# On a conjecture of A. Magnus concerning the asymptotic behavior of the recurrence coefficients of the generalized Jacobi polynomials

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## Abstract

In 1995 Magnus [15] posed a conjecture about the asymptotics of the recurrence coefficients of orthogonal polynomials with respect to the weights on $[-1, 1]$ of the form

$$(1-x)^{\alpha} (1+x)^{\beta} |x_0 - x|^{\gamma} \times \begin{cases} B, & \text{for } x \in [-1, x_0), \\ A, & \text{for } x \in [x_0, 1], \end{cases}$$

with $A, B > 0$, $\alpha, \beta, \gamma > -1$, and $x_0 \in (-1, 1)$. We show rigorously that Magnus' conjecture is correct even in a more general situation, when the weight above has an extra factor, which is analytic in a neighborhood of $[-1, 1]$ and positive on the interval. The proof is based on the steepest descendent method of Deift and Zhou applied to the non-commutative Riemann-Hilbert problem characterizing the orthogonal polynomials. A feature of this situation is that the local analysis at $x_0$ has to be carried out in terms of confluent hypergeometric functions.

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# 1. Introduction and statement of results

## 1.1. Introduction

A. Magnus considered in [15] a weight function which is smooth and positive on the whole interval of orthogonality up to a finite number of points where algebraic singularities occur. His primary goal was to investigate the influence of these singular points on the asymptotic behavior of the recurrence coefficients of the corresponding orthogonal polynomials (generalized Jacobi polynomials). Based on numerical evidence, he conjectured explicit formulas for the asymptotics of these coefficients (as the degree of the polynomial grows) for the weights of the form

$$(1-x)^\alpha (1+x)^\beta |x_0 - x|^\gamma \times \begin{cases} B, & \text{for } x \in [-1, x_0), \\ A, & \text{for } x \in [x_0, 1], \end{cases} \tag{1}$$

with $A, B > 0$ and $\alpha, \beta, \gamma > -1$, and $x_0 \in (-1, 1)$. This weight combines at a single point both an algebraic singularity and a jump.

So far, Magnus' conjecture has been confirmed rigorously in some special cases (see below); our main goal is to establish it in its full generality, and even to extend it further. Namely, we consider polynomials that are orthogonal on a finite interval $[-1, 1]$ with respect to a modified Jacobi weight of the form

$$w_{c,\gamma}(x) = (1-x)^\alpha (1+x)^\beta |x_0 - x|^\gamma h(x)\, \Xi_c(x), \qquad x \in [-1, 1], \tag{2}$$

where $x_0 \in (-1, 1)$, $\alpha, \beta, \gamma > -1$, $h$ is real analytic and strictly positive on $[-1, 1]$, and $\Xi_c$ is a step-like function, equal to 1 on $[-1, x_0)$ and $c^2 > 0$ on $[x_0, 1]$.

The proof is based on the nonlinear steepest descent analysis of Deift and Zhou, introduced in [7] and further developed in [2, 6, 9], which is based on the Riemann–Hilbert characterization of orthogonal polynomials due to Fokas, Its, and Kitaev [10]. A crucial contribution to this approach is [14], where the complete asymptotic expansion for the orthogonal polynomials with respect to the Jacobi weight modified by a real analytic and strictly positive function was obtained (in notation (1), $A = B$ and $\gamma = 0$). The first application of this technique to weights with a jump discontinuity is due to [13], where the authors considered an exponential weight on $\mathbb{R}$ with a jump at the origin.

Let $P_n(x) = P_n(x; w_{c,\gamma})$ be the monic polynomial of degree $n$ orthogonal with respect to the weight $w_{c,\gamma}$ on $[-1, 1]$,

$$\int_{-1}^{1} P_n(x; w_{c,\gamma}) x^k w_{c,\gamma}(x)\, dx = 0, \qquad \text{for } k = 0, 1, \dots, n-1.$$

It is well known (see e.g. [16]) that $\{P_n\}$ satisfy the three-term recurrence relation

$$P_{n+1}(z) = (z - b_n)P_n(z) - a_n^2 P_{n-1}(z). \tag{3}$$

The central result of this paper is:

2

**Theorem 1.** *The recurrence coefficients $a_n$ and $b_n$ of orthogonal polynomials corresponding to the generalized Jacobi weight (2) have a complete asymptotic expansion of the form*

$$a_n = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{A_k(n)}{n^k}, \quad b_n = -\sum_{k=1}^{\infty} \frac{B_k(n)}{n^k},$$

