A Survey on Isometries Between Lipschitz Spaces

M. G. Cabrera-Padilla, A. Jiménez-Vargas and Moisés Villegas-Vallecillos

Abstract The famous Banach–Stone theorem, which characterizes surjective linear isometries between $C(X)$ spaces as certain weighted composition operators, has motivated the study of isometries defined on different function spaces (see [33, 34]). The research on surjective linear isometries between spaces of Lipschitz functions is a subject of long tradition which goes back to the sixties with the works of de Leeuw [61] and Roy [81], and followed by those by Mayer-Wolf [67], Weaver [97], Araujo and Dubarbie [3], and Botelho, Fleming and Jamison [8]. This topic continues to attract the attention of some authors (see [44, 52, 62]). In the setting of Lipschitz spaces, we present a survey on non-necessarily surjective linear isometries and codimension 1 linear isometries [55], vector-valued linear isometries [56], local isometries and generalized bi-circular projections [54], 2-local isometries [52, 57], projections and averages of isometries [12] and hermitian operators [13, 14]. We also raise some open problems on bilinear isometries and approximate isometries in the same context.

Keywords Lipschitz function · Linear isometry · Weighted composition operator · Banach–Stone type theorem

1 Introduction

Given two metric spaces (X, d_X) and (Y, d_Y), let us recall that a map $f: X \to Y$ is said to be Lipschitz if there exists a constant $k \geq 0$ such that $d_Y(f(x), f(w)) \leq k \cdot d_X(x, w)$ for all $x, w \in X$. In this case, the number

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$$\mathrm{Lip}(f) = \sup\left\{\frac{d_Y(f(x), f(w))}{d_X(x, w)} : x, w \in X,\ x \neq w\right\}$$

is called the Lipschitz constant of f. If f is bijective and both f and f^{-1} are Lipschitz, it is said that f is a Lipschitz homeomorphism.

For a metric space X and a Banach space E over $\mathbb{K}$ (the field of real or complex numbers), the Lipschitz space $\mathrm{Lip}(X, E)$ is the Banach space of all bounded Lipschitz maps from X to E, with the maximum norm defined by

$$\|f\| = \max\{\|f\|_\infty, \mathrm{Lip}(f)\},$$

where

$$\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}.$$

In some results of this survey, we will consider the vector space of all bounded Lipschitz maps from X to E equipped with another classical norm, the so-called sum norm:

$$\|f\|_s = \|f\|_\infty + \mathrm{Lip}(f).$$

An important subset of $\mathrm{Lip}(X, E)$ is the closed subspace $\mathrm{lip}(X, E)$ consisting of all functions $f \in \mathrm{Lip}(X, E)$ verifying the following condition:

$$\forall \varepsilon > 0,\ \exists \delta > 0 : x, w \in X,\ 0 < d_X(x, w) < \delta \quad \Rightarrow \quad \|f(x) - f(w)\| < \varepsilon\, d_X(x, w).$$

The elements of $\mathrm{lip}(X, E)$ are called little Lipschitz functions. Given a metric space (X, d) and a number $\alpha \in]0, 1[$, the set X with the distance d^α is named Hölder metric space and it is usual to denote it by X^α. The maps in $\mathrm{Lip}(X^\alpha, E)$ are known as Hölder functions. The space $\mathrm{lip}(X^\alpha, E)$ contains $\mathrm{Lip}(X, E)$, but there exist $\mathrm{lip}(X, E)$ spaces containing only constant maps.

The spaces $\mathrm{Lip}(X) := \mathrm{Lip}(X, \mathbb{K})$ and $\mathrm{lip}(X) := \mathrm{lip}(X, \mathbb{K})$, endowed with the pointwise product, are unital algebras whose unit is $\mathbf{1}_X$, the map which is constantly equal to 1 on X. For a complete information on Lipschitz spaces, we refer the reader to the monograph [99] by Nik Weaver.

This paper essentially gathers our contribution to the study about isometries between Lipschitz spaces, published over the past years in a series of papers [12–14, 52, 54–57].

In order to describe in a precise way its contents, we need to introduce a bit of nomenclature and some notation.

Throughout this paper and, unless otherwise stated, E and F will be nonzero normed spaces, and the symbols B_E, S_E and $\mathrm{Ext}(B_E)$ will stand for the closed unit ball, the unit sphere of E and the set of extreme points of B_E, respectively. The space of bounded linear operators from E to F will be represented by $\mathcal{L}(E, F)$ and we will write $\mathcal{L}(E) = \mathcal{L}(E, E)$ and $E^* = \mathcal{L}(E, \mathbb{K})$ for short. For a bounded metric space X, we denote its diameter by $\mathrm{diam}(X)$.

Given a topological space X, we will use $C(X, E)$, $C_b(X, E)$ and $C_0(X, E)$ to denote the spaces of continuous functions, bounded and continuous functions and continuous functions vanishing at the infinity from X to E, respectively. When $E = \mathbb{K}$, we will usually write $C(X)$, $C_b(X)$ and $C_0(X)$.

For two sets X and Y and two linear spaces E and F, let $\mathcal{M}$ and $\mathcal{N}$ be two families of functions from X to E and from Y into F, respectively. A map $T: \mathcal{M} \to \mathcal{N}$ is said to be a weighted composition operator on a subset $Y_0 \subset Y$ if there exist two maps φ from Y_0 to X and $\widehat{T}$ from Y_0 to the space of all linear operators from E to F such that

$$T(f)(y) = \widehat{T}(y)(f(\varphi(y))), \quad \forall y \in Y_0, \ \forall f \in \mathcal{M}.$$

The maps φ and $\widehat{T}$ are said to be the symbol and the weight of T, respectively. In the case $E = F = \mathbb{K}$, a weighted composition operator on Y_0 is of the form

$$T(f)(y) = \tau(y) f(\varphi(y)), \quad \forall y \in Y_0, \ \forall f \in \mathcal{M},$$

for a symbol $\varphi: Y_0 \to X$ and a weight $\tau: Y_0 \to \mathbb{K}$.

We have divided this survey into a series of sections which collect our main results about different types of isometries on Lipschitz spaces and some operators related to those isometries.

The first section is devoted to linear isometries between spaces of scalar-valued Lipschitz functions. The study of these maps has its origin in the 1960s, when Roy [81] described the surjective linear isometries T on $\mathrm{Lip}(X)$ when X is a compact, connected space whose diameter is less than or equal to 1. Namely, he proved that T is a weighted composition operator whose symbol is a surjective isometry on X and whose weight is a unimodular constant function.

In 1969, Vasavada [92] extended this result by establishing that if X and Y are compact r-connected metric spaces for some $r \in]0, 1[$ with their diameters being less than or equal to 2, then any surjective linear isometry from $\mathrm{Lip}(X)$ to $\mathrm{Lip}(Y)$ comes from a surjective isometry from Y into X as in the aforementioned description.

Let us recall that a metric space X is r-connected for some $r \in \mathbb{R}^+$ if it is not possible to decompose it into two nonempty subsets A and B such that $d(A, B) \geq r$.

The condition that metric spaces have diameters less than or equal to 2 is not too restrictive, since, if (X, d) is a metric space and X' is the set X endowed with the distance $d'(x, y) = \min\{2, d(x, y)\}$, then $\mathrm{diam}(X') \leq 2$ and $\mathrm{Lip}(X')$ is isometrically isomorphic to $\mathrm{Lip}(X)$ [98, Proposition 1.7.1].

We must also mention the works by Novinger [75] and Mayer-Wolf [67] in which surjective linear isometries between $\mathrm{Lip}(X^\alpha)$ spaces, with X being compact, were classified. Jarosz and Pathak [48] also analyzed surjective linear isometries on $\mathrm{Lip}(X)$ and $\mathrm{lip}(X^\alpha)$, even though their most relevant contribution was made with respect to the sum norm. Recently, Hatori and Oi [44] have continued this study on the isometry group of $\mathrm{Lip}(X)$ with the sum norm.

On the other hand, Weaver [97] developed a technique to eliminate the compactness hypothesis on metric spaces and obtained the same description as Vasavada for surjective linear isometries $T: \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ when X and Y are complete, 1-connected and whose diameters are less than or equal to 2.

Onto linear isometries on $\text{Lip}(X)$ have been considerably studied in contrast with the case of into linear isometries. This fact is surprising if we compare it with the formidable literature on nonsurjective isometries in the context of $C(X)$ spaces. In this setting, the most important result is the famous Holsztyński theorem [45]. He proved that if T is a linear isometry from $C(X)$ into $C(Y)$, then there exist a closed subset Y_0 of Y, a surjective continuous map φ from Y_0 into X and a function $\tau \in C(Y)$ with $\|\tau\|_\infty = 1$ and $|\tau(y)| = 1$ for all $y \in Y_0$ such that T is a weighted composition operator on Y_0 with symbol φ and weight τ. Holsztyński theorem was extended to $C_0(X)$ spaces by Jeang and Wong [49], and to subspaces of continuous functions by Araujo and Font [4]. It has been also extended to the setting of C*-algebras, JB*-triples, and real C*-algebras and JB*-triples by Chu and Wong [20], Chu and Mackey [19], and Apazoglou and Peralta [2], respectively.

We show in Sect. 2 that Holsztyński theorem has a natural formulation in the context of $\text{Lip}(X)$ spaces. Given two compact metric spaces X and Y, our main result (Theorem 1) states that every linear isometry T from $\text{Lip}(X)$ to $\text{Lip}(Y)$ such that $T(\mathbf{1}_X)$ is a contraction ($\text{Lip}(T(\mathbf{1}_X)) < 1$) admits a representation in terms of a weighted composition operator on a closed subset Y_0 of Y, whose symbol $\varphi\colon Y_0 \to X$ is a surjective Lipschitz map with $\text{Lip}(\varphi) \leq \max\{1, \text{diam}(X)/2\}$ and whose weight $\tau\colon Y \to \mathbb{K}$ is a Lipschitz map with $|\tau| = 1$ on Y_0. Moreover, we evidence that Y_0 is the largest subset of Y in which T admits this representation as a weighted composition operator (Corollary 1). In general, our result does not hold when $T(\mathbf{1}_X)$ is not a contraction. A simple counterexample can be found in [97].

In the proof of our theorem, we use techniques of extreme points as it was done in [24, 75, 81] and, because of it, we start Sect. 2 with a description of the extreme points of the unit ball of $\text{Lip}(X)^*$.

Our Lipschitz version of Holsztyński theorem will be used along the rest of sections in order to study surjective isometries, codimension 1 isometries and local isometries.

In the surjective case, we show that every linear isometry T from $\text{Lip}(X)$ onto $\text{Lip}(Y)$ such that $T(\mathbf{1}_X)$ is a contraction, is a weighted composition operator in which the symbol $\varphi\colon Y \to X$ is a Lipschitz homeomorphism and the weight $\tau\colon Y \to S_{\mathbb{K}}$ is a Lipschitz function verifying that $d(\varphi(y), \varphi(z)) = d(y, z)$ and $\tau(y) = \tau(z)$ for any $y, z \in Y$ with $d(y, z) < 2$ (Theorem 2).

In Sect. 3, we use Theorem 1 to investigate about codimension 1 linear isometries between $\text{Lip}(X)$ spaces.

In the context of spaces of continuous functions, codimension 1 linear isometries and, in particular, isometric shift operators were studied by several authors by using Holsztyński theorem. We cite, for instance, the works by Gutek, Hart, Jamison and Rajagopalan [40] and Farid and Varadarajan [27].

Making use of a similar technique, we classify codimension 1 linear isometries $T\colon \text{Lip}(X) \to \text{Lip}(Y)$ such that $\text{Lip}(T(\mathbf{1}_X)) < 1$ into two types (Definition 4). We say that T is of type I if there exists an isolated point $p \in Y$ such that T is a weighted composition operator on $Y\backslash\{p\}$ whose symbol function is a surjective Lipschitz map and whose weight function is a unimodular Lipschitz map. We say that T is of type II if it admits the same representation in all Y. These two types are not self-excluding

(Proposition 6). There are also codimension 1 linear isometries of type I which are not of type II and vice versa (Propositions 7 and 8). Moreover, we prove that if T is of type I, then its symbol is a Lipschitz homeomorphism.

Section 4 is focused on studying the algebraic reflexivity of the isometry group of $\mathrm{Lip}(X)$. In the last decades several works on this topic have appeared in the context of operator algebras. The main question consists of investigating when the local action of some set of transformations (derivations, automorphisms or isometries) completely determines the mentioned set. Consult, for instance, the paper by Brešar and Šemrl [15] on local automorphisms on $\mathcal{L}(H)$ where H is a Hilbert space. Nevertheless, the algebraic reflexivity can be defined by considering any normed space instead of an operator algebra. Given a normed space E and a nonempty subset $\mathcal{S}$ of $\mathcal{L}(E)$, denote

$$\mathrm{ref}_{\mathrm{al}}(\mathcal{S}) = \{\Phi \in \mathcal{L}(E) \colon \forall e \in E \ \ \exists \Phi_e \in \mathcal{S} \text{ with } \Phi(e) = \Phi_e(e)\}\,.$$

We say that $\mathcal{S}$ is algebraically reflexive if $\mathrm{ref}_{\mathrm{al}}(\mathcal{S}) = \mathcal{S}$. In particular, if $\mathcal{G}(E)$ is the set of all surjective linear isometries on E, the elements of $\mathrm{ref}_{\mathrm{al}}(\mathcal{G}(E))$ are called local linear isometries, and E is said to be iso-reflexive if $\mathcal{G}(E)$ is algebraically reflexive.

Molnár [69] proved that $\mathcal{L}(H)$ is iso-reflexive when H is an infinite-dimensional separable Hilbert space and, in a joint work with Zalar [73], they investigated the algebraic reflexivity of the isometry group of other important Banach spaces. For example, they showed that $C(X, \mathbb{C})$ is iso-reflexive if X is a compact Hausdorff space verifying the first axiom of countability. Their theorem is not true without this condition [16, Example 2 and Theorem 9].

