leads to a contradiction). In particular, we deduce that

$$u(x) \le u(x_{\nu(x_0),\eta}), \ x \in \Sigma_{\nu(x_0),\eta}, \ \eta \in (\eta_0 - \delta_0/2, \eta_0).$$

Thus, for every $x \in \Omega_{\delta_0/2}$, there exists $\Sigma_{\nu(x_0),\eta}$ such that $x \in \Sigma_{\nu(x_0),\eta}$, $x_{\nu(x_0),\eta} \notin \Omega_{\delta_0/2}$ and $u(x) \le u(x_{\nu(x_0),\eta})$ which implies that

$$\max_{\overline{\Omega}_{\delta_0/2}} u \le \max_{\Omega \setminus \Omega_{\delta_0/2}} u,$$

and this completes the proof. $\square$

4. Proof of the main results

Proof of Theorem 1.1. Let $\lambda \in (0, \lambda_1)$ and let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution to (P_λ). First of all, Theorem 3.7 implies that $\|u\|_{L^\infty(\Omega)} = \|u\|_{L^\infty(\Omega')}$, where $\Omega' = \Omega \setminus \overline{\Omega}_{\delta_0/2}$. On the other hand, from Corollary 3.5 and the Sobolev embeddings it follows that $\lambda \|u\|_{L^\infty(\Omega')} \le C$ for some $C > 0$ independent of u and λ. Therefore, (1.4) holds. $\square$

Proof of Theorem 1.2. It is a consequence of Theorem 2.1 and Theorem 1.1. The proof is standard, see [4]. $\square$

Proof of Theorem 1.3. We divide the proof in four steps, in the first one we prove the singular behavior of unbounded sequences of bounded solutions. The second step describes the behavior of solutions to (1.6) near the boundary. The third step deals with the uniqueness of the ergodic constant and in the fourth step we prove the uniqueness up to addition of constants of solution to (1.6).

Step 1: Singular behavior. Let $\omega \subset\subset \Omega$. Bearing (3.15) in mind, there exists $u \in W^{1,\infty}(\omega)$ such that, passing to a subsequence, $\lambda_n u_n \to u$ in $W^{1,\infty}(\omega)$. On the other hand, from Theorem 3.1, we deduce that

$$\lambda_n |\nabla u_n| \to 0 \quad \text{in } L^r(\omega), \text{ for every } r \in [1, \infty).$$

The uniqueness of the limits implies that $u \equiv c_0$ for some constant $c_0 \geq 0$.

Let us denote $v_n = u_n - \|u_n\|_{L^\infty(\Omega)}$ and let $x_n \in \Omega$ be such that $u_n(x_n) = \|u_n\|_{L^\infty(\Omega)}$. We claim that $\{v_n\}$ is bounded in $L^s(\omega)$ for every $s \in (1, \infty)$. Indeed, Theorem 3.7 implies that $x_n \in \Omega'$ for every n, where $\Omega' = \Omega \setminus \overline{\Omega}_{\delta_0/2}$. Since Ω is convex, there exists an open convex set ω' such that $\Omega' \cup \omega \subset\subset \omega' \subset\subset \Omega$. Hence, for every $x \in \omega'$ we have

$$v_n(x) = u_n(x) - u_n(x_n) = \int_0^1 \frac{d}{dt} u_n\,(tx + (1-t)x_n)\,dt$$

$$= \int_0^1 (x - x_n) \cdot \nabla u_n\,(tx + (1-t)x_n)\,dt.$$

In particular, using Hölder inequality,

$$|v_n(x)|^s \leq |x - x_n|^s \int_0^1 |\nabla u_n\,(tx + (1-t)x_n)\,|^s dt.$$

Now we take $R > 0$ large enough, independent of n, such that $\omega' \subset B_R(x_n)$ for every $n \in \mathbb{N}$ and we extend by zero the function ∇u_n outside ω'. Thus,

$$\begin{aligned}
\int_\omega |v_n(x)|^s dx &\leq \int_{\omega'} |v_n(x)|^s dx \\
&\leq \int_{\omega'} |x - x_n|^s \left(\int_0^1 |\nabla u_n\,(tx + (1-t)x_n)\,|^s dt \right) dx \\
&\leq \int_{B_R(x_n)} |x - x_n|^s \left(\int_0^1 |\nabla u_n\,(tx + (1-t)x_n)\,|^s dt \right) dx \\
&= \int_0^1 \left(\int_{B_R(x_n)} |x - x_n|^s |\nabla u_n\,(tx + (1-t)x_n)\,|^s dx \right) dt.
\end{aligned}$$