*as $n \to \infty$, where for every $k \in \mathbb{N}$ the coefficients $A_k(n)$ and $B_k(n)$ are bounded in $n$. In particular,*

$$A_1(n) = -\frac{\sqrt{1-x_0^2}}{2} \sqrt{\frac{\gamma^2}{4} + \frac{\log^2 c}{\pi^2}} \cos\left[2n \arccos x_0 + 2\frac{\log c}{\pi} \log\left(4n\sqrt{1-x_0^2}\right) - \Theta\right], \quad (4)$$

$$B_1(n) = -\sqrt{1-x_0^2} \sqrt{\frac{\gamma^2}{4} + \frac{\log^2 c}{\pi^2}} \cos\left[(2n+1) \arccos x_0 + 2\frac{\log c}{\pi} \log\left(4n\sqrt{1-x_0^2}\right) - \Theta\right], \tag{5}$$

*where*

$$\Theta = \left(\alpha + \frac{\gamma}{2}\right)\pi - (\alpha + \beta + \gamma)\arccos x_0 - 2\arg\Gamma\left(\frac{\gamma}{2} - i\frac{\log c}{\pi}\right)$$
$$- \arg\left(\frac{\gamma}{2} - i\frac{\log c}{\pi}\right) - \frac{\sqrt{1-x_0^2}}{\pi} \fint_{-1}^{1} \frac{\log h(t)}{\sqrt{1-t^2}} \frac{dt}{t - x_0}, \tag{6}$$

*and $\fint$ denotes the integral understood in terms of its principal value.*

**Remark 1.** We can rewrite the result of this theorem as

$$a_n = \frac{1}{2} - \frac{M}{n} \cos\left[2n\arccos x_0 - 2\mu \log\left(4n\sqrt{1-x_0^2}\right) - \Theta\right] + \mathcal{O}\left(\frac{1}{n^2}\right),$$

$$b_n = -\frac{2M}{n} \cos\left[(2n+1)\arccos x_0 - 2\mu \log\left(4n\sqrt{1-x_0^2}\right) - \Theta\right] + \mathcal{O}\left(\frac{1}{n^2}\right),$$

as $n \to \infty$, where

$$\mu = -\frac{\log c}{\pi}, \quad M = \frac{\sqrt{1-x_0^2}}{2} \sqrt{\frac{\gamma^2}{4} + \mu^2},$$

and $\Theta$ defined by (6).

A comparison of these formulas with those in [15] (setting $h(x) \equiv B$ and $c^2 = A/B$) shows that Magnus' conjecture on the asymptotic behavior of the recurrence coefficients holds true for weights of the form (1). Observe that this is a slight extension of the original statement of Magnus: (i) we allow for an extra real analytic and strictly positive factor $h$ in the weight, and (ii) we can replace the error term $o(1/n)$ in [15] by a more precise $\mathcal{O}(1/n^2)$.

Theorem 1 generalizes some previous known results about the asymptotics of the recurrence coefficients. To mention a few, weight $w_{1,0}$ was considered in [14], $w_{1,\gamma}$ is a particular case of the weight studied in [17], and $w_{c,0}$ was matter of attention in [11].

The proof is based on the steepest descent analysis of the non-commutative Riemann-Hilbert problem characterizing the orthogonal polynomials $P_n$. In theory, this approach allows to compute all coefficients $A_k$ and $B_k$ in (4)–(5). However, the computations are cumbersome and their complexity increases with $k$.

Most of the steps of the steepest descent analysis below are standard and well described in the literature, see e.g. [5, 8, 14]. The main feature of the situation treated here in comparison with the classical Jacobi weight is the singularity of the weight of orthogonality at $x_0$. The local behavior of $P_n$'s at $x_0$ is described in terms of the confluent hypergeometric functions. Such functions appeared already in the Riemann-Hilbert analysis for the weight $w_{c,0}$ in [11] and [13], and will work also in the general situation studied here. However, the parameter describing the family of these functions is complex; its real part depends on the degree of the algebraic singularity $\gamma$ and its imaginary part is a function of the size of the jump $c^2$ in the weight $w_{c,\gamma}$. Also, for $c = 1$ these confluent hypergeometric functions degenerate into the Bessel functions, in correspondence with [14].

Interestingly enough, the confluent hypergeometric functions appear in the scaling limit (as the number of particles goes to infinity) of the correlation functions of the pseudo-Jacobi ensemble, see [3]. This ensemble corresponds to a sequence of weights of the form

$$(1 + x^2)^{-n-\operatorname{Re}(s)} e^{2\operatorname{Im}(s)\arg(1+ix)}, \quad x \in \mathbb{R}, \tag{7}$$

where $n$ is the degree of the polynomial and $s$ a complex parameter. The connection between both problems becomes apparent if we perform the inversion $x \mapsto 1/x$ in (7); this creates at the origin an algebraic singularity with the exponent $\operatorname{Re}(s)$ and a jump depending on $\operatorname{Im}(s)$.