Motivated by these results, we prove in Sect. 4 that $\mathrm{Lip}(X)$ is iso-reflexive (Theorem 4). Furthermore, we use this property of $\mathrm{Lip}(X)$ to get the algebraic reflexivity of some sets of isometries and projections on $\mathrm{Lip}(X)$.

Let us recall that, given a metric space X, it is said that an isometry $\varphi \colon X \to X$ is involutive if $\varphi^2 = \mathrm{Id}$, where Id is the identity map on X. In particular, if E is a normed space, involutive linear isometries of E are usually given the name of isometric reflections of E.

On the other hand, a (linear) projection $P \colon E \to E$ is bi-circular if $P + \lambda(\mathrm{Id} - P)$ is an isometry for all $\lambda \in S_{\mathbb{K}}$, and it is generalized bi-circular if $P + \lambda(\mathrm{Id} - P)$ is an isometry for some $\lambda \in S_{\mathbb{K}} \setminus \{1\}$. Bi-circular projections were first studied by Stachó and Zalar [84] in 2004. Shortly after, Fosner, Ilisevic and Li [37] extended this notion by introducing the generalized bi-circular projections.

We prove that the sets of isometric reflections and generalized bi-circular projections on $\mathrm{Lip}(X)$ are algebraically reflexive (Corollaries 6 and 7). To this end, we first give a description of such operators. We show that every isometric reflection of $\mathrm{Lip}(X)$ is, either a composition operator whose symbol is an involutive isometry on X, or its inverse (Corollary 5), and that every generalized bi-circular projection on $\mathrm{Lip}(X)$ is the average of the identity map and an isometric reflection (Proposition 9).

Generalized bi-circular projections were studied by Botelho and Jamison [9, 10] for Lipschitz spaces. We must point out that Dutta and Rao [25] investigated the algebraic reflexivity of the sets of isometric reflections and generalized bi-circular projections on $C(X)$ where X is a compact Hausdorff space verifying the first axiom of countability.

Section 5 is devoted to the study of projections on $\mathrm{Lip}(X)$ which can be expressed as the average of two or three surjective linear isometries, when X is a compact 1-connected metric space with diameter at most 2. We show that generalized bi-circular projections on $\mathrm{Lip}(X)$ are the only projections on $\mathrm{Lip}(X)$ which can be represented as the average of two surjective linear isometries (Theorem 5). In order to achieve this objective, we first characterize when the average of two isometries is a projection on $\mathrm{Lip}(X)$ (Proposition 11). In [7, 11], similar results were obtained for such projections on the spaces $C(X, \mathbb{C})$ and $C(X, E)$ where E is a strictly convex space.

Our method will be extended to investigate when the average of three isometries is a projection on $\mathrm{Lip}(X)$ (Theorem 6). The concept of n-circular projection allows us to establish that the average P of two (three) surjective linear isometries on $\mathrm{Lip}(X)$ is a projection if and only if P is either a trivial projection or a 2-circular (3-circular, respectively) projection (Theorem 7).

All the results that we mentioned above on the algebraic reflexivity correspond to linear structures. Taking into account the important achievements obtained, it is natural to want to extend this research line to more general structures where linearity is not a premise. However, the locality condition satisfied by the elements of $\mathrm{ref}_{\mathrm{al}}(\mathcal{S})$ is too weak to get reasonable results without considering linearity.

In 1997, Šemrl [82] introduced the concept of 2-locality for automorphisms and derivations. Given an algebra $\mathcal{A}$, a map $\Phi\colon \mathcal{A} \to \mathcal{A}$ (which is not assumed to be linear) is a 2-local automorphism (respectively, a 2-local derivation) if for any $a, b \in \mathcal{A}$ there exists an automorphism (respectively, a derivation) $\Phi_{a,b}\colon \mathcal{A} \to \mathcal{A}$ such that $\Phi(a) = \Phi_{a,b}(a)$ and $\Phi(b) = \Phi_{a,b}(b)$. Šemrl proved in this paper that if H is an infinite-dimensional separable Hilbert space, then every 2-local automorphism on $\mathcal{L}(H)$ is an automorphism, and he established a similar statement for 2-local derivations.

Molnár [70] extended it to isometries in 2002. Given a Banach space E, a map $T\colon E \to E$ is said to be a 2-local isometry if for every $x, y \in E$ there exists a surjective linear isometry $\Phi_{x,y}\colon E \to E$, depending on x and y, such that $T(x) = \Phi_{x,y}(x)$ and $T(y) = \Phi_{x,y}(y)$ (a priori, it is not assumed that T is linear or surjective). Molnár [70] showed that every 2-local isometry on $\mathcal{L}(H)$ is a surjective linear isometry and posed the problem of studying 2-local isometries on other Banach spaces.

In recent years numerous papers on 2-locality in the setting of spaces of operators have been published (see for example [63, 71, 72]). In function algebras, the first result in this research line was presented by György [41] in 2001. He showed that if X is a σ-compact first countable Hausdorff space, then every 2-local isometry on $C_0(X, \mathbb{C})$ is a surjective linear isometry. Furthermore, Hatori, Miura, Oka and Takagi [42] considered 2-local isometries and 2-local automorphisms on uniform algebras, including certain algebras of holomorphic functions of one and two variables.

We devote Sect. 6 to the study of 2-local isometries on $\mathrm{Lip}(X)$. We consider the space $\mathrm{Lip}(X)$ equipped indistinctly with the maximum norm and the sum norm. It is said that the isometry group of $\mathrm{Lip}(X)$ is canonical if every surjective linear isometry on $\mathrm{Lip}(X)$ can be expressed as a weighted composition operator, where the weight is a unimodular constant and the symbol is a surjective isometry on X. We prove that if X is a separable bounded metric space and the isometry group of $\mathrm{Lip}(X)$ is canonical, then each 2-local isometry on $\mathrm{Lip}(X)$ is a surjective linear isometry (Theorem 9). Moreover, we give a complete description of all 2-local isometries of $\mathrm{Lip}(X)$ when X is bounded (Theorem 8).

In the final part of the section, we present 2-locality for the spaces of vector-valued Lipschitz maps. Under convenient conditions on a compact metric space X and a Banach space E, we provide a description of (standard) 2-local isometries on the Banach space $\mathrm{Lip}(X, E)$ in terms of a generalized composition operator (Theorems 12 and 13) and study in which cases every (standard) 2-local isometry on $\mathrm{Lip}(X, E)$ is linear and surjective (Theorems 14 and 15).

Another approach in the study of 2-local isometries on uniform algebras and Lipschitz algebras was introduced by Li, Peralta, Wang and Wang in [62]. They focused on the notion of weak-2-local isometry, established spherical variants of the Gleason–Kahane–Żelazko [39, 58] and Kowalski–Słodkowski theorems [60], and applied them to prove that every weak-2-local isometry between two uniform algebras is a linear map. In this way, they resolved a couple of problems raised in [42]. Some contributions also are given for weak-2-local isometries on Lipschitz algebras.

Linear isometries between spaces of vector-valued Lipschitz maps are dealed in Sect. 7. In 1950, Jerison [50] extended the Banach–Stone theorem to $C(X, E)$ spaces and Cambern [17] generalized in 1978 Jerison's theorem considering non necessarily surjective linear isometries $T : C(X, E) \to C(Y, F)$, where X, Y are compact Hausdorff spaces and E, F are normed spaces with F strictly convex.

Our purpose in this section is to state Cambern and Jerison theorems in the context of Lipschitz spaces. If X, Y are compact metric spaces and E is a strictly convex normed space, then every linear isometry T from $\mathrm{Lip}(X, E)$ to $\mathrm{Lip}(Y, E)$ fixing some constant function with modulus one is a weighted composition operator on a nonempty closed subset $Y_0 \subset Y$, whose symbol is a surjective Lipschitz map $\varphi : Y_0 \to X$ with $\mathrm{Lip}(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$ and whose weight is a Lipschitz map $\widehat{T} : Y \to \mathcal{L}(E)$ with $||\widehat{T}(y)|| = 1$ for all $y \in Y$ (Theorem 17).

If, in addition, T is surjective and X and Y have diameters less than or equal to 2, then $Y_0 = Y$, φ is a surjective isometry and $\widehat{T}(y)$ is a surjective linear isometry for all $y \in Y$ (Corollary 9).

In the surjective case, Araujo and Dubarbie [3] generalized our results by considering (non-necessarily compact) complete metric spaces, and weakening the additional condition imposed to the isometry T. Furthermore, Botelho, Fleming and Jamison [8] also tackled this problem by removing the hypothesis of strict convexity on E and F, but they added other constraints.

Sections 8 and 9 focus on hermitian operators between Lipschitz spaces. The notion of an hermitian operator on a Banach space goes back to the early work of Lumer [65] and also Vidav [94]. Hermitian operators have played an important role in the characterization of surjective isometries of various Banach spaces, but they are also interesting in themselves. For certain Banach spaces, hermitian operators are trivial, which means that they are real scalar multiples of the identity. See [33, 34] for a complete information on hermitian operators.

In Sect. 8, we show that the spaces $\mathrm{Lip}(X)$ and $\mathrm{lip}(X^\alpha)$ $(0 < \alpha < 1)$, equipped with the sum norm, support only trivial hermitian operators when X is a compact metric space (Theorem 20). This generalizes some known results [6] on hermitian operators between Lipschitz function spaces on [0, 1]. We also deal with the natural connection between hermitian operators and bi-circular projections and our result implies that the only such projections on $\mathrm{Lip}(X)$ and $\mathrm{lip}(X^\alpha)$ with $0 < \alpha < 1$ are the trivial projections, 0 and Id.

In Sect. 9, we investigate the class of hermitian operators on Lipschitz spaces $\mathrm{Lip}(X, E)$ on a compact and 2-connected metric space and with values on a complex Banach space, following a scheme employed by Fleming and Jamison in [32] in the characterization of the hermitian operators on $C(X, E)$. This approach relies on the existence of semi-inner products on such spaces which are compatible with the norm. To be more precise, we start by embedding $\mathrm{Lip}(X, E)$ isometrically into a space of vector valued continuous functions defined on a compact space. Then we construct a semi-inner product on $\mathrm{Lip}(X, E)$ compatible with the norm. This approach allows us to describe the hermitian operators as multiplication operators via a hermitian operator on E. In particular we conclude that the space of all scalar valued Lipschitz functions only supports trivial hermitian operators (Theorem 21). These results yield the form for the normal and the adjoint abelian operators on this setting (Theorem 22).

Recently, Hatori and Oi [43] studied hermitian operators on a Banach algebra of Lipschitz maps $\mathrm{Lip}(X, A)$, where X is a compact metric space which is not necessarily 2-connected, with values in a uniform algebra A. It is proved that if A is a uniform algebra on a compact Hausdorff space Y, then T is a hermitian operator on $\mathrm{Lip}(X, A)$ if and only if there is a real-valued function $f \in A$ such that T is the corresponding left multiplication operator $M_{\mathbf{1}_X \otimes f}$.

In Sect. 10, some open problems will be posed. Just as a very succinct presentation of them, we would like to comment that we intend to give a Banach–Stone type representation for two types of isometries that have not been studied until now in the context of Lipschitz spaces: linear isometries of finite codimension (Problem 1) and bilinear isometries (Problem 2). Concerning to approximate isometries, we also propose the study of problems of Hyers–Ulam type stability on the approximation, perhaps in the uniform norm, of an ε-nonexpansive map or an ε-approximate isometry on $\mathrm{Lip}(X)$ by a nonexpansive map or an isometry, respectively (Problem 3).

Problem 4 is a particularization of the so-called Tingley's problem in the setting of Lipschitz spaces. This problem asks on the possibility of extending a surjective isometry between the unit spheres of two normed spaces, and is related to the Mazur–Ulam's property. It is a subject with a certain tradition, the first work of Tingley dates back to 1987, and a renewed interest has attracted a widen audience since the first contribution by Ding [22] in 2002. More recently, Tan [85–87] in 2011–12, Peralta and Tanaka [79] in 2016, Fernández and Peralta [29–31] in 2017, and Tanaka [88–90] in 2014–17, have given it a new and vigorous impulse.

To finish the exposition of open questions, we propose in Problem 6 the study of contractive and bicontractive projections on Lip(X) to determine under what conditions on the metric space X, the space Lip(X) belongs to the class of Banach spaces in which every bicontractive projection is the mean of the identity mapping and an involutive isometry. Our interest in this issue is justified by our previous study of generalized bi-circular projections on Lip(X). In fact, we ask in Problem 5 if the average of $n > 4$ surjective linear isometries on Lip(X) (where X is a compact 1-connected metric space with diam$(X) \leq 2$) is a projection if and only if it is either a trivial projection or a n-circular projection.

2 Scalar-Valued Linear Isometries

The main objective of this section is to present a version for Lip(X)-spaces of the renowned Holztyński theorem [45] on linear isometries (not necessarily surjective) between $C(X)$-spaces. This aim is achieved at Theorem 1.

We begin by describing the set of extreme points of the unit ball of Lip$(X)^*$. Let us recall the notion of extreme point.

Definition 1 Let E be a normed space and let C be a subset of E. A point $e \in C$ is said to be an extreme point of C if given $t \in]0, 1[$ and $x, y \in C$ such that $e = tx + (1-t)y$, then $e = x = y$. The set of all extreme points of C will be denoted by Ext(C). It is said that E is strictly convex if Ext$(B_E) = S_E$.

A simple application of the Hahn–Banach theorem, the Banach–Alaoglu theorem and the Krein–Milman theorem yields the following well-known fact.

Proposition 1 *Let E be a normed space and let M be a vector subspace of E. For each $\eta \in$ Ext(B_{M^*}) there exists $\varphi \in$ Ext(B_{E^*}) such that $\varphi|_M = \eta$.*

Let us recall that a Hausdorff topological space X is completely regular if given a closed subset $C \subset X$ and $x_0 \in X \backslash C$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \in C$.