We perform now the change of variable $z = tx + (1-t)x_n$ in the last integral and we reach

$$
\begin{aligned}
\int_{\omega} |v_n(x)|^s dx \leq & \int_0^1 \left(\int_{B_{tR}(x_n)} \frac{|z-x_n|^s}{t^s} |\nabla u_n(z)|^s \frac{1}{t^N} dz \right) dt \\
= & \int_0^1 \left(\int_0^{tR} \frac{1}{t^{N+s}} \left(\int_{|z-x_n|=r} r^s |\nabla u_n(z)|^s d\sigma \right) dr \right) dt \\
= & \int_0^R \left(\int_{r/R}^1 \frac{1}{t^{N+s}} \left(\int_{|z-x_n|=r} r^s |\nabla u_n(z)|^s d\sigma \right) dt \right) dr \\
= & \frac{1}{N+s-1} \int_0^R \left(\frac{R^{N+s-1}}{r^{N+s-1}} - 1 \right) \left(\int_{|z-x_n|=r} r^s |\nabla u_n(z)|^s d\sigma \right) dr \\
= & \frac{1}{N+s-1} \int_{B_R(x_n)} \left(\frac{R^{N+s-1} - |z-x_n|^{N+s-1}}{|z-x_n|^{N-1}} \right) |\nabla u_n(z)|^s dz.
\end{aligned}
$$

Now we use Hölder inequality for some $1 < \alpha < 1 + \frac{1}{N-1}$ and, taking into account the fact that $\frac{R^{N+s-1}-|y|^{N+s-1}}{|y|^{N-1}} \in L^{\alpha}(B_R(0))$ we deduce that

$$
\begin{aligned}
\int_{\omega} |v_n(x)|^s dx \leq & \frac{1}{N+s-1} \left(\int_{B_R(x_n)} \left(\frac{R^{N+s-1} - |z-x_n|^{N+s-1}}{|z-x_n|^{N-1}} \right)^{\alpha} dz \right)^{\frac{1}{\alpha}} \cdot \\
& \cdot \left(\int_{B_R(x_n)} |\nabla u_n(z)|^{\frac{s\alpha}{\alpha-1}} dz \right)^{1-\frac{1}{\alpha}} \\
= & \frac{1}{N+s-1} \left(\int_{B_R(0)} \left(\frac{R^{N+s-1} - |y|^{N+s-1}}{|y|^{N-1}} \right)^{\alpha} dy \right)^{\frac{1}{\alpha}} \cdot \\
& \cdot \left(\int_{B_R(x_n)} |\nabla u_n(z)|^{\frac{s\alpha}{\alpha-1}} dz \right)^{1-\frac{1}{\alpha}} \\
= & C \left(\int_{B_R(x_n)} |\nabla u_n(z)|^{\frac{s\alpha}{\alpha-1}} dz \right)^{1-\frac{1}{\alpha}} = C \left(\int_{\omega'} |\nabla u_n(z)|^{\frac{s\alpha}{\alpha-1}} dz \right)^{1-\frac{1}{\alpha}}.
\end{aligned}
$$

Therefore, the claim follows by virtue of Theorem 3.1.

Next, since $|\nabla v_n| = |\nabla u_n|$ and $\Delta v_n = \Delta u_n = -\lambda_n u_n - \mu(x)|\nabla u_n|^q - f(x)$, we have by Corollary 3.5 that $\{v_n\}$ is bounded in $W^{2,p}_{\text{loc}}(\Omega)$. Then, there exists a function $v \in W^{1,\infty}_{\text{loc}}(\Omega)$ such that $v_n \to v$ in $W^{1,\infty}_{\text{loc}}(\Omega)$. Moreover, since $\max_{x\in\Omega} v_n(x) = 0$ we have that $\max_{x\in\Omega} v(x) = 0$. It is now straightforward to pass to the limit (locally) in the equation for v_n.