In the next Section we state the Riemann-Hilbert problem for the orthogonal polynomials and perform the steepest descent analysis; as a result, Theorem 1 is proved in Section 3. For the sake of brevity, the description of the standard steps is rather sketchy; an interested reader may find the missing details in the literature cited above, and especially in [11]. However, the local parametrix (the Riemann-Hilbert problem in a neighborhood of the singularity) at $x_0$ is analyzed in full detail. The same problem has appeared very recently in an independent work of Deift, Its and Krasovsky [4] on the asymptotics of Toeplitz and Hankel determinants.

## 2. The steepest descent analysis

### 2.1. The Riemann-Hilbert problem and first transformations

Following Fokas, Its and Kitaev [10], we characterize the orthogonal polynomials in terms of the unique solution $\mathbf{Y}$ of the following $2 \times 2$ matrix valued Riemann-Hilbert (RH) problem: for $n \in \mathbb{N}$,

4

(Y1) $\mathbf{Y}$ is analytic in $\mathbb{C} \setminus [-1, 1]$.

(Y2) On $(-1, x_0) \cup (x_0, 1)$, $\mathbf{Y}$ possesses continuous boundary values $\mathbf{Y}_+$ (from the upper half plane) and $\mathbf{Y}_-$ (from the lower half plane), and

$$\mathbf{Y}_+(x) = \mathbf{Y}_-(x) \begin{pmatrix} 1 & w_{c,\gamma}(x) \\ 0 & 1 \end{pmatrix}.$$

(Y3) $\mathbf{Y}(z) = (\mathbf{I} + \mathcal{O}(1/z)) z^{n\sigma_3}$, as $z \to \infty$, where all terms are $2 \times 2$ matrices, $\mathbf{I}$ is the identity and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix.[1]

(Y4) if $(\zeta, s) \in \{(-1, \beta), (x_0, \gamma), (1, \alpha)\}$ then for $z \to \zeta$, $z \in \mathbb{C} \setminus [-1, 1]$,

$$\mathbf{Y}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z - \zeta|^s \\ 1 & |z - \zeta|^s \end{pmatrix}, & \text{if } s < 0; \\[2mm] \mathcal{O}\begin{pmatrix} 1 & \log|z - \zeta| \\ 1 & \log|z - \zeta| \end{pmatrix}, & \text{if } s = 0; \\[2mm] \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } s > 0. \end{cases}$$

Standard arguments (see e.g. [14]) show that this RH problem has a unique solution given by

$$\mathbf{Y}(z, n) = \begin{pmatrix} P_n(z) & \mathcal{C}(P_n w_{c,\gamma})(z) \\ -2\pi i k_{n-1}^2 P_{n-1}(z) & -2\pi i k_{n-1}^2 \mathcal{C}(P_{n-1} w_{c,\gamma})(z) \end{pmatrix}, \tag{8}$$

where $P_n$ is the monic orthogonal polynomial of degree $n$ with respect to $w_{c,\gamma}$, $p_n(x) = p_n(x; w_{c,\gamma})$ is the corresponding orthonormal polynomial,

$$p_n(x) = k_n P_n(x),$$

where $k_n > 0$ is the leading coefficient of $p_n$, and $\mathcal{C}(\cdot)$ is the Cauchy transform on $[-1, 1]$ defined by

$$\mathcal{C}(f)(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{f(x)}{x - z} dx.$$

Note that $\mathbf{Y}$ and other matrices introduced hereafter depend on $n$; however, to simplify notation we omit the explicit reference to $n$.

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[1] In what follows, for $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$, $a^{b\sigma_3}$ we use the notation

$$a^{b\sigma_3} \stackrel{\text{def}}{=} \begin{pmatrix} a^b & 0 \\ 0 & 1/a^b \end{pmatrix}.$$

The first transformations of the Deift-Zhou steepest descendent method are standard, and up to slight variations match those described in [11, subsection 2.2]. Hence, we will omit the details, highlighting basically the differences with the cited reference. The reader should keep in mind also that in [11] the analysis is made for a singularity fixed at $x_0 = 0$; however, extending it to the general case of $x_0 \in (-1, 1)$ is straightforward.