For instance, every metric space X is completely regular. Note that the Lipschitz map $f : X \to [0, 1]$, defined by $f(x) = \max\{0, 1 - d(x, C)/d(x_0, C)\}$, satisfies the required conditions.

The next result shows that it is possible to compactify a completely regular topological space so that bounded continuous functions can be extended and their norms can be preserved.

Proposition 2 ([21, 6.2. Stone–Čech compactification]) *Let X be a completely regular topological space. Then there exist a compact Hausdorff space βX (the Stone–Čech compactification of X) and a continuous map $\triangle : X \to \beta X$ such that*

1. *$\triangle : X \to \triangle(X)$ is a homeomorphism.*
2. *$\triangle(X)$ is dense in βX.*

3. *If* $f \in C_b(X)$*, then there exists a unique continuous map* $\beta f : \beta X \to \mathbb{K}$ *such that* $\beta f \circ \triangle = f$.

Moreover, if Ω *is a compact Hausdorff space and* $\pi : X \to \Omega$ *is a continuous map for which the aforementioned properties hold, then* βX *is homeomorphic to* Ω *and* π *is the composition of* $\triangle$ *with the corresponding homeomorphism.*

Our next step is to define the so-called de Leeuw's map.

Definition 2 Let X be a metric space and consider the set

$$\widetilde{X} = \left\{(x, y) \in X^2 : x \neq y\right\}.$$

Then, for each map $f \in \mathrm{Lip}(X)$, we define $\widetilde{f} : \widetilde{X} \to \mathbb{K}$ as follows

$$\widetilde{f}(x, y) = \frac{f(x) - f(y)}{d(x, y)}, \quad \forall (x, y) \in \widetilde{X}.$$

The map $f \mapsto \widetilde{f}$ from $\mathrm{Lip}(X)$ into $C_b(\widetilde{X})$ is called the de Leeuw's map.

Since $\widetilde{X}$ is a metric space with the distance given by

$$((x, y), (x_0, y_0)) \mapsto d(x, x_0) + d(y, y_0), \quad \forall (x, y), (x_0, y_0) \in \widetilde{X},$$

then $\widetilde{X}$ is completely regular. Thus, we can consider the Stone–Čech compactification of $\widetilde{X}$, $\beta\widetilde{X}$. So, for any map $f \in \mathrm{Lip}(X)$ there exists a unique extension of $\widetilde{f}$, $\beta\widetilde{f} \in C(\beta\widetilde{X})$, such that $\mathrm{Lip}(f) = ||\widetilde{f}||_\infty = ||\beta\widetilde{f}||_\infty$. Note that if X is a compact space, then the disjoint union $X \cup \beta\widetilde{X}$ is a compact Hausdorff topological space endowed with the topology

$$\mathcal{T} = \left\{U \cup V : U \text{ is open in } X \text{ and } V \text{ is open in } \beta\widetilde{X}\right\}.$$

This construction identifies $\mathrm{Lip}(X)$ with a closed subspace of continuous maps:

Lemma 1 *Let X be a compact metric space. Then* $\Phi : \mathrm{Lip}(X) \to C\left(X \cup \beta\widetilde{X}\right)$ *given by*

$$\Phi(f)(w) = \begin{cases} f(w) & \text{if } w \in X, \\ \left(\beta\widetilde{f}\right)(w) & \text{if } w \in \beta\widetilde{X}, \end{cases}$$

for all $f \in \mathrm{Lip}(X)$*, is a linear isometry.*

For each $w \in X \cup \beta\widetilde{X}$, we define $\delta_w \in C\left(X \cup \beta\widetilde{X}\right)^*$ and $\widetilde{\delta}_w \in \mathrm{Lip}(X)^*$ as

$$\delta_w(g) = g(w), \quad \forall g \in C\left(X \cup \beta\widetilde{X}\right), \quad \widetilde{\delta}_w(f) = \Phi(f)(w), \quad \forall f \in \mathrm{Lip}(X).$$

It is plain that $|\widetilde{\delta}_w(f)| \leq \|f\|$ for all $f \in \mathrm{Lip}(X)$, and so $\widetilde{\delta}_w \in B_{\mathrm{Lip}(X)^*}$.

It is well-known [24, p. 441] that the extreme points of $B_{\Phi(\mathrm{Lip}(X))^*}$ are functionals of the form $\tau\cdot\delta_w|_{\Phi(\mathrm{Lip}(X))}$ where $\tau\in S_{\mathbb{K}}$ and $w\in X\cup\beta\widetilde{X}$. From this, it is possible to deduce the following

Proposition 3 *If X is a compact metric space, then every extreme point of $B_{\mathrm{Lip}(X)^*}$ is of the form $\tau\cdot\widetilde{\delta}_w$ where $\tau\in S_{\mathbb{K}}$ and $w\in X\cup\beta\widetilde{X}$.*

For our purposes we need to present part of the converse statement: $\widetilde{\delta}_w$ is an extreme point of $B_{\mathrm{Lip}(X)^*}$ when $w\in X$. This fact was proved by Roy [81, Lemma 1.2] by using a result due to de Leeuw [61, Lemma 3.2].

Proposition 4 *Let X be a metric space and $x\in X$. Then the functional $\widetilde{\delta}_x\in\mathrm{Lip}(X)^*$ given by $\widetilde{\delta}_x(f)=f(x)$ for all $f\in\mathrm{Lip}(X)$ is an extreme point of $B_{\mathrm{Lip}(X)^*}$.*

Before formulating our Lipschitz version of Holztyński theorem, we introduce an auxiliary result which makes the proof easier.

Lemma 2 ([95, Lema 2.8]) *Let X and Y be compact metric spaces and let $T\colon\mathrm{Lip}(X)\to\mathrm{Lip}(Y)$ be a linear isometry. Suppose that there exist a nonempty subset $Y_0\subset Y$ and a map $\varphi\colon Y_0\to X$ such that*

$$|T(\mathbf{1}_X)(y)|=1,\quad T(f)(y)=T(\mathbf{1}_X)(y)\,f(\varphi(y)),\quad\forall f\in\mathrm{Lip}(X),\ \forall y\in Y_0.$$

Then φ is a Lipschitz map and $\mathrm{Lip}(\varphi)\le\max\{1,\mathrm{diam}(X)/2\}$. Furthermore, if $y,z\in Y_0$ and $d(y,z)<2$, then $d(\varphi(y),\varphi(z))\le d(y,z)$.

Let us recall that a contraction is a Lipschitz map whose Lipschitz constant is less than 1. After this preparation, we can formulate the announced result.

Theorem 1 ([55, Theorem 2.4]) *Let X and Y be compact metric spaces and let $T\colon\mathrm{Lip}(X)\to\mathrm{Lip}(Y)$ be a linear isometry. Suppose that $T(\mathbf{1}_X)$ is a contraction. Then there exist a nonempty closed subset $Y_0\subset Y$, a surjective Lipschitz map $\varphi\colon Y_0\to X$ with $\mathrm{Lip}(\varphi)\le\max\{1,\mathrm{diam}(X)/2\}$ and a function $\tau\in\mathrm{Lip}(Y)$ with $\|\tau\|=1$ and $|\tau(y)|=1$ for all $y\in Y_0$ such that*

$$T(f)(y)=\tau(y)f(\varphi(y)),\quad\forall y\in Y_0,\ \forall f\in\mathrm{Lip}(X).$$

In Theorem 1, we cannot remove the condition $\mathrm{Lip}(T(\mathbf{1}_X))<1$. Weaver [97] gives an example of this fact by considering the metric space $X=\{p,q\}$, where $d(p,q)=1$, and defining $T\colon\mathrm{Lip}(X)\to\mathrm{Lip}(X)$ as $T(f)(p)=f(p)$ and $T(f)(q)=f(p)-f(q)$ for all $f\in\mathrm{Lip}(X)$.

The next result shows that the triple (Y_0,τ,φ) associated to the isometry T in Theorem 1 possesses a universal property.

Corollary 1 ([55, Corollary 2.5]) *Let X and Y be compact metric spaces and $T\colon\mathrm{Lip}(X)\to\mathrm{Lip}(Y)$ a linear isometry. Suppose that $T(\mathbf{1}_X)$ is a contraction and that Y_0, τ and φ are defined as in Theorem 1. If $Y_0'\subset Y$, and $\tau'\colon Y_0'\to\mathbb{K}$ and $\varphi'\colon Y_0'\to X$ are two Lipschitz maps such that*

$$T(f)(y) = \tau'(y) f(\varphi'(y)), \quad \forall y \in Y_0', \ \forall f \in \mathrm{Lip}(X);$$

then $Y_0' \subset Y_0$, $\tau' = \tau|_{Y_0'}$ *and* $\varphi' = \varphi|_{Y_0'}$.

Now, we study the onto case.

Lemma 3 ([95, Lema 2.11]) *Let X and Y be compact metric spaces, $T: \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ a surjective linear isometry and $Y_1 = \{y \in Y : T(\mathbf{1}_X)(y) \neq 0\}$. Then $Y_1 \neq \emptyset$, $|T(\mathbf{1}_X)(y)| = 1$ for all $y \in Y_1$ and there exists an injective Lipschitz map $\varphi: Y_1 \to X$ such that*

$$T(f)(y) = T(\mathbf{1}_X)(y) f(\varphi(y)), \quad \forall y \in Y_1, \ \forall f \in \mathrm{Lip}(X).$$

Next, we improve this lemma by using Theorem 1 and the following notion.

Definition 3 Let X be a metric space, $r > 0$ and let C be a subset of X. C is said to be r-connected if it is not possible to find nonempty sets $A, B \subset C$ such that $C = A \cup B$ and $d(A, B) \geq r$.

Based on this concept, we define on X the following equivalence relation: $x \sim z$ if and only if there exist $x_1, \ldots, x_k, x_{k+1} \in X$ such that $x_1 = x$, $x_{k+1} = z$ and $d(x_j, x_{j+1}) < r$ for all $j \in \{1, \ldots, k\}$. It can easily be checked that the equivalence classes of this relation are maximal r-connected subsets on X. This equivalence classes of the relation $\sim$ are called r-connected components of X, and it is straightforward to prove that they are (pairwise disjoint) open and closed subsets of X.

We get the mentioned improvement of Lemma 3:

Theorem 2 ([95, Teorema 2.14], compare to [55, Theorem 3.1]) *Let X and Y be compact metric spaces and let T be a linear isometry from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ such that $T(\mathbf{1}_X)$ is a contraction. Then there exist a Lipschitz map $\tau: Y \to S_{\mathbb{K}}$ and a Lipschitz homeomorphism $\varphi: Y \to X$ with $\mathrm{Lip}(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$ and $\mathrm{Lip}(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)/2\}$ such that*

$$T(f)(y) = \tau(y) f(\varphi(y)), \quad \forall y \in Y, \ \forall f \in \mathrm{Lip}(X).$$

Moreover, if $y, z \in Y$ and $d(y, z) < 2$, then $d(\varphi(y), \varphi(z)) = d(y, z)$ and $\tau(y) = \tau(z)$.

Under the conditions given in Theorem 2, if we also suppose that T is unital, then we get that T is an algebra isomorphism. More specifically:

Corollary 2 ([95, Corolario 2.15], see also [55, Corollary 3.2]) *Let X and Y be compact metric spaces and $T: \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ a surjective linear isometry such that $T(\mathbf{1}_X) = \mathbf{1}_Y$. Then there exist a Lipschitz homeomorphism φ from Y to X such that $d(\varphi(y), \varphi(z)) = d(y, z)$ for any $y, z \in Y$ with $d(y, z) < 2$, $\mathrm{Lip}(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$, $\mathrm{Lip}(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)/2\}$ and $T(f) = f \circ \varphi$ for all $f \in \mathrm{Lip}(X)$.*

If $\alpha \in S_{\mathbb{K}}$ and $\varphi\colon Y \to X$ is a surjective isometry, it is easy to prove that $T\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ defined as $T(f) = \alpha \cdot (f \circ \varphi)$ for all $f \in \mathrm{Lip}(X)$ is a surjective linear isometry. Conversely, we have the following consequence of Lemma 3.

Corollary 3 ([95, Corolario 2.16], see also [55, Theorem 3.3]) *Let X and Y be compact metric spaces whose diameters are less than 2 and let $T\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a surjective linear isometry such that $T(\mathbf{1}_X)$ is a contraction. Then there exist $\alpha \in S_{\mathbb{K}}$ and a surjective isometry $\varphi\colon Y \to X$ such that $T(f) = \alpha \cdot (f \circ \varphi)$ for all $f \in \mathrm{Lip}(X)$.*

If we replace the hypothesis $\mathrm{Lip}(T(\mathbf{1}_X)) < 1$ in Theorem 2 by the condition that the metric spaces are 1-connected, we can get, as a consequence, the following result by Weaver [98, Theorem 2.6.7]. Note that conditions $\mathrm{diam}(X) \leq 2$ and $\mathrm{diam}(Y) \leq 2$ are not restrictive, since we can apply Vasavada's reduction [98, Proposition 1.7.1].

Corollary 4 ([95, Corolario 2.17]) *Let X and Y be compact 1-connected metric spaces whose diameters are less than or equal to 2 and let $T\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a surjective linear isometry. Then there exist $\alpha \in S_{\mathbb{K}}$ and a surjective isometry $\varphi\colon Y \to X$ such that $T(f) = \alpha \cdot (f \circ \varphi)$ for all $f \in \mathrm{Lip}(X)$.*

3 Codimension 1 Linear Isometries

Let us recall that a linear map $T\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ has codimension 1 if $T(\mathrm{Lip}(X))$ is a maximal vector subspace of $\mathrm{Lip}(Y)$.