We will next justify that (1.5) holds. More precisely, we will prove that

$$\lim_{n\to\infty} \int_\Omega \mu(x)|\nabla u_n|^q \varphi_1 = \infty. \tag{4.1}$$

If this is true, by Brezis-Cabré lemma [8] (see also [28]) we have

$$u_n \geq C\varphi_1 \int_\Omega (\lambda_n u_n + \mu(x)|\nabla u_n|^q + f(x))\varphi_1 dx \quad \text{in } \Omega,$$

and (1.5) follows directly. Let us prove now that (4.1) holds. Otherwise, and bearing in mind Proposition 3.4, we deduce that $\{(-\Delta u_n)\varphi_1\}$ is bounded in $L^1(\Omega)$, taking possibly a subsequence. Therefore, Proposition 2.2 in [18] implies that $\{u_n\}$ is bounded in $L^r_{\text{loc}}(\Omega)$ for some $r > 1$. This contradicts the fact that $\{u_n\}$ diverges locally uniformly (observe that $u_n = v_n + \|u_n\|_{L^\infty(\Omega)}$ and v_n is locally bounded).

Let us finally show that $v \to -\infty$ as $x \to \partial\Omega$. To this aim, observe first that

$$-\Delta v_n \leq a|\nabla v_n|^q + b, \quad x \in \Omega_{\delta_0}, \tag{4.2}$$

where

$$a = \|\mu\|_{L^\infty(\Omega)}, \quad b = \sup_{n\in\mathbb{N}} \lambda_n \|u_n\|_{L^\infty(\Omega)} + \sup_{x\in\overline{\Omega}_{\delta_0}} f(x).$$

We claim that there exists a function that blows up near $\partial\Omega$ and satisfies the reverse inequality (4.2) in some $\Omega_\delta \subset \Omega_{\delta_0}$, in such a way that this function lies above v_n. In order to prove the claim, we argue as in [29]. First, we take $\delta_0 > 0$ in (H_2) to be small enough so that

$$d \in \mathcal{C}^2(\overline{\Omega}_{\delta_0}), \quad |\nabla d| = 1 \text{ in } \Omega_{\delta_0}, \tag{4.3}$$

where, recall, $d(x) = \text{dist}(x, \partial\Omega)$. This can be done thanks to the $\mathcal{C}^2$ regularity of $\partial\Omega$ (see Lemma 14.16 in [20], for instance). Now we distinguish between the two cases $q < 2$ and $q = 2$. Assuming in the first place that $q < 2$, let us consider the function

$$\psi_n(x) = M_n - \sigma\left(d(x) + \frac{1}{m}\right)^{-\alpha}$$

defined in Ω_{δ_0}, where

$$\alpha = \frac{2-q}{q-1}, \quad M_n = \sigma\left(\delta + \frac{1}{m}\right)^{-\alpha} + \max_{d(x)=\delta} v_n(x),$$

and $m, \sigma, \delta > 0$ are constants to be chosen suitably but independently of n. Direct calculations show that, in Ω_{δ_0}, the following is satisfied

$$\begin{aligned} &\Delta\psi_n + a|\nabla\psi_n|^q + b \\ &= -\alpha\sigma\left(d(x)+\frac{1}{m}\right)^{-(\alpha+2)}\left(\alpha+1-a(\alpha\sigma)^{q-1}-\left(d(x)+\frac{1}{m}\right)\Delta d(x)\right)+b, \end{aligned} \tag{4.4}$$

where we have used (4.3) and the fact that $\alpha+2 = q(\alpha+1)$. Now, recalling that $\Delta d \in L^\infty(\Omega_{\delta_0})$, we choose σ, δ and $1/m$ small enough so that

$$\alpha+1-a(\alpha\sigma)^{q-1}-\left(d(x)+\frac{1}{m}\right)\Delta d(x) > 0, \quad x \in \Omega_\delta.$$