This section is devoted to describe codimension 1 linear isometries T from $\mathrm{Lip}(X)$ to $\mathrm{Lip}(Y)$ such that $T(\mathbf{1}_X)$ is a contraction. With this aim, we make use of Theorem 1. First of all, we see that these maps can be of two types.

Proposition 5 ([55, Theorem 4.1]) *Let X and Y be compact metric spaces and let $T\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a codimension 1 linear isometry such that $T(\mathbf{1}_X)$ is a contraction. Then there exist a subset $Y_0 \subset Y$, a surjective Lipschitz map $\varphi\colon Y_0 \to X$ and a Lipschitz function $\tau\colon Y_0 \to S_{\mathbb{K}}$ such that*

$$T(f)(y) = \tau(y) f(\varphi(y)), \quad \forall y \in Y_0,\ \forall f \in \mathrm{Lip}(X).$$

Furthermore, either $Y_0 = Y$ or $Y_0 = Y \backslash \{p\}$ for some isolated point $p \in Y$.

Proposition 5 allows us to make the following classification.

Definition 4 Let X and Y be compact metric spaces and let $T\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a codimension 1 linear isometry such that $T(\mathbf{1}_X)$ is a contraction. We say that:

1. T is of type I if there exists an isolated point $p \in Y$ such that T is a weighted composition operator on $Y \backslash \{p\}$ whose symbol function is a surjective Lipschitz map $\varphi\colon Y \backslash \{p\} \to X$ and its weight function is a Lipschitz map $\tau\colon Y \backslash \{p\} \to S_{\mathbb{K}}$.

2. T is of type II if it is a weighted composition operator on Y whose symbol function is a surjective Lipschitz map $\varphi\colon Y \to X$ and whose weight map is a Lipschitz function $\tau\colon Y \to S_{\mathbb{K}}$.

These two types are not disjoint as the next result shows:

Proposition 6 ([55, Proposition 4.2]) *Let Y be a compact metric space and suppose that there exist two points $y_0, p \in Y$ such that $y_0 \neq p$ and $d(y, y_0) \leq d(y, p)$ for all $y \in Y\backslash\{p\}$ (in particular, this happens when $\operatorname{diam}(Y\backslash\{p\}) \leq d(p, Y\backslash\{p\})$). Then p is an isolated point of Y and the map $T : \operatorname{Lip}(Y\backslash\{p\}) \to \operatorname{Lip}(Y)$ defined by*

$$T(f)(y) = f(y), \quad \forall y \in Y\backslash\{p\}, \quad T(f)(p) = f(y_0), \quad \forall f \in \operatorname{Lip}(Y\backslash\{p\}),$$

is a codimension 1 linear isometry with $\operatorname{Lip}(T(\mathbf{1}_X)) < 1$ which is simultaneously of type I and type II.

Next we give a method for constructing codimension 1 linear isometries of type I which are not of type II.

Proposition 7 ([55, Proposition 4.3]) *Let X and Y be compact metric spaces and let $\alpha \in S_{\mathbb{K}}$. Suppose that there exist $p \in Y$ such that $1 < d(p, Y\backslash\{p\})$ and a surjective isometry $\varphi\colon Y\backslash\{p\} \to X$. Then the map $T : \operatorname{Lip}(X) \to \operatorname{Lip}(Y)$ defined by*

$$T(f)(y) = \alpha f(\varphi(y)), \quad \forall y \in Y\backslash\{p\}, \quad T(f)(p) = 0, \quad \forall f \in \operatorname{Lip}(X),$$

is a codimension 1 linear isometry with $\operatorname{Lip}(T(\mathbf{1}_X)) < 1$ of type I but it is not of type II.

There are also codimension 1 linear isometries of type II which are not of type I:

Proposition 8 *Consider the metric spaces $X = [0, 2]$ and $Y = [0, 1] \cup [4, 5]$ equipped with their usual distances. Let $\varphi\colon Y \to X$ and $\tau : Y \to S_{\mathbb{K}}$ be the maps given by*

$$\varphi(y) = \begin{cases} y & \text{if } y \in [0, 1], \\ y - 3 & \text{if } y \in [4, 5]; \end{cases} \qquad \tau(y) = \begin{cases} 1 & \text{if } y \in [0, 1], \\ -1 & \text{if } y \in [4, 5]. \end{cases}$$

The map $T : \operatorname{Lip}(X) \to \operatorname{Lip}(Y)$ defined by

$$T(f)(y) = \tau(y) f(\varphi(y)), \quad \forall y \in Y, \ \forall f \in \operatorname{Lip}(X),$$

is a codimension 1 linear isometry. Observe that T is of type II, but it is not of type I because Y has no isolated points.

Next, we study some properties of the map φ. In particular, we show that φ is a Lipschitz homeomorphism when T is of type I.

Theorem 3 ([55, Proposition 4.6]) *Let X and Y be compact metric spaces and let $T: \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a codimension 1 linear isometry with $\mathrm{Lip}(T(\mathbf{1}_X)) < 1$. Take Y_0, φ and τ as in Proposition 5. The following statements hold:*

1. *Given $x \in X$, $\varphi^{-1}(\{x\})$ has at most two elements.*
2. *If there exists a point $x_0 \in X$ such that $\varphi^{-1}(\{x_0\})$ has two distinct points, then, for all $x \in X\backslash\{x_0\}$, $\varphi^{-1}(\{x\})$ consists of an only element.*
3. *If T is of type I, then, for the corresponding isolated point $p \in Y$, we have that $\varphi: Y\backslash\{p\} \to X$ is a Lipschitz homeomorphism.*

4 Local Isometries

This section is a contribution to an interesting current topic, the study of normed spaces whose isometry group is determined by the local behavior of its elements.

Definition 5 Let E be a normed space and let $\mathcal{S}$ be a nonempty subset of $\mathcal{L}(E)$. We denote by $\mathrm{ref}_{\mathrm{al}}(\mathcal{S})$ the set of operators $T \in \mathcal{L}(E)$ verifying the following property:

$$\forall e \in E \;\; \exists\, \Phi_e \in \mathcal{S}\colon \;\; T(e) = \Phi_e(e).$$

It is said that $\mathcal{S}$ is algebraically reflexive if $\mathrm{ref}_{\mathrm{al}}(\mathcal{S}) = \mathcal{S}$.

We say that E is iso-reflexive if $\mathcal{G}(E)$ is algebraically reflexive, where $\mathcal{G}(E)$ stands for the set of all surjective linear isometries from E to E.

The elements of $\mathrm{ref}_{\mathrm{al}}(\mathcal{G}(E))$ are called locally surjective linear isometries or local isometries. Thus, E is iso-reflexive if and only if every locally surjective linear isometry is surjective.

Using those descriptions of into linear isometries (Theorem 1) and onto linear isometries (Corollary 4) between $\mathrm{Lip}(X)$-spaces, we can state the following result on the algebraic reflexivity of $\mathrm{Lip}(X)$.

Theorem 4 ([95, Teorema 2.26]) *Let X be a compact 1-connected metric space. Then $\mathrm{Lip}(X)$ is iso-reflexive.*

Now, we use the iso-reflexivity of $\mathrm{Lip}(X)$ in order to study the algebraic reflexivity of some subsets of isometries and projections on $\mathrm{Lip}(X)$. Let us introduce these sets. We use the symbol Id_A to denote the identity map on a nonempty set A, or Id if there is no confusion.

Definition 6 Let X be a metric space and let E be a normed space.

1. An isometry $\varphi: X \to X$ is said to be involutive if φ^2 is the identity on X.
2. An isometric reflection of E is an involutive linear isometry on E.
3. A linear map $P: E \to E$ is a projection if $P^2 = P$.

4. It is said that a projection $P\colon E \to E$ is bi-circular if $\alpha P + \gamma\,(\mathrm{Id} - P)$ is an isometry for any $\alpha, \gamma \in S_{\mathbb{K}}$. It is clear that P is bi-circular if and only if $P + \lambda \cdot (\mathrm{Id} - P)$ is an isometry for all $\lambda \in S_{\mathbb{K}}$.
5. A projection $P\colon E \to E$ is called generalized bi-circular if there exists $\lambda \in S_{\mathbb{K}} \setminus \{1\}$ such that $P + \lambda \cdot (\mathrm{Id} - P)$ is an isometry.
6. A projection $P\colon E \to E$ with $\|P\| \leq 1$ is said to be contractive. If $\|P\| \leq 1$ and $\|\mathrm{Id} - P\| \leq 1$, we say that P is bi-contractive.

From Corollary 4, we can easily get the description of the isometric reflections of $\mathrm{Lip}(X)$ as weighted composition operators:

Corollary 5 ([95, Corolario 2.28]) *Let X be a compact 1-connected metric space whose diameter is less than or equal to 2. A map $T : \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ is an isometric reflection if and only if there exist a constant $\tau \in \{-1, 1\}$ and an involutive isometry φ on X such that*

$$T(f)(x) = \tau \cdot f(\varphi(x)), \quad \forall x \in X,\ \forall f \in \mathrm{Lip}(X).$$

Corollary 6 ([95, Corolario 2.29]) *Let X be a compact 1-connected metric space. Then the set of isometric reflections of $\mathrm{Lip}(X)$ is algebraically reflexive.*

Now, we proceed in the same way with generalized bi-circular projections. Firstly, we give a description of them:

Proposition 9 ([95, Proposición 2.30]) *Let X be a compact 1-connected metric space whose diameter is less than or equal to 2. A map $P\colon \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ is a generalized bi-circular projection if and only if there exist a number $\tau \in \{-1, 1\}$ and an involutive isometry $\varphi\colon X \to X$ such that*

$$P(f)(x) = \frac{1}{2}\left[f(x) + \tau \cdot f(\varphi(x))\right], \quad \forall x \in X,\ \forall f \in \mathrm{Lip}(X).$$

Secondly, we show the algebraic reflexivity of the set of generalized bi-circular projections:

Corollary 7 ([95, Corolario 2.31]) *Let X be a compact 1-connected metric space. Then the set of all generalized bi-circular projections on $\mathrm{Lip}(X)$ is algebraically reflexive.*

The following result is an immediate consequence of Proposition 9.

Corollary 8 ([95, Corolario 2.32]) *Let X be a compact 1-connected metric space. Every generalized bi-circular projection on $\mathrm{Lip}(X)$ is a bi-contractive projection.*

All the results of this section were stated with similar proofs in [54] for $\mathrm{Lip}(X)$ endowed with the sum norm, when X is a compact metric space.

5 Projections and Averages of Linear Isometries

We have just shown that every generalized bi-circular projection on $\mathrm{Lip}(X)$ can be expressed as the average of two isometries. Now, we prove that such projections are the only projections on $\mathrm{Lip}(X)$ having this property. In order to achieve these aim, we first characterize when the average of two isometries is a projection on $\mathrm{Lip}(X)$.

First of all, we study the projections on $\mathrm{Lip}(X)$ which can be expressed as convex combinations of two surjective linear isometries. Let I_1 and I_2 be two surjective linear isometries on $\mathrm{Lip}(X)$ given by

$$I_k(f)(x) = \tau_k f(\varphi_k(x)), \quad \forall f \in \mathrm{Lip}(X),\ \forall x \in X \qquad (k = 1, 2),$$

where $\tau_k \in \mathbb{K}$ with $|\tau_k| = 1$ and $\varphi_k \colon X \to X$ is a surjective isometry.

Our initial objective is to find conditions on the constants τ_k, the functions φ_k and the parameter $0 < \lambda < 1$ in order to $\lambda I_1 + (1 - \lambda)I_2$ is a projection on $\mathrm{Lip}(X)$:

Proposition 10 ([12, Proposition 2.1]) *Let P be a projection on* $\mathrm{Lip}(X)$ *and let* $0 < \lambda < 1$. *If $P = \lambda I_1 + (1 - \lambda)I_2$, then*

(i) $\tau_1 = \tau_2 = 1$, *or* $\tau_1 = -\tau_2$ *and* $\lambda = 1/2$.
(ii) *If* $\varphi_1(x) \neq \varphi_2(x)$, *then either* $\varphi_1(x) = x$ *or* $\varphi_2(x) = x$.
(iii) *If* $x = \varphi_1(x) \neq \varphi_2(x)$, *then* $\varphi_1(\varphi_2(x)) = \varphi_2(x)$, $\varphi_2^2(x) = x$, $\lambda = 1/2$, $\tau_1 = 1$ *and* $\tau_2^2 = 1$.
(iv) *If* $x = \varphi_2(x) \neq \varphi_1(x)$, *then* $\varphi_2(\varphi_1(x)) = \varphi_1(x)$, $\varphi_1^2(x) = x$, $\lambda = 1/2$, $\tau_2 = 1$ *and* $\tau_1^2 = 1$.

Now, we characterize the operators $(I_1 + I_2)/2$ which are projections on $\mathrm{Lip}(X)$.

Proposition 11 ([12, Proposition 2.2]) *The operator* $(I_1 + I_2)/2$ *is a projection on* $\mathrm{Lip}(X)$ *if and only if one of the following statements holds:*

1. $\tau_1 = \tau_2 = 1$ *and for each* $x \in X$ *it is verified that:*
   a. $x = \varphi_1(x) = \varphi_2(x)$, *or*
   b. $x = \varphi_1(x) \neq \varphi_2(x)$, $\varphi_1(\varphi_2(x)) = \varphi_2(x)$ *and* $\varphi_2^2(x) = x$, *or*
   c. $x = \varphi_2(x) \neq \varphi_1(x)$, $\varphi_2(\varphi_1(x)) = \varphi_1(x)$ *and* $\varphi_1^2(x) = x$.
2. $\tau_1 = -\tau_2$ *and* $\varphi_1(x) = \varphi_2(x)$ *for all* $x \in X$, *namely* $((I_1 + I_2)/2)(f)(x) = 0$, *for all* $f \in \mathrm{Lip}(X)$.
3. $\tau_1 = 1$, $\tau_2 = -1$ *and each* $x \in X$ *satisfies:*
   a. $\varphi_1(x) = \varphi_2(x)$, *or*
   b. $x = \varphi_1(x) \neq \varphi_2(x)$, $\varphi_1(\varphi_2(x)) = \varphi_2(x)$ *and* $\varphi_2^2(x) = x$.
4. $\tau_1 = -1$, $\tau_2 = 1$ *and each* $x \in X$ *verifies:*
   a. $\varphi_1(x) = \varphi_2(x)$, *or*
   b. $x = \varphi_2(x) \neq \varphi_1(x)$, $\varphi_2(\varphi_1(x)) = \varphi_1(x)$ *and* $\varphi_1^2(x) = x$.

We know that a generalized bi-circular projection on $\mathrm{Lip}(X)$ is the average of the identity map and an involutive isometry. By applying Proposition 11 yields:

Theorem 5 ([12, Theorem 2.3]) *A linear projection on* $\mathrm{Lip}(X)$ *is the average of two surjective linear isometries if and only if it is a generalized bi-circular projection.*

Following a process similar to the one above, we next characterize the projections on $\mathrm{Lip}(X)$ which can be expressed as the average of three surjective linear isometries.

Theorem 6 ([12, Theorem 3.8]) *For* $k = 1, 2, 3$*, let* I_k *be a surjective linear isometry on* $\mathrm{Lip}(X)$ *given by*

$$I_k(f)(x) = \tau_k f(\varphi_k(x)) \quad (f \in \mathrm{Lip}(X),\ x \in X) \quad (k = 1, 2, 3),$$

where τ_k *is a unimodular scalar and* φ_k *is a surjective isometry on* X*, and let* Q *be the average of* I_1*,* I_2 *and* I_3*. Then* Q *is a projection on* $\mathrm{Lip}(X)$ *if and only if there exist a* $\tau \in \mathbb{K}$ *with* $\tau^3 = 1$ *and a surjective isometry* φ *on* X *with* $\varphi^3 = \mathrm{Id}$ *such that*

$$Q(f)(x) = \frac{f(x) + \tau f(\varphi(x)) + \tau^2 f(\varphi^2(x))}{3}$$

for all $f \in \mathrm{Lip}(X)$ *and* $x \in X$*.*

The previous theorem motivates the following concept.

Definition 7 Let $n \in \mathbb{N}$ with $n \geq 2$. It is said that a continuous linear operator Q on $\mathrm{Lip}(X)$ is a n-circular projection if there exist a scalar $\tau \in \mathbb{K}$ such that $\tau^n = 1$ and a surjective isometry φ on X such that $\varphi^n = \mathrm{Id}$ and $\varphi^k \neq \mathrm{Id}$ for $k = 1, \ldots, n-1$ satisfying

$$Q(f)(x) = \frac{\sum_{k=0}^{n-1} \tau^k f(\varphi^k(x))}{n},$$

for every $f \in \mathrm{Lip}(X)$ and $x \in X$. We take $\varphi^0 = \mathrm{Id}$.

Theorems 5 and 6 can be restated as in the following theorem. We refer to a projection as being trivial if it is equal to either the zero or the identity operators.

Theorem 7 ([12, Theorem 4.2]) *Let* X *be a compact* 1*-connected metric space with* $\mathrm{diam}(X) \leq 2$.

1. *The average of two surjective isometries on* $\mathrm{Lip}(X)$ *is a projection if and only if it is either a trivial projection or a* 2*-circular projection.*
2. *The average of three surjective isometries on* $\mathrm{Lip}(X)$ *is a projection if and only if it is either a trivial projection or a* 3*-circular projection.*

We end this section with two remarks motivated by the previous results.

Remark 1 It can be checked that if X is a compact 1-connected metric space with $\text{diam}(X) \le 2$, then 3-circular projections on $\text{Lip}(X)$ cannot be expressed as the average of two surjective linear isometries on $\text{Lip}(X)$.

Remark 2 It is well-known that generalized bi-circular projections are bicontractive [37]. Observe that 3-circular projections are not necessarily bicontractive. In fact, let $X = \{a, b, c\}$ equipped with the metric $d(a, b) = d(b, c) = d(a, c) = 2$. Consider $P = (\text{Id} + R + R^2)/3$ with $R(f) = f \circ \varphi$ and φ a period 3 isometry on X ($\varphi(a) = b$, $\varphi(b) = c$ and $\varphi(c) = a$). Then $\text{Id} - P = (2\text{Id} - R - R^2)/3$. Consider $f \in \text{Lip}(X)$ such that $f(\varphi(a)) = f(\varphi^2(a)) = -1$ and $f(a) = 1$. Note that $\|f\| = 6/5$ and $\|(\text{Id} - P)(f)\| = 23/15$, hence $\|\text{Id} - P\| > 1$.

6 2-Local Isometries

The notion of locally surjective linear isometry can be generalized as follows: on the one hand, linearity can be suppressed; on the other hand, we can consider several points (more than one) in which the map coincides with a surjective linear isometry. Taking this into account, it is not surprising to introduce the following definition.

Definition 8 Let E and F be normed spaces. We say that a map $\Phi\colon E \to F$ is a 2-local isometry if for any $x, y \in E$, there exists a surjective linear isometry $\Phi_{x,y}\colon E \to F$, depending, in general, on x and y, such that $\Phi(x) = \Phi_{x,y}(x)$ and $\Phi(y) = \Phi_{x,y}(y)$.

It is said that a Banach space E is 2-iso-reflexive if every 2-local isometry from E to E is linear and surjective.

This concept was introduced by Molnár in [70], where he showed that the space of linear bounded operators on a Hilbert space is 2-iso-reflexive. Numerous papers on 2-iso-reflexivity of Banach spaces have appeared since then (cf. [1, 41, 42, 51, 62]).

In the following results, we consider the space $\text{Lip}(X)$ endowed with the maximum norm $\max\left\{\|f\|_\infty, \text{Lip}(f)\right\}$ or with the sum norm $\|f\|_\infty + \text{Lip}(f)$. It is easy to check that if τ is a unimodular scalar and φ is a surjective isometry on X, then the weighted composition operator given by

$$\Phi(f) = \tau \cdot (f \circ \varphi), \quad \forall f \in \text{Lip}(X),$$

is a surjective linear isometry on $\text{Lip}(X)$. If every surjective linear isometry on $\text{Lip}(X)$ can be expressed in this way, it is said that the isometry group of $\text{Lip}(X)$ is canonical. The isometry group of $\text{Lip}(X)$ is not canonical in general (see an example in [97]). However, Hatori and Oi [44] showed that the isometry group of $\text{Lip}(X)$ with the sum norm is canonical when X is compact (see also [48]). On the other hand, the same conclusion was obtained for the maximum norm of $\text{Lip}(X)$ by Roy [81] when X is compact and connected with diameter at most 1; and, independently, by Vasavada [92] when X is compact and satisfies certain separation conditions.

First of all, we give a description of 2-local isometries on $\mathrm{Lip}(X)$ in terms of a weighted composition operator.

Theorem 8 ([57, Theorem 2.1]) *Let X be a bounded metric space and let Φ be a 2-local isometry on $\mathrm{Lip}(X)$ whose isometry group is canonical. Then there exist a subset X_0 of X, a unimodular scalar τ and a Lipschitz bijection $\varphi\colon X_0 \to X$ such that*

$$\Phi(f)|_{X_0} = \tau \cdot (f \circ \varphi), \quad \forall f \in \mathrm{Lip}(X).$$

Clearly, every 2-local isometry on a Banach space is an isometry, so the main problem is to study if 2-local isometries are linear and surjective. We can give a positive answer for 2-local isometries on $\mathrm{Lip}(X)$. More precisely, when X is, in addition, separable, we show that $X_0 = X$ and φ is a Lipschitz homeomorphism, and thus Φ is a surjective linear isometry on $\mathrm{Lip}(X)$. For this aim, we use the following result which is interesting in itself. It is a Lipschitz version of a result due to Győry [41] for continuous functions. A detailed reading of its proof shows that its adaptation to Lipschitz functions is far from being simple.

Proposition 12 ([57, Proposition 3.2]) *Let X be a metric space and let $R = \{r_n : n \in \mathbb{N}\}$ be a countable set of distinct points of X. Then there exist two Lipschitz maps $f, g\colon X \to [0, 1]$ satisfying that f has a strict local maximum in each point of R and*

$$\{z \in X : (f(z), g(z)) = (f(r_n), g(r_n))\} = \{r_n\}, \quad \forall n \in \mathbb{N}.$$

We have all the ingredients to prove the most important result on 2-locality on $\mathrm{Lip}(X)$.

Theorem 9 ([57, Theorem 3.3]) *Let X be a separable bounded metric space. If the isometry group of* $\mathrm{Lip}(X)$ *is canonical, then* $\mathrm{Lip}(X)$ *is 2-iso-reflexive.*

Now, let us recall the following concept.

Definition 9 If E and F are normed spaces, it is said that a map $\Phi\colon E \to F$ is a weak 2-local isometry if for any $x, y \in E$ and $\varphi \in F^*$, there exists a surjective linear isometry $\Phi_{x,y,\varphi}\colon E \to F$, depending in general on x, y, φ, such that $\varphi\Phi(x) = \varphi\Phi_{x,y,\varphi}(x)$ and $\varphi\Phi(y) = \Phi_{x,y,\varphi}(y)$.

In [62], the authors stated a spherical version of the Kowalski–Slodkowski theorem [60], and applied this result to prove the following fact:

Theorem 10 ([62, Theorem 3.5 and Corollaries 3.8 and 3.9]) *Let X be a metric space and assume that the isometry group of* $\mathrm{Lip}(X)$ *is canonical. Then every weak 2-local isometry on* $\mathrm{Lip}(X)$ *is linear. As a consequence, if X is in addition compact, then* $\mathrm{Lip}(X)$ *is 2-iso-reflexive.*

The final part of this section is devoted to the study of 2-locality on the space of vector-valued Lipschitz maps $\mathrm{Lip}(X, E)$ equipped with the maximum norm. To achieve our goals it is necessary to get a description of the surjective linear isometries of such spaces. In our case, we make use of the following result by Botelho, Fleming and Jamison.

Theorem 11 ([8, Theorem 10]) *Let X and Y be pathwise-connected compact metric spaces, let E and F be smooth reflexive Banach spaces, and let* $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ *be a surjective linear isometry. Then, there exist a Lipschitz homeomorphism* $\varphi : Y \to X$ *with* $\mathrm{Lip}(\varphi) \leq \max\{1, \mathrm{diam}(X)\}$ *and* $\mathrm{Lip}(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)\}$, *and a Lipschitz map* $V : Y \to \mathrm{Iso}(E, F)$ *such that*

$$T(f)(y) = V(y)(f(\varphi(y))), \quad \forall y \in Y, \ f \in \mathrm{Lip}(X, E).$$

Araujo and Dubarbie [3] introduced a subclass of surjective linear isometries admitting an expression similar to the one given in Theorem 11, which is also adequate for our purposes.

Given X and Y two metric spaces, a bijection $\varphi : Y \to X$ is said to preserve distances less than 2 if $d(\varphi(x), \varphi(y)) = d(x, y)$ when $d(x, y) < 2$. We denote by $\mathrm{Iso}_{<2}(Y, X)$ the set of all bijections $\varphi : Y \to X$ such that φ and φ^{-1} preserve distances which are less than 2. Note that every element $\varphi \in \mathrm{Iso}_{<2}(Y, X)$ is a Lipschitz homeomorphism when X and Y are bounded.

Definition 10 ([3, Definition 2.3]) Let X and Y be metric spaces and let E and F be normed spaces. We say that a map $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is a standard isometry if there exist a bijection $\varphi \in \mathrm{Iso}_{<2}(Y, X)$ and a map $J : Y \to \mathrm{Iso}(E, F)$ which is constant in each 2-component of Y such that

$$T(f)(y) = J(y)(f(\varphi(y))), \quad \forall y \in Y, \ f \in \mathrm{Lip}(X, E).$$

In fact, J is a Lipschitz map.

Note that every standard isometry is a surjective linear isometry. Theorem 3.1 in [3] provides us a condition under which both kinds of isometries coincide.

Definition 11 Let X and Y be metric spaces and let E and F be normed spaces. A map $\Delta : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is said to be a 2-local standard isometry if for any $f, g \in \mathrm{Lip}(X, E)$, there exists a standard isometry $T_{f,g} : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$, depending in general on f and g, such that $\Delta(f) = T_{f,g}(f)$ and $\Delta(g) = T_{f,g}(g)$.

We say that $\mathrm{Lip}(X, E)$ is 2-standard-iso-reflexive if every 2-local standard isometry on $\mathrm{Lip}(X, E)$ is linear and surjective.

Under certain conditions on a compact metric space X and a normed space E, we can give a description of all 2-local isometries and all 2-local standard isometries on $\mathrm{Lip}(X, E)$ as a generalized composition operator, and study when the space $\mathrm{Lip}(X, E)$ is 2-iso-reflexive or 2-standard-iso-reflexive.

Let us recall that a normed space E is smooth if for every point $e \in S_E$, there exists a unique functional $e^* \in S_{E^*}$ such that $e^*(e) = 1$. It is known that a space E is smooth if and only if its norm is Gâteaux-differentiable at every point of S_E.

We establish a Lipschitz version of [1, Theorem 6].

Theorem 12 ([52, Theorem 2.3]) *Let X and Y be pathwise-connected compact metric spaces, let E be a smooth reflexive Banch space which is 2-iso-reflexive, and let $\Delta\colon \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, E)$ be a 2-local isometry. Then there exist a subset Y_0 of Y, a Lipschitz bijection ψ from Y_0 into X, and a continuous map $y \in Y_0 \mapsto V(y) \in \mathrm{Iso}(E)$ from Y_0 to $\mathrm{Iso}(E)$ endowed with the strong operator topology (SOT for short) such that*

$$\Delta(f)(y) = V(y)(f(\psi(y))), \quad \forall y \in Y_0,\ f \in \mathrm{Lip}(X, E).$$

Similar reasoning given in the aforementioned theorem can be used to describe 2-local standard isometries.