Combining this last inequality with (4.4) and taking δ and $1/m$ even smaller if necessary, we arrive at

$$-\Delta\psi_n \geq a|\nabla\psi_n|^q + b, \quad x \in \Omega_\delta.$$

Moreover, using that $\max_{d(x)=\delta} u_n(x)$ tends to infinity as n tends to infinity, it is easy to see that $\psi_n \geq v_n$ on $\partial\Omega_\delta$. Thus, Theorem 2.2 implies that $\psi_n \geq v_n$ in Ω_δ. Since $\{M_n\}$ is bounded (as a consequence of the uniform local boundedness of $\{v_n\}$), there exists a constant $C \in \mathbb{R}$, independent of n, such that

$$v_n \leq C - \sigma\left(d(x)+\frac{1}{m}\right)^{-\alpha}, \quad x \in \Omega_\delta.$$

Taking limits as $n \to \infty$, and as $m \to \infty$ afterwards, yields

$$v \leq C - \sigma d(x)^{-\alpha}, \quad x \in \Omega_\delta,$$

which confirms our claim in the case $q < 2$. Considering now the case $q = 2$, we define in Ω_{δ_0} the function

$$\psi_n(x) = M_n + \sigma\log\left(d(x)+\frac{1}{m}\right),$$

where

$$M_n = -\sigma\log\left(\delta+\frac{1}{m}\right) + \max_{d(x)=\delta} v_n(x),$$

and, again, $m, \sigma, \delta > 0$ are constants to be chosen suitably but independently of n. As above, direct calculations show that, in Ω_{δ_0}, the following is satisfied

$$\begin{aligned} &\Delta\psi_n + a|\nabla\psi_n|^2 + b \\ &= -\sigma\left(d(x)+\frac{1}{m}\right)^{-2}\left(1-a\sigma-\left(d(x)+\frac{1}{m}\right)\Delta d(x)\right)+b. \end{aligned}$$

It is now clear that the proof of the claim in this case follows analogously to the case $q < 2$.

Step 2: Behavior near the boundary for the ergodic problem. Before dealing with the uniqueness of the ergodic constant c_0 and the uniqueness of solution to (1.6) up to additions of constants, we start by justifying that every locally Lipschitz solution to (1.6) has the same behavior close to $\partial\Omega$. For that, we follow the arguments of Theorem 2.15 in [25].

We point out that (1.7) is equivalent to the fact that, for any $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0, \delta_0)$ such that

$$(b-\varepsilon)d(x)^\sigma < \mu(x) < (b+\varepsilon)d(x)^\sigma, \quad x \in \Omega_{\delta_\varepsilon}. \tag{4.5}$$

So let us take $\varepsilon > 0$ and choose

$$\gamma_\varepsilon = \varepsilon + 1 - \left(\frac{b}{b+\varepsilon}\right)^{\frac{1}{q-1}}.$$

Denote also

$$\alpha = \frac{2-q+\sigma}{q-1}, \quad \ell = \frac{1}{\alpha}\left(\frac{\alpha+1}{b}\right)^{\frac{1}{q-1}}.$$

Observe that,

$$(\alpha\ell(1-\gamma_\varepsilon))^{q-1}(b+\varepsilon) = (\alpha+1)\left(1-\left(1+\frac{\varepsilon}{b}\right)^{\frac{1}{q-1}}\varepsilon\right)^{q-1}. \tag{4.6}$$

For every $\delta \in (0, \delta_\varepsilon)$, consider the function

$$w_\varepsilon = M_\varepsilon - \frac{(1-\gamma_\varepsilon)\ell}{(d(x)+\delta)^\alpha},$$

where M_ε is a constant that will be determined. Bearing in mind (4.3), (4.5) and the fact that $\sigma = (\alpha+1)q - (\alpha+2)$, one may easily check that, in $\Omega_{\delta_\varepsilon}$, the following is satisfied

$$\begin{aligned}
&\Delta w_\varepsilon + c_0 + \mu(x)|\nabla w_\varepsilon|^q + f(x) \le c_0 + \|f\|_{L^\infty(\Omega_{\delta_0})} \\
&+ \frac{\alpha\ell(1-\gamma_\varepsilon)}{(d(x)+\delta)^{\alpha+2}}\Big((\alpha\ell(1-\gamma_\varepsilon))^{q-1}(b+\varepsilon) + \|\Delta d\|_{L^\infty(\Omega_{\delta_0})}(d(x)+\delta) - (\alpha+1)\Big).
\end{aligned}$$