Theorem 13 ([52, Theorem 2.4]) *Let X and Y be compact metric spaces, let E be a smooth and reflexive Banach space which is also 2-iso-reflexive, and let $\Delta\colon \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, E)$ be a 2-local standard isometry. Then there exist a subset Y_0 of Y, a Lipschitz bijection ψ from Y_0 into X, and a continuous map V from Y_0 to $(\mathrm{Iso}(E), \mathrm{SOT})$ such that*

$$\Delta(f)(y) = V(y)(f(\psi(y))), \quad \forall y \in Y_0,\ f \in \mathrm{Lip}(X, E).$$

Theorem 12 and Proposition 12 provide us the tools to apply the proof method introduced by Győry in [41].

Theorem 14 ([52, Theorem 2.6]) *Let X and Y be pathwise-connected compact metric spaces, let E be a smooth and reflexive Banach space which is 2-iso-reflexive, and let $\Delta\colon \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, E)$ a 2-local isometry. Then there exist a Lipschitz homeomorphism ψ from Y into X and a continuous map V from Y to $(\mathrm{Iso}(E), \mathrm{SOT})$ such that*

$$\Delta f(y) = V(y)(f(\psi(y))), \quad \forall y \in Y,\ f \in \mathrm{Lip}(X, E).$$

Furthermore, Δ is a surjective linear isometry.

Thanks to Theorem 13, the arguments used at the proof of the previous theorem also work in the case of 2-local standard isometries.

Theorem 15 ([52, Theorem 2.7]) *Let X and Y be compact metric spaces, let E be a smooth and reflexive which is 2-iso-reflexive, and let $\Delta\colon \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, E)$ be a 2-local standard isometry. Then there exist a Lipschitz homeomorphism ψ from Y to X and a continuous map V from Y into $(\mathrm{Iso}(E), \mathrm{SOT})$ such that*

$$\Delta f(y) = V(y)(f(\psi(y))), \quad \forall y \in Y,\ f \in \mathrm{Lip}(X, E).$$

Moreover, Δ is a surjective linear isometry.

In Theorems 14 and 15, it was shown that, under certain hypotheses on X and E, the space $\mathrm{Lip}(X, E)$ is 2-iso-reflexive and 2-standard-iso-reflexive, respectively. One of these requirements is that E is 2-iso-reflexive. In fact, this condition is necessary to get the conclusion of these theorems.

Theorem 16 ([52, Theorem 2.8]) *Let X be a metric space and let E be a Banach space. If $\mathrm{Lip}(X, E)$ is 2-iso-reflexive or 2-standard-iso-reflexive, then E is 2-iso-reflexive.*

Finally, it is interesting to point out that our results also hold if, in their hypotheses, we consider a 2-local standard isometry Δ from $\mathrm{Lip}(X, E)$ into $\mathrm{Lip}(Y, F)$, where E and F are smooth and reflexive Banach spaces for which each 2-local isometry from E into F is linear and surjective. In that case, the map V in the conclusion takes its values on $\mathrm{Iso}(E, F)$.

7 Vector-Valued Linear Isometries

We give now a complete description of linear isometries between $\mathrm{Lip}(X, E)$-spaces satisfying some mild conditions. So we establish a Lipschitz version of a known theorem by Cambern [17] on linear isometries between $C(X, E)$-spaces.

Theorem 17 ([56, Theorem 2.1]) *Let X and Y be compact metric spaces, let E be a strictly convex normed space and let $T: \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, E)$ be a linear isometry. Suppose that $T(\mathbf{1}_X \otimes e) = \mathbf{1}_Y \otimes e$ for some $e \in S_E$. Then there exist a closed nonempty subset Y_0 of Y, a surjective Lipschitz map $\varphi: Y_0 \to X$ with $\mathrm{Lip}(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$ and a Lipschitz map $\widehat{T}: Y \to \mathcal{L}(E)$ with $\|\widehat{T}(y)\| = 1$ for all $y \in Y$ such that*

$$T(f)(y) = \widehat{T}(y)(f(\varphi(y))), \quad \forall y \in Y_0, \ \forall f \in \mathrm{Lip}(X, E).$$

The above condition, $T(\mathbf{1}_X \otimes e) = \mathbf{1}_Y \otimes e$ for some $e \in S_E$, is not too restrictive if one studies some known results in the scalar case. Our condition in this case means $T(\mathbf{1}_X) = \mathbf{1}_Y$ and notice that connectedness assumptions on the metric spaces in [81, Lemma 1.5] and [97, Lemma 6] imply that the function $T(\mathbf{1}_X)$ is constant.

Jerison's theorem [50] on surjective linear isometries between $C(X, E)$-spaces also has a natural formulation for Lipschitz maps:

Theorem 18 ([56, Theorem 3.1]) *Let X and Y be compact metric spaces, E a strictly convex normed space and $T: \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, E)$ a surjective linear isometry such that $T(\mathbf{1}_X \otimes e) = \mathbf{1}_Y \otimes e$ for some $e \in S_E$. Then there exist a Lipschitz homeomorphism $\varphi: Y \to X$ with $\mathrm{Lip}(\varphi) \leq \max\{1, \mathrm{diam}(X)/2\}$ and $\mathrm{Lip}(\varphi^{-1}) \leq \max\{1, \mathrm{diam}(Y)/2\}$, and a Lipschitz map $\widehat{T}: Y \to \mathcal{L}(E)$ such that $\widehat{T}(y)$ is an isometry from E onto itself for all $y \in Y$ and*

$$T(f)(y) = \widehat{T}(y)(f(\varphi(y))), \quad \forall y \in Y, \ \forall f \in \mathrm{Lip}(X, E).$$

As a direct consequence of Theorem 18, we obtain the following:

Corollary 9 ([56, Corollary 3.2]) *Let X and Y be two compact metric spaces whose diameters are less than or equal to 2, let E be a strictly convex normed space and $T\colon \mathrm{Lip}(X,E)\to \mathrm{Lip}(Y,E)$ a surjective linear isometry such that $T(\mathbf{1}_X\otimes e)=\mathbf{1}_Y\otimes e$ for some $e\in S_E$. Then there exist a surjective isometry $\varphi\colon Y\to X$ and a Lipschitz map $\widehat{T}\colon Y\to \mathcal{L}(E)$ such that $\widehat{T}(y)$ is a surjective isometry from E to E for all $y\in Y$ and we have that*

$$T(f)(y)=\widehat{T}(y)(f(\varphi(y))),\quad \forall y\in Y,\ \forall f\in \mathrm{Lip}(X,E).$$

When E is a Hilbert space, Theorems 17 and 18 can be improved:

Corollary 10 ([56, Corollary 3.3]) *Let X and Y be compact metric spaces and let E be a Hilbert space. Let T be a linear isometry from $\mathrm{Lip}(X,E)$ into $\mathrm{Lip}(Y,E)$ such that $T(1_X\otimes e)$ is a constant function for some $e\in S_E$. Then there exists a Lipschitz map φ from a closed subset Y_0 of Y to X with $\mathrm{Lip}(\varphi)\le\max\{1,\mathrm{diam}(X)/2\}$, and a Lipschitz map $\widehat{T}$ from Y into $\mathcal{L}(E)$ with $||\widehat{T}(y)||=1$ for all $y\in Y$, such that*

$$T(f)(y)=\widehat{T}(y)(f(\varphi(y))),\quad \forall y\in Y_0,\ \forall f\in \mathrm{Lip}(X,E).$$

If, in addition, T is surjective, then $Y_0=Y$, φ is a Lipschitz homeomorphism with $\mathrm{Lip}(\varphi^{-1})\le\max\{1,\mathrm{diam}(Y)/2\}$ and, for each $y\in Y$, $\widehat{T}(y)$ is a surjective isometry.

Some versions of the preceding results for isometries from $\mathrm{Lip}(X,E)$ to $\mathrm{Lip}(Y,F)$ with $E\neq F$ can be consulted in [95].

8 Scalar-Valued Hermitian Operators

Let A be a complex Banach algebra with unity I and let A^* be its dual space. Given $a\in A$, recall that the algebraic numerical range $V(a)$ is given by

$$V(a)=\left\{F(a)\colon F\in A^*,\ \|F\|=F(I)=1\right\}.$$

An element $a\in A$ is said to be hermitian if $V(a)\subset\mathbb{R}$. It is known that $a\in A$ is hermitian if and only if $\|\exp(ita)\|=1$ for all $t\in\mathbb{R}$.

Let E be a complex Banach space and $\mathcal{L}(E)$ the Banach algebra of all bounded linear operators on E equipped with the operator norm. It is well-known that an operator $T\in\mathcal{L}(E)$ is hermitian if and only if $\exp(itT)$ is an isometry for each $t\in\mathbb{R}$. The set of hermitian operators on E is a real subspace of $\mathcal{L}(E)$ which contains all operators of the form $\lambda\mathrm{Id}$, where λ is a real number. A hermitian operator is said to be trivial if it is a real multiple of the identity operator.

We study in this section the hermitian operators defined on spaces $\mathrm{Lip}(X)$ and $\mathrm{lip}(X^\alpha)$ with $\alpha\in]0,1[$, equipped with the sum norm. In [6], it was shown that hermitian operators on Lip([0, 1]) and $\text{lip}([0,1]^\alpha)$ are trivial. We generalize here this result by showing that the same property holds for the spaces Lip(X) and $\text{lip}(X^\alpha)$ with $\alpha \in]0, 1[$, whenever X is a compact metric space.

The proof of our main theorem requires some preliminary lemmas. Firstly, we need to have a good representation for surjective linear isometries on such spaces.

Theorem 19 ([14, Theorem 2.1]) *Let X be a compact metric space. Then $T\colon \text{Lip}(X) \to \text{Lip}(X)$ is a surjective linear isometry such that $T(\mathbf{1}_X) = \mathbf{1}_X$ if and only if there exists a surjective isometry $\varphi\colon X \to X$ such that T is of the form $T(f) = f \circ \varphi$ for all $f \in \text{Lip}(X)$. The same characterization holds for a surjective linear isometry T on $\text{lip}(X^\alpha)$ $(0 < \alpha < 1)$ such that $T(\mathbf{1}_X) = \mathbf{1}_X$.*

Secondly, we characterize the hermitian elements of such spaces.

Lemma 4 ([14, Lemma 2.2]) *Let X be a compact metric space and $h \in \text{Lip}(X)$ (or $\text{lip}(X^\alpha)$, $0 < \alpha < 1$). Then h is a hermitian element in $\text{Lip}(X)$ (or $\text{lip}(X^\alpha)$) if and only if h is a real constant function.*

Following an idea of de Leeuw [61], we embed the Banach spaces Lip(X) and $\text{lip}(X^\alpha)$ $(0 < \alpha < 1)$ isometrically into some suitable spaces of complex-valued continuous functions.

Let X be a compact metric space. Since $\widetilde{X}$ is completely regular, we can consider $\beta\widetilde{X}$, the Stone-Čech compactification of $\widetilde{X}$. Let $C(X \cup \beta\widetilde{X})$ denote the Banach space of all complex-valued continuous functions on $X \cup \beta\widetilde{X}$, under the norm

$$\|f\| = \left\| f|_X \right\|_\infty + \|f|_{\beta\widetilde{X}}\|_\infty \qquad (f \in C(X \cup \beta\widetilde{X})),$$

and let $C_0(X \cup \widetilde{X})$ denote the Banach space of all complex-valued continuous functions on $X \cup \widetilde{X}$ vanishing at the infinity, endowed with the norm

$$\|f\| = \left\| f|_X \right\|_\infty + \left\| f|_{\widetilde{X}} \right\|_\infty \qquad (f \in C_0(X \cup \widetilde{X})).$$

Let us recall that the Riesz representation theorem states that the map $\mu \mapsto F_\mu$, given by

$$F_\mu(f) = \int_{X\cup\beta\widetilde{X}} f\, d\mu \qquad (f \in C(X \cup \beta\widetilde{X})),$$

defines an isometric isomorphism from the Banach space $\mathcal{M}(X \cup \beta\widetilde{X})$ of all complex-valued regular Borel measures on $X \cup \beta\widetilde{X}$ equipped with the norm of total variation:

$$\|\mu\| = |\mu|\,(X \cup \beta\widetilde{X}) \qquad (\mu \in \mathcal{M}(X \cup \beta\widetilde{X}))$$

onto the dual space of $(C(X \cup \beta\widetilde{X}), \|\cdot\|_\infty)$. Similarly, the map $\nu \mapsto G_\nu$ defined by

$$G_\nu(f) = \int_{X\cup\widetilde{X}} f\, d\nu \qquad (f \in C_0(X \cup \widetilde{X}))$$

is an isometric isomorphism from the Banach space $\mathcal{M}(X \cup \widetilde{X})$ with the norm:

$$\|\nu\| = |\nu|\,(X \cup \widetilde{X}) \qquad (\nu \in \mathcal{M}(X \cup \widetilde{X}))$$

onto the dual space of $(C_0(X \cup \widetilde{X}), \|\cdot\|_\infty)$.