Thus, taking (4.6) into account, we can choose ε and δ_ε small enough so that

$$-\Delta w_\varepsilon \ge c_0 + \mu(x)|\nabla w_\varepsilon|^q + f(x), \quad x \in \Omega_{\delta_\varepsilon}.$$

If we consider now an arbitrary solution $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ to (1.6), and take

$$M_\varepsilon = \sup_{d(x)=\delta_\varepsilon} u(x) + \frac{(1-\gamma_\varepsilon)\ell}{\delta_\varepsilon^\alpha},$$

then one easily checks that $w_\varepsilon \ge u$ on $\partial\Omega_{\delta_\varepsilon}$. Thus, Theorem 2.2 implies that

$$w_\varepsilon = -\frac{(1-\gamma_\varepsilon)\ell}{(d(x)+\delta)^\alpha} + M_\varepsilon \geq u, \quad x \in \Omega_{\delta_\varepsilon}.$$

After taking limits when δ tends to zero, we get

$$\limsup_{d(x)\to 0} u(x)d(x)^\alpha \leq -(1-\gamma_\varepsilon)\ell.$$

Arguing similarly, but comparing now with the subsolution

$$-\frac{(1+\kappa_\varepsilon)\ell}{(d(x)-\delta)^\alpha} + \inf_{d(x)=\delta_\varepsilon} u(x), \quad \text{with } \kappa_\varepsilon = \left(\frac{b}{b-\varepsilon}\right)^{\frac{1}{q-1}} - 1 + \varepsilon,$$

one also gets the inequality

$$\liminf_{d(x)\to 0} u(x)d(x)^\alpha \geq -(1+\kappa_\varepsilon)\ell.$$

Passing to the limit when ε tends to zero yields the following behavior near the boundary for any solution $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ to (1.6):

$$\lim_{d(x)\to 0} u(x)d(x)^\alpha = -\ell. \tag{4.7}$$

Step 3: Uniqueness of the ergodic constant. Having (4.7) in hand, the proof of the uniqueness of c_0 assuming $f \in L^\infty(\Omega)$ is classical (see Step 3 of the proof of Theorem VI.1 in [24]). In any case, we include the details for completeness.

Arguing by contradiction, let $c_1 > c_2$ be constants such that problem (1.6) with $c_0 = c_1$ admits a solution $u_1 \in W^{1,\infty}_{\text{loc}}(\Omega)$ and (1.6) with $c_0 = c_2$ admits a solution $u_2 \in W^{1,\infty}_{\text{loc}}(\Omega)$. Let $\theta \in (0,1)$. Observe that

$$\begin{aligned}
&\Delta(\theta u_1) + \mu(x)|\nabla(\theta u_1)|^q + \theta(f(x)+c_1) \\
&\qquad = \theta\left(\Delta u_1 + \mu(x)|\nabla u_1|^q \theta^{q-1} + f(x) + c_1\right) \leq 0, \quad x \in \Omega.
\end{aligned}$$

Therefore,

$$\begin{aligned}
-\Delta(\theta u_1) &\geq \mu(x)|\nabla(\theta u_1)|^q + \theta(f(x)+c_1) \\
&= \mu(x)|\nabla(\theta u_1)|^q + f(x) + \theta c_1 - (1-\theta)f(x) \\
&\geq \mu(x)|\nabla(\theta u_1)|^q + f(x) + c_2 + [\theta c_1 - c_2 - (1-\theta)\|f\|_{L^\infty(\Omega)}],
\end{aligned}$$

$x \in \Omega$.