For each $f \in \mathrm{Lip}(X)$ or $f \in \mathrm{lip}(X^\alpha)$, $0 < \alpha < 1$, we set $\widetilde{f}\colon \widetilde{X} \to \mathbb{C}$ to be the map given by

$$\widetilde{f}(x, y) = \frac{f(x) - f(y)}{d(x, y)^\alpha}, \quad \forall (x, y) \in \widetilde{X},$$

where $\alpha = 1$ when $f \in \mathrm{Lip}(X)$. It is easy to show that $\widetilde{f}$ is continuous on $\widetilde{X}$ and $||\widetilde{f}||_\infty = p_\alpha(f)$ $(0 < \alpha \leq 1)$. Hence there exists a unique continuous function $\beta\widetilde{f}$ on $\beta\widetilde{X}$ such that $(\beta\widetilde{f})|_{\widetilde{X}} = \widetilde{f}$ and $||\beta\widetilde{f}||_\infty = ||\widetilde{f}||_\infty$. Furthermore, if $f \in \mathrm{lip}(X^\alpha)$, then $\widetilde{f}$ vanishes at infinity on $\widetilde{X}$. The maps $\Phi\colon \mathrm{Lip}(X) \to C(X \cup \beta\widetilde{X})$ and $\Psi\colon \mathrm{lip}(X^\alpha) \to C_0(X \cup \widetilde{X})$, defined by

$$\Phi(f)(w) = \begin{cases} f(w) & \text{if } w \in X, \\ \beta\widetilde{f}(w) & \text{if } w \in \beta\widetilde{X}, \end{cases}$$

and

$$\Psi(f)(w) = \begin{cases} f(w) & \text{if } w \in X, \\ \widetilde{f}(w) & \text{if } w \in \widetilde{X}, \end{cases}$$

are isometric linear embeddings from $\mathrm{Lip}(X)$ with the norm $\|\cdot\|_s$ into $C(X \cup \beta\widetilde{X})$, and from $\mathrm{lip}(X^\alpha)$ with the norm $\|\cdot\|_s$ into $C_0(X \cup \widetilde{X})$, respectively.

Using these identifications, the Hahn–Banach theorem and the Riesz representation theorem provide the following descriptions of the dual spaces of $\mathrm{Lip}(X)$ and $\mathrm{lip}(X^\alpha)$.

Lemma 5 ([14, Lemma 2.3]) *Let X be a compact metric space.*

1. *For each* $F \in \mathrm{Lip}(X)^*$, *there exists* $\mu \in \mathcal{M}(X \cup \beta\widetilde{X})$ *with* $\|F\| \leq \|\mu\|$ *satisfying*

$$F(f) = \int_{X \cup \beta\widetilde{X}} \Phi(f)(w)\, d\mu(w), \quad \forall f \in \mathrm{Lip}(X).$$

2. *Let* $\alpha \in (0, 1)$. *For each* $G \in \mathrm{lip}(X^\alpha)^*$, *there exists* $\nu \in \mathcal{M}(X \cup \widetilde{X})$ *with* $\|G\| \leq \|\nu\|$ *such that*

$$G(f) = \int_{X \cup \widetilde{X}} \Psi(f)(w)\, d\nu(w), \quad \forall f \in \mathrm{lip}(X^\alpha).$$

Finally, we need some properties of hermitian operators on our Lipschitz spaces.

Lemma 6 ([14, Lemmas 3.2 and 3.3]) *Let* $A(X)$ *denote either* $\mathrm{Lip}(X)$ *or* $\mathrm{lip}(X^\alpha)$, $0 < \alpha < 1$. *Recall that* $\alpha = 1$ *in the case* $A(X) = \mathrm{Lip}(X)$. *If* $T : A(X) \to A(X)$ *is a hermitian bounded linear operator, then the following statements hold:*

(i) *There exists* $\lambda \in \mathbb{R}$ *such that* $T(\mathbf{1}_X) = \lambda \mathbf{1}_X$.
(ii) *For each* $t \in \mathbb{R}$, $\exp(it(T - \lambda \mathrm{Id}))$ *is a surjective linear isometry on* $A(X)$ *fixing* $\mathbf{1}_X$.
(iii) *For each* $t \in \mathbb{R}$, *there exists a surjective isometry* φ_t *on* X *such that*

$$\exp(it(T - \lambda \mathrm{Id}))(f)(x) = f(\varphi_t(x)), \qquad \forall f \in A(X),\ \forall x \in X.$$

(iv) $\{\varphi_t\}_{t\in\mathbb{R}}$ *is a one-parameter group of surjective isometries on* X *such that, for each* $x \in X$, *the map* $t \mapsto \varphi_t(x)$ *from* $\mathbb{R}$ *to* X *is continuous.*
(v) *For every* $f \in A(X)$,

$$\lim_{t\to 0}(f \circ \varphi_t - f)(x) = 0, \qquad \forall x \in X,$$

and

$$\lim_{t\to 0} \frac{(f \circ \varphi_t - f)(x) - (f \circ \varphi_t - f)(y)}{d(x, y)^\alpha} = 0, \qquad \forall (x, y) \in \widetilde{X}.$$

(vi) *For every* $f \in A(X)$,

$$\lim_{t\to 0} \beta \widetilde{f_t}(w) = 0, \qquad \forall w \in \beta \widetilde{X},$$

where, for each $t \in \mathbb{R}$, f_t *denotes the function* $f \circ \varphi_t - f$.

Using the previous results, we may prove our main theorem which describes all the hermitian operators on $\mathrm{Lip}(X)$ or $\mathrm{lip}(X^\alpha)$ with $0 < \alpha < 1$.

Theorem 20 ([14, Theorem 3.1]) *Let* X *be a compact metric space. A bounded linear operator* $T : \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ *is hermitian if and only if* T *is a real multiple of the identity operator on* $\mathrm{Lip}(X)$. *An analogous assertion holds for* $T : \mathrm{lip}(X^\alpha) \to \mathrm{lip}(X^\alpha)$ *with* $0 < \alpha < 1$.

We also mention the natural connection between hermitian operators and the class of bi-circular projections. Jamison [47] showed that these projections are exactly the hermitian projections. Our result implies that the only bi-circular projections on $\mathrm{Lip}(X)$ and $\mathrm{lip}(X^\alpha)$ with $0 < \alpha < 1$ are the trivial projections, 0 and Id.

9 Vector-Valued Hermitian Operators

Our objective in this section is to characterize the hermitian bounded operators on $\mathrm{Lip}(X, E)$, equipped with the maximum norm, with X a compact and 2-connected metric space and E a complex Banach space with norm $\|\cdot\|_E$.

We now review the definition of semi-inner product on a complex Banach space, as presented in [65, 66]. Given a complex Banach space E, a function $[\cdot,\cdot]_E\colon E\times E\to\mathbb{C}$ is called a semi-inner product if, for every $x, y, z \in E$ and $\lambda\in\mathbb{C}$, the following properties hold:

1. $[x+y,z]_E=[x,z]_E+[y,z]_E$,
2. $[\lambda x,y]_E=\lambda[x,y]_E$,
3. $[x,x]_E>0$ for $x\neq 0$,
4. $|[x,y]_E|^2\leq[x,x]_E[y,y]_E$.

A semi-inner product $[\cdot,\cdot]_E$ is said to be compatible with the norm $\|\cdot\|_E$ if $[x,x]_E=\|x\|_E^2$ for every $x\in E$. The existence of semi-inner products compatible with the norm follows from the Hahn–Banach theorem which guarantees the existence of duality maps $u\mapsto\varphi_u$ from E into E^* which satisfy $\|\varphi_u\|=1$ and $\varphi_u(u)=\|u\|_E$. Such a duality map yields a semi-inner product by defining $[u,v]_E=\varphi_v(u)$. Such maps are not unique and so there are several semi-inner products compatible with the existing norm unless the unit ball of E is smooth. We denote the sets of hermitian bounded operators on E by $H(E)$, respectively. See [33] for these results.

A bounded operator T on E is hermitian if and only if there exists a semi-inner product $[\cdot,\cdot]_E$ compatible with the norm such that $[Tx,x]_E\in\mathbb{R}$ for every $x\in E$. It is important to mention that if T is hermitian, then for every semi-inner product $[\cdot,\cdot]$ on E compatible with the norm, $[Tx,x]\in\mathbb{R}$ for every $x\in E$, cf. [33].

Note that the Stone–Čech compactification $\beta(\widetilde{X}\times B(E^*))$ of $\widetilde{X}\times B(E^*)$ is a compact space containing $\widetilde{X}\times B(E^*)$ as a dense subspace. For each $f\in\text{Lip}(X,E)$, the bounded continuous mapping $\widetilde{f}\colon\widetilde{X}\times B(E^*)\to\mathbb{C}$, given by

$$\widetilde{f}((x,y),\varphi)=\varphi\left(\frac{f(x)-f(y)}{d(x,y)}\right),$$

has a unique continuous extension $\beta(\widetilde{f})\colon\beta(\widetilde{X}\times B(E^*))\to\mathbb{C}$ such that $||\beta(\widetilde{f})||_\infty=||\widetilde{f}||_\infty$.

We now consider the isometric embedding

$$\Gamma\colon\text{Lip}(X,E)\to C(X\cup\beta(\widetilde{X}\times B(E^*)),E\oplus_\infty\mathbb{C})$$

given by

$$\begin{aligned} f\to\Gamma(f)\colon X\cup\beta(\widetilde{X}\times B(E^*))&\to E\oplus_\infty\mathbb{C}\\ x\in X&\to(f(x),0),\\ \xi\in\beta(\widetilde{X}\times B(E^*))&\to(0,\beta(\widetilde{f})(\xi)).\end{aligned}$$

Standard techniques show that Γ is a linear isometry. For each $g\in\text{Lip}(X,E)$, we define

$$P_g=\left\{t\in X\cup\beta(\widetilde{X}\times B(E^*))\colon\ \|\Gamma(g)(t)\|_{E\oplus_\infty\mathbb{C}}=\|g\|\right\}.$$

We first choose a semi-inner product on E, $[\cdot,\cdot]_E$, compatible with the norm. Then we define the following semi-inner product $[\cdot,\cdot]_{E\oplus_\infty\mathbb{C}}$ by

$$[(u_0,\lambda_0),(u_1,\lambda_1)]_{E\oplus_\infty\mathbb{C}} = \begin{cases} [u_0,u_1]_E & \text{if } \|(u_1,\lambda_1)\|_{E\oplus_\infty\mathbb{C}} = \|u_1\|_E, \\ \lambda_0\overline{\lambda_1} & \text{if } \|(u_1,\lambda_1)\|_{E\oplus_\infty\mathbb{C}} \neq \|u_1\|_E. \end{cases}$$

This semi-inner product is compatible with the norm on $E \oplus_\infty \mathbb{C}$. It is easy to see that this semi-inner product induces the following semi-inner products:

$$[(u,0),(v,0)]_{E\oplus_\infty\mathbb{C}} = [u,v]_E,$$
$$[(0,\lambda_0),(0,\lambda_1)]_{E\oplus_\infty\mathbb{C}} = \lambda_0\overline{\lambda_1},$$

on the respective component spaces $\{(u,0)\colon u\in E\}$ and $\{(0,\lambda)\colon \lambda\in\mathbb{C}\}$, compatible with the existing norms.

Let $\psi\colon \mathrm{Lip}(X,E)\to\cup_{g\in\mathrm{Lip}(X,E)}P_g$ be a mapping such that $\psi(g)\in P_g$ for each $g\in\mathrm{Lip}(X,E)$. We now define

$$[f,g]_\psi = [\Gamma(f)(\psi(g)),\Gamma(g)(\psi(g))]_{E\oplus_\infty\mathbb{C}} \qquad (f,g\in\mathrm{Lip}(X,E)).$$

This is a semi-inner product in $\mathrm{Lip}(X,E)$ compatible with the norm on $\mathrm{Lip}(X,E)$, since

$$[f,f]_\psi = [\Gamma(f)(\psi(f)),\Gamma(f)(\psi(f))]_{E\oplus_\infty\mathbb{C}} = \|\Gamma(f)(\psi(f))\|^2_{E\oplus_\infty\mathbb{C}} = \|f\|^2$$

for all $f\in\mathrm{Lip}(X,E)$. Given $v\in E$, the symbol $\mathbf{v}$ represents the constant function on X everywhere equal to v.

Lemma 7 ([13, Lemma 2.1]) *Let X be a compact metric space, E a complex Banach space and T a hermitian bounded operator on* $\mathrm{Lip}(X,E)$*. Then the function $A\colon X\to\mathcal{L}(E)$, given by $A(x)(v)=T(\mathbf{v})(x)$ for all $x\in X$ and $v\in E$, is Lipschitz on X and with values in $H(E)$.*

Proposition 13 ([13, Proposition 2.2]) *Let X be a compact metric space, E a complex Banach space and T a hermitian bounded operator on* $\mathrm{Lip}(X,E)$*. If $f\in\mathrm{Lip}(X,E)$ and $x_0\in X$ are such that $f(x_0)=0$, then $T(f)(x_0)=0$.*

Using the two preceding results, we can give the following description of the bounded hermitian operators on $\mathrm{Lip}(X,E)$.

Proposition 14 ([13, Proposition 2.3]) *Let X be a compact metric space, E a complex Banach space and T a bounded operator on* $\mathrm{Lip}(X,E)$*. If T is hermitian, then there exists a mapping $A\in\mathrm{Lip}(X,H(E))$ such that $T(f)(x)=A(x)(f(x))$ for every $f\in\mathrm{Lip}(X,E)$ and $x\in X$.*

We now are ready to characterize such operators.

Theorem 21 ([13, Theorem 2.4]) *Let X be a compact and 2-connected metric space, E a complex Banach space and $T: \mathrm{Lip}(X, E) \to \mathrm{Lip}(X, E)$ a bounded operator. Then T is hermitian if and only if there exists a hermitian bounded operator $A: E \to E$ such that $T(f)(x) = A(f(x))$ for every $f \in \mathrm{Lip}(X, E)$ and $x \in X$.*

Taking into account that the metric space X is compact and that the 2-connected components of X are open sets in X, the next corollary follows straightforwardly from Theorem 21 and Proposition 14.