Now we choose θ close enough to 1 so that

$$-\Delta(\theta u_1) \geq \mu(x)|\nabla(\theta u_1)|^q + f(x) + c_2, \quad x \in \Omega.$$

Thus, θu_1 is a supersolution to the equation satisfied by u_2. Moreover, taking (4.7) into account, we deduce that

$$\lim_{d(x)\to 0}(\theta u_1(x) - u_2(x))d(x)^\alpha = \ell(1-\theta) > 0,$$

and in consequence,

$$\lim_{d(x)\to 0}(\theta u_1(x) - u_2(x)) = +\infty.$$

Then, Theorem 2.2 implies that $\theta u_1 \geq u_2$ in Ω. Taking limits as θ tends to 1, we conclude that $u_1 \geq u_2$ in Ω. But this is not possible since, for any $k \in \mathbb{R}$, the function $u_1 + k$ is also a solution to (1.6) with $c_0 = c_1$. Thus, following the arguments above, we would have that $u_1 + k \geq u_2$ in Ω for any $k \in \mathbb{R}$, a contradiction.

Step 4: Uniqueness of solution. We finish the proof by showing that any two solutions in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ to (1.6) differ by a constant. Again, the proof is essentially contained in [24] (it is the Step 4 of the proof of Theorem VI.1) and (4.7) is strongly used once more.

Let $u_1, u_2 \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ be two solutions to (1.6). Let $C \in (0, \ell)$. Similarly as in Step 2, it can be checked that

$$\Delta\left(-\frac{C}{d(x)^\alpha}\right) + c_0 + \mu(x)\left|\nabla\left(-\frac{C}{d(x)^\alpha}\right)\right|^q + f(x) \leq 0, \quad x \in \Omega_\delta,$$

for every $\delta \in (0, \delta_0)$ small enough. Then, by virtue of the convexity of the function $\xi \mapsto |\xi|^q$, one has that the function

$$\tilde{u}_1 = \theta u_1 + (1-\theta)\left(-\frac{C}{d(x)^\alpha}\right)$$

satisfies

$$-\Delta\tilde{u}_1 \geq c_0 + \mu(x)|\nabla\tilde{u}_1|^q + f(x), \quad x \in \Omega_\delta.$$

Moreover, given $\varepsilon \in (0, \delta)$, it follows that

$$\begin{aligned}-\Delta(\tilde{u}_1 - u_2) &\geq \mu(x)(|\nabla\tilde{u}_1|^q - |\nabla u_2|^q)\\ &\geq \mu(x)q|\nabla u_2|^{q-2}\nabla u_2\nabla(\tilde{u}_1 - u_2), \quad x \in \Omega_\delta \setminus \overline{\Omega}_\varepsilon.\end{aligned}$$

Since $\mu(x)q|\nabla u_2|^{q-2}\nabla u_2 \in L^\infty(\Omega \setminus \overline{\Omega}_\varepsilon)$, the maximum principle implies that

$$\min_{\overline{\Omega_\delta\setminus\Omega_\varepsilon}}(\tilde{u}_1 - u_2) = \min_{\partial(\overline{\Omega_\delta\setminus\Omega_\varepsilon})}(\tilde{u}_1 - u_2). \tag{4.8}$$

On the other hand, from (4.7) we derive

$$\begin{aligned}&\lim_{d(x)\to 0}\left(d(x)^\alpha(\tilde{u}_1 - u_2)\right)\\ &= \lim_{d(x)\to 0}\left(\theta d(x)^\alpha u_1 - (1-\theta)C - d(x)^\alpha u_2\right) = (\ell - C)(1-\theta) > 0.\end{aligned}$$

Thus, $\lim_{d(x)\to 0}(\tilde{u}_1 - u_2) = +\infty$. Therefore, letting ε tend to zero in (4.8) we get

$$\min_{\overline{\Omega}_\delta\setminus\partial\Omega}(\tilde{u}_1-u_2)=\min_{\{d(x)=\delta\}}(\tilde{u}_1-u_2).$$

Since δ does not depend on θ, we may let θ tend to one in the previous equality, so that

$$\inf_{\Omega_\delta}(u_1-u_2)=\min_{\{d(x)=\delta\}}(u_1-u_2).$$

Furthermore, again by convexity, one has

$$-\Delta(u_1-u_2)\geq\mu(x)q|\nabla u_2|^{q-2}\nabla u_2\nabla(u_1-u_2),\quad x\in\Omega.$$

Considering the previous inequality in $\Omega\setminus\overline{\Omega}_\delta$, it follows from the maximum principle that

$$\min_{\Omega\setminus\Omega_\delta}(u_1-u_2)=\min_{\{d(x)=\delta\}}(u_1-u_2).$$

In conclusion, u_1-u_2 reaches a global minimum on $\{d(x)=\delta\}$. In particular, there exists $x_0\in\Omega\setminus\overline{\Omega}_{\delta/2}$ such that

$$u_1(x_0)-u_2(x_0)=\min_{\Omega\setminus\overline{\Omega}_{\delta/2}}(u_1-u_2).$$

The strong maximum principle applied in $\Omega\setminus\overline{\Omega}_{\delta/2}$ implies that u_1-u_2 must be constant in $\Omega\setminus\overline{\Omega}_{\delta/2}$. Since δ can be taken arbitrarily small, we conclude that u_1-u_2 is constant in Ω. □

Data availability

No data was used for the research described in the article.

Acknowledgments

Grant PID2021-122122NB-I00 funded by MCIN/AEI/10.13039/501100011033 and by “ERDF A way of making Europe”. Junta de Andalucía, Consejería de Transformación Económica, Industria, Conocimiento y Universidades-Unión Europea grant P18-FR-667. Second and third author supported by Junta de Andalucía FQM-116. First author supported by Junta de Andalucía FQM-194 and CDTIME. Third author supported by Junta de Andalucía, Consejería de Transformación Económica, Industria, Conocimiento y Universidades grant UAL2020-FQM-B2046.

The authors would like to thank the reviewers of the manuscript as their comments and suggestions have contributed deeply to the improvement not only in clarity but also in the quality of the results obtained.

References

[1] Asadollah Aghajani, Craig Cowan, Shiu Hong Lui, Singular solutions of elliptic equations involving nonlinear gradient terms on perturbations of the ball, J. Differ. Equ. 264 (4) (2018) 2865–2896.

[2] Nour Eddine Alaa, Michel Pierre, Weak solutions of some quasilinear elliptic equations with data measures, SIAM J. Math. Anal. 24 (1) (1993) 23–35.