Corollary 11 ([13, Corollary 2.6]) *Let X be a compact metric space, E a complex Banach space, $T: \mathrm{Lip}(X, E) \to \mathrm{Lip}(X, E)$ a hermitian bounded operator and $X_1, \ldots, X_m$ the 2-connected components of X. Then there exist m hermitian bounded operators $A_1, \ldots, A_m: E \to E$ such that*

$$T(f)(x) = \sum_{j=1}^{m} A_j(\chi_j(f)(x)), \qquad \forall x \in X, \ \forall f \in \mathrm{Lip}(X, E),$$

where, for each $j \in \{1, \ldots, m\}$, $\chi_j(f)(x) = f(x)$ if $x \in X_j$ and $\chi_j(f)(x) = 0$ otherwise.

We now state the result for the scalar case which follows as a particular case of Theorem 21.

Corollary 12 ([13, Corollary 2.7]) *Let X be a compact and 2-connected metric space and T a bounded operator on $\mathrm{Lip}(X)$. Then T is hermitian if and only if T is a real multiple of the identity operator on $\mathrm{Lip}(X)$.*

We make now some remarks on adjoint abelian and normal operators on $\mathrm{Lip}(X, E)$. We start with the definitions of adjoint abelian and normal operators as presented in [83] and in [59].

Definition 12 Let E be a complex Banach space and let $T: E \to E$ be a bounded operator.

1. T is adjoint abelian if and only if there exists a semi-inner product $[\cdot, \cdot]$ compatible with the norm of E such that $[Tx, y] = [x, Ty]$ for all $x, y \in E$.
2. T is normal if and only if there exist two hermitian and commuting operators T_0 and T_1 on E such that $T = T_0 + iT_1$.

The results presented before imply the following.

Theorem 22 ([13, Theorem 3.2]) *Let X be a compact and 2-connected metric space, E a complex Banach space and T a bounded operator on $\mathrm{Lip}(X, E)$.*

1. *If T is an adjoint abelian hermitian operator, then there exist an adjoint abelian hermitian operator A on E such that $T(f) = Af$ for every $f \in \mathrm{Lip}(X, E)$.*
2. *T is normal if and only if there exist commuting hermitian operators A and B on E such that $T(f) = Af + iBf$ for all $f \in \mathrm{Lip}(X, E)$.*

10 Open Problems

According to the antecedents and the current state of the research lines previously showed in this paper, we next present some concrete problems which are pretended to be studied in the near future.

Our previous study of linear isometries on Lipschitz spaces suggests the immediate approach of a couple of problems. We studied codimension 1 linear isometries on Lip(X) in [55]. Araujo and Font tackled in [5] a related issue for finite codimension linear isometries between $C(X, E)$-spaces. On the other hand, the study of isometries of spaces of Lipschitz maps is restricted to the case in which these maps are defined between the spaces Lip(X, E) but, apparently, very few is known when these maps are defined between subspaces of Lip(X, E).

Problem 1 Making use of our Lipschitz version of Cambern theorem (17) for the first question and the extreme point method for the second one, we pose:

1. To give a complete description of linear isometries from Lip(X, E) into Lip(Y, F), whose ranges have finite codimension.
2. To establish Banach–Stone type theorems for linear isometries defined between subspaces of vector-valued Lipschitz maps.

A problem which has not been addressed yet is the study of bilinear isometries between Lipschitz spaces. Let us recall that a bilinear map T from $C(X, E) \times C(Y, E)$ to $C(Z, E)$ is an isometry if

$$\|T(f, g)\|_\infty = \|f\|_\infty \|g\|_\infty, \quad \forall (f, g) \in C(X, E) \times C(Y, E).$$

Some examples of these bilinear isometries can be found in reference [80].

In [74], Moreno and Rodríguez proved the following bilinear version of the well-known Holsztyński theorem on (not necessarily surjective) linear isometries of the spaces $C(X)$ (see [45] and also [4]): if $T : C(X) \times C(Y) \to C(Z)$ is a bilinear isometry, then there exist a closed subset Z_0 of Z, a surjective continuous map h from Z_0 to $X \times Y$ and a norm one map $a \in C(Z)$ such that

$$T(f, g)(z) = a(z)(f(\pi_X(h(z))), g(\pi_Y(h(z)))), \quad \forall z \in Z_0, \ \forall (f, g) \in C(X) \times C(Y).$$

The proof of this fact is essentially based on Stone–Weierstrass theorem.

In [35], Font and Sanchís extended these results to certain subspaces of scalar-valued continuous functions, where this theorem cannot been applied. Furthermore, they studied in [36] conditions under which a Banach–Stone type representation for bilinear isometries on $C(X, E)$ can be obtained, which we can see below.

Given X, Y, Z compact Hausdorff spaces and E_1, E_2, E_3 Banach spaces, we say that a map $T : C(X, E_1) \times C(Y, E_2) \to C(Z, E_3)$ is stable on constants if, for any $(f, g) \in C(X, E_1) \times C(Y, E_2)$ and $z \in Z$, it is satisfied that

$$\|T(f, \mathbf{1}_X \otimes e_2)(z)\| = \|T(f, \mathbf{1}_X \otimes u_2)(z)\|$$

for all $e_2, u_2 \in S_{E_2}$, and

$$\|T(\mathbf{1}_X \otimes e_1, g)(z)\| = \|T(\mathbf{1}_X \otimes u_1, g)(z)\|$$

for every $e_1, u_1 \in S_{E_1}$. We denote by $\mathrm{Bil}(E_1 \times E_2, E_3)$ the space of all continuous bilinear maps from $E_1 \times E_2$ into E_3, equipped with the strong operator topology.

Font and Sanchís proved that if $T: C(X, E_1) \times C(Y, E_2) \to C(Z, E_3)$ is a stable-on-constants isometry and E_3 is strictly convex, then there exist a set $Z_0 \subset Z$, a continuous function $\omega: Z_0 \to \mathrm{Bil}(E_1 \times E_2, E_3)$ and a surjective continuous map $h: Z_0 \to X \times Y$ such that

$$T(f, g)(z) = \omega(z)(f(\pi_X(h(z))), g(\pi_Y(h(z)))), \ \forall z \in Z_0, \ \forall (f, g) \in C(X, E_1) \times C(Y, E_2).$$

In some sense, they showed that the stability of T on constants is a necessary condition. The aforementioned result extends, on the one hand, the theorem by Moreno and Rodríguez [74] taking $E_1 = E_2 = E_3 = \mathbb{K}$, and, on the other hand, the vector-valued version of Holsztyński theorem proved by Cambern [17] supposing that Y is a singleton and E_2 is the field $\mathbb{K}$.

Naturally, the notion of bilinear isometry can be defined between other distinguished subspaces of continuous functions and, this way, it is possible to give rise to

Problem 2 Study the Banach–Stone type representation of bilinear isometries defined between $\mathrm{Lip}(X, E)$ spaces.

Other kind of isometries which can be interesting to study is the class of approximate isometries. In [93], Vestfrid introduced the concept of ε-nonexpansive map as follows. Given two metric spaces (X, d_X) and (Y, d_Y) and $\varepsilon > 0$, it is said that $f: X \to Y$ is ε-nonexpansive if

$$d_Y(f(x), f(y)) \leq d_X(x, y) + \varepsilon, \quad \forall x, y \in X.$$

We say that f is a ε-isometric map or a ε-isometry if

$$|d_Y(f(x), f(y)) - d_X(x, y)| \leq \varepsilon, \quad \forall x, y \in X.$$

An ε-isometry defined between normed spaces is called standard if $f(0) = 0$.

A notorious result, proved by Mazur and Ulam [68] in 1932, states that if $f: X \to Y$ is a surjective isometry between two real Banach spaces, then f is affine. Hyers and Ulam [46] showed in 1945 that for every standard surjective ε-isometry $f: X \to Y$ between two real Hilbert spaces, there exists a surjective linear isometry $g: X \to Y$ such that $\|f(x) - g(x)\| \leq 10\varepsilon$ for all $x \in X$. The reader is referred to the introduction of [93] for a wide discussion on some extensions of these results and others related topics.

In his work, Vestfrid obtained Hyers–Ulam type results for maps from $C(X)$ into $C(Y)$. His main theorem assures that if X is a compact Hausdorff space, Y is a metrizable compact space and $T: C(X) \to C(Y)$ is a standard ε-isometry, then there exists an isometry (which is not necessarily linear) $S: C(X) \to C(Y)$ such that $\|T(f) - S(f)\|_\infty \leq 5\varepsilon$ for all $f \in C(X)$. If, in addition, for each proper closed subset S of Y, there exists $f \in C(X)$ such that $|T(f)(z)| < \|T(f)\|_\infty - 3.5\varepsilon$ for all $z \in S$, then S can be chosen so that it is linear. This assertion does not hold, for instance, for the ℓ_p-norm with $1 < p < \infty$. The proof is based on some results concerning the approximation of ε-nonexpansive maps from a metric space M to $\ell_\infty(S)$ where S is an arbitrary set, or to $C_b(S)$ where S is a topological space, by means of nonexpansive maps.

The previous results motivate the approach of the following problems in the setting of spaces of Lipschitz functions:

Problem 3 Study the Hyers–Ulam type stability with respect to, either the uniform norm, or the maximum or the sum Lipschitz norms, of both (not necessarily surjective) ε-nonexpansive maps and standard ε-isometries from $\mathrm{Lip}(X, E)$ into $\mathrm{Lip}(Y, F)$.

Now we present some questions which are also open for $C(X)$-spaces: given compact metric spaces X and Y,

1. Is there any constant c such that for each (continuous) ε-nonexpansive map T from $\mathrm{Lip}(X)$ to $\mathrm{Lip}(Y)$, there exists a nonexpansive map S from $\mathrm{Lip}(X)$ into $\mathrm{Lip}(Y)$ whose distance to T is less than or equal to $c\varepsilon$?
2. Is there any constant c such that for each ε-isometry $T\colon \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$, there exists an isometry S from $\mathrm{Lip}(X)$ to $\mathrm{Lip}(Y)$ whose distance to T is less than or equal to $c\varepsilon$?
3. If the answer to any of the previous questions is positive, what is the best possible constant c?

The Mazur–Ulam theorem [68] has a wide amount of applications in different areas of Mathematics. In 1972, Mankiewicz proved a local version of this theorem showing that every bijective isometry between convex sets in normed linear spaces with nonempty interiors admits a unique extension to a bijective affine isometry between the corresponding spaces. Thus this result states that it is not necessary to have a surjective isometry defined between the totality of the spaces X and Y to identify them through an affine transformation, it is enough to make an isometric identification of their respective closed unit balls.

One of the most modern variants of the Mazur–Ulam theorem is due to a result by Tingley [91] in 1987. The result, known as Tingley's problem, asks when a surjective isometry $T\colon S_X \to S_Y$ can be extended to a real linear isometry from X into Y. It is known that this problem has a positive answer for certain classical Banach spaces such as sequence spaces, continuous function algebras and spaces of measurable functions [85, 87, 96]. The work [100] can give a general vision on the advances in this problem.

The list of spaces for which Tingley's question has a positive answer has been increased by the recent works of Fernández, Peralta and Tanaka and it includes the space of compact operators, compact C*-algebras and weakly compact JB*-triples [29, 79], von Neumann factors of type I, atomic von Neumann algebras and atomic JBW*-triples [30, 31], spaces of trace class operators [28] and positive operator algebras [77]. See [78] for a compilation of recent results on Tingley's problem in operator algebras.

The Mazur–Ulam property is intrinsically related to Tingley's problem. According to [18], it is said that a Banach space Z satisfies the Mazur–Ulam property if every surjective isometry from S_Z into S_Y, where Y is any Banach space, admits an (unique) extension to a real surjective linear isometry from Z to Y.

A pioneer contribution due to Ding [23] shows that $c_0(\mathbb{N}, \mathbb{R})$ satisfies the Mazur–Ulam property. The list of real Banach spaces having this property includes $c(\Gamma, \mathbb{R})$, $c_0(\Gamma, \mathbb{R})$ and $\ell_\infty(\Gamma, \mathbb{R})$ where Γ is an infinite set endowed with the discrete topology, and $C(K, \mathbb{R})$ where K is a compact metric space (see [26, 64] and their references). In [85–87], it was shown that the space $L^p((\Omega, \Sigma, \mu), \mathbb{R})$ of all real measurable functions on a σ-finite measure space (Ω, Σ, μ) satisfies the Mazur–Ulam property for all $1 \leq p \leq \infty$. In the complex case, the problem for the spaces $c_0(\Gamma, \mathbb{C})$ and $\ell_\infty(\Gamma, \mathbb{C})$ was closed in [53] and [76], respectively. All this literature justifies our interest in tackling the following:

Problem 4 Study Tingley's problem and the Mazur–Ulam property for the spaces of Lipschitz functions.

Our previous study on projections motivates the last two problems of this section. Theorem 7 claims that, under certain constraints, the average of two (three) isometries on $\mathrm{Lip}(X)$ is a nontrivial projection if and only if it is a 2-circular (respectively, 3-circular) projection, and, therefore, it is natural to arise the following question:

Problem 5 Let X be a compact 1-connected metric space with $\mathrm{diam}(X) \leq 2$ and $n \geq 2$. Is the average of n pairwise distinct isometries on $\mathrm{Lip}(X)$ a projection if and only if it is a trivial projection or a n-circular projection?

It is well-known by a celebrated work by Friedman and Russo [38] that if X is a compact Hausdorff metric space, then a linear projection $P : C(X) \to C(X)$ is bicontractive if and only if $P = (1/2)(\mathrm{Id} + T)$, where T is an involutive isometry on $C(X)$.

Problem 6 Study contractive and bicontractive projections on $\mathrm{Lip}(X)$ in order to determine under which conditions on the metric space X, the space $\mathrm{Lip}(X)$ belongs to the class of Banach spaces E in which every bicontractive projection $P : E \to E$ is of the form $P = (1/2)(\mathrm{Id} + T)$, where $T : E \to E$ is an involutive isometry.

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