[3] David Arcoya, Colette De Coster, Louis Jeanjean, Kazunaga Tanaka, Remarks on the uniqueness for quasilinear elliptic equations with quadratic growth conditions, J. Math. Anal. Appl. 420 (1) (2014) 772–780.
[4] David Arcoya, Colette De Coster, Louis Jeanjean, Kazunaga Tanaka, Continuum of solutions for an elliptic problem with critical growth in the gradient, J. Funct. Anal. 268 (8) (2015) 2298–2335.
[5] Guy Barles, François Murat, Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions, Arch. Ration. Mech. Anal. 133 (1) (1995) 77–101.
[6] Lucio Boccardo, François Murat, Jean-Pierre Puel, Résultats d'existence pour certains problèmes elliptiques quasilinéaires, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) 11 (2) (1984) 213–235.
[7] Lucio Boccardo, François Murat, Jean-Pierre Puel, L^∞ estimate for some nonlinear elliptic partial differential equations and application to an existence result, SIAM J. Math. Anal. 23 (2) (1992) 326–333.
[8] Haïm Brezis, Xavier Cabré, Some simple nonlinear PDE's without solutions, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 1 (2) (1998) 223–262.
[9] José Carmona, Tommaso Leonori, Salvador López-Martínez, Pedro J. Martínez-Aparicio, Quasilinear elliptic problems with singular and homogeneous lower order terms, Nonlinear Anal. 179 (2019) 105–130.
[10] Marco Cirant, Alessandro Goffi, On the problem of maximal L^q-regularity for viscous Hamilton-Jacobi equations, Arch. Ration. Mech. Anal. 240 (3) (2021) 1521–1534.
[11] Colette De Coster, Antonio J. Fernández, Existence and multiplicity for elliptic p-Laplacian problems with critical growth in the gradient, Calc. Var. Partial Differ. Equ. 57 (3) (2018) 89, 42.
[12] Colette De Coster, Antonio J. Fernández, Existence and multiplicity for an elliptic problem with critical growth in the gradient and sign-changing coefficients, Calc. Var. Partial Differ. Equ. 59 (3) (2020) 97, 34.
[13] Colette De Coster, Antonio J. Fernández, Louis Jeanjean, A priori bounds and multiplicity of solutions for an indefinite elliptic problem with critical growth in the gradient, J. Math. Pures Appl. 9 (132) (2019) 308–333.
[14] Colette De Coster, Louis Jeanjean, Multiplicity results in the non-coercive case for an elliptic problem with critical growth in the gradient, J. Differ. Equ. 262 (10) (2017) 5231–5270.
[15] Djairo G. de Figueiredo, Pierre-Louis Lions, Roger D. Nussbaum, A priori estimates and existence of positive solutions of semilinear elliptic equations, J. Math. Pures Appl. (9) 61 (1) (1982) 41–63.
[16] Vincenzo Ferone, François Murat, Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small, Nonlinear Anal. 42 (7, Ser. A: Theory Methods) (2000) 1309–1326.
[17] Vincenzo Ferone, Maria Rosaria Posteraro, Jean-Michel Rakotoson, L^∞-estimates for nonlinear elliptic problems with p-growth in the gradient, J. Inequal. Appl. 3 (2) (1999) 109–125.
[18] Marek Fila, Philippe Souplet, Fred B. Weissler, Linear and nonlinear heat equations in L^q_δ spaces and universal bounds for global solutions, Math. Ann. 320 (1) (2001) 87–113.
[19] Basilis Gidas, Wei Ming Ni, Louis Nirenberg, Symmetry and related properties via the maximum principle, Commun. Math. Phys. 68 (3) (1979) 209–243.
[20] David Gilbarg, Neil S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
[21] Alessandro Goffi, On the Optimal l^q-Regularity for Viscous Hamilton-Jacobi Equations with Sub-Quadratic Growth in the Gradient, 2022.
[22] Nathalie Grenon, François Murat, Alessio Porretta, A priori estimates and existence for elliptic equations with gradient dependent terms, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 13 (1) (2014) 137–205.
[23] Louis Jeanjean, Boyan Sirakov, Existence and multiplicity for elliptic problems with quadratic growth in the gradient, Commun. Partial Differ. Equ. 38 (2) (2013) 244–264.
[24] Jean-Michel Lasry, Pierre-Louis Lions, Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem, Math. Ann. 283 (4) (1989) 583–630.
[25] Tommaso Leonori, Alessio Porretta, Large solutions and gradient bounds for quasilinear elliptic equations, Commun. Partial Differ. Equ. 41 (6) (2016) 952–998.
[26] Pierre-Louis Lions, Quelques remarques sur les problèmes elliptiques quasilinéaires du second ordre, J. Anal. Math. 45 (1985) 234–254.
[27] Pierre-Louis Lions, Recorded video of séminaire de mathématiques appliquées at collége de france, November 14, 2014.
[28] Salvador López-Martínez, A singularity as a break point for the multiplicity of solutions to quasilinear elliptic problems, Adv. Nonlinear Anal. 9 (1) (2020) 1351–1382.
[29] Alessio Porretta, The "ergodic limit" for a viscous Hamilton-Jacobi equation with Dirichlet conditions, Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl. 21 (1) (2010) 59–78.
[30] James Serrin, A symmetry problem in potential theory, Arch. Ration. Mech. Anal. 43 (1971) 304–318.

[31] Philippe Souplet, A priori estimates and bifurcation of solutions for an elliptic equation with semidefinite critical growth in the gradient, Nonlinear Anal. 121 (2015) 412–423.
[32] Giovanni Maria Troianiello, Elliptic Differential Equations and Obstacle Problems, The University Series in Mathematics, Plenum Press, New York, 1987.