We start by defining

$$\mathbf{T}(z) \stackrel{\text{def}}{=} 2^{n\sigma_3} \mathbf{Y}(z) \varphi(z)^{-n\sigma_3}, \tag{9}$$

where

$$\varphi(z) = z + \sqrt{z^2 - 1} \tag{10}$$

is the conformal map from $\mathbb{C} \setminus [-1, 1]$ onto the exterior of the unit circle, with the branch of $\sqrt{z^2 - 1}$ analytic in $\mathbb{C} \setminus [-1, 1]$ and that behaves like $z$ as $z \to \infty$.

In order to perform the second transformation we need to extend the definition of the weight of orthogonality to a neighborhood of the interval $[-1, 1]$. By assumptions, $h$ is a holomorphic function in a neighborhood $U$ of $[-1, 1]$, and positive on this interval. For any Jordan arc $\Gamma$, intersecting $[-1, 1]$ transversally at $x_0$ and dividing $U$ into two connected components, we denote by $\Sigma_5$ its intersection with the upper half plane, and by $\Sigma_6$ its intersection with the lower half plane, oriented as shown in Figure 1. Contours $\Sigma_5 \cup \Sigma_6 \cup \mathbb{R}$ divide $U$ into four open domains ("quadrants"), that we denote by $Q_{\pm}^{L,R}$ as depicted. Finally, let $Q^L$ (resp., $Q^R$) be the connected component of $U \setminus \Gamma$ containing $-1$ (resp., $+1$).
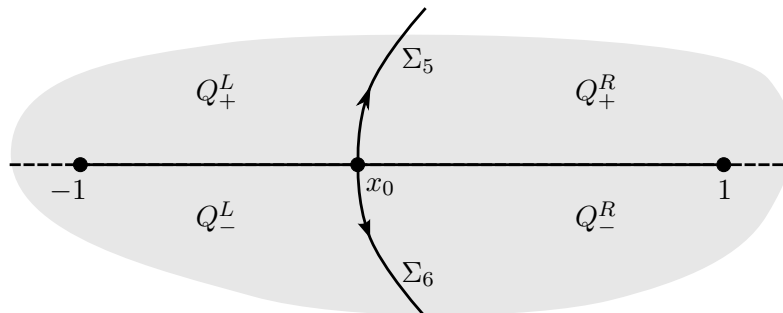


Figure 1: Division of the neighborhood of $[-1, 1]$ in four regions.

Now we set

$$w(z) \stackrel{\text{def}}{=} h(z) (1 - z)^\alpha (1 + z)^\beta \times \begin{cases} (x_0 - z)^\gamma, & z \in Q^L \setminus (-\infty, -1], \\ (z - x_0)^\gamma, & z \in Q^R \setminus [1, +\infty), \end{cases} \tag{11}$$

where the principal branches of the power functions are taken. In this way, $w$ is defined and holomorphic in $\widetilde{U} \stackrel{\text{def}}{=} U \setminus ((-\infty, -1] \cup [1, +\infty) \cup \Gamma)$, and $w(x) > 0$ for $x \in (-1, 1) \setminus x_0$. Setting

$$\Xi_c(z) = \begin{cases} 1 & z \in Q^L \\ c^2 & z \in Q^R, \end{cases}$$

we extend also

$$w_{c,\gamma}(z) \stackrel{\text{def}}{=} w(z)\, \Xi_c(z), \tag{12}$$

to a holomorphic function in $\widetilde{U}$.

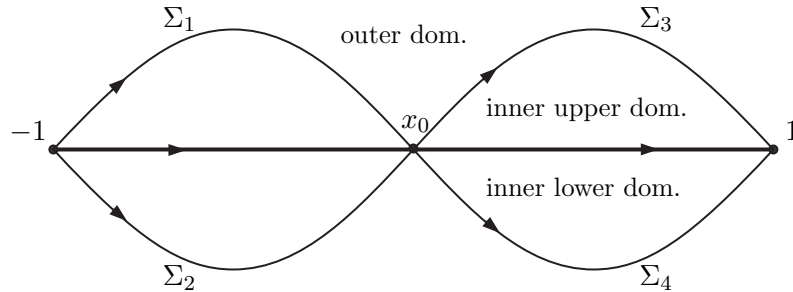We describe now the next transformation consisting in opening of lenses or contour deformation.



Figure 2: First lens opening.

We build the four contours $\Sigma_i$ lying in $\widetilde{U}$ (except for their end points) such that $\Sigma_1$ and $\Sigma_3$ are in the upper half plane, and $\Sigma_1$ and $\Sigma_2$ are in the left half plane, and oriented "from $-1$ to $1$" but now through $x_0$ (see Fig. 2). This construction defines three domains: the inner upper domain, bounded by $[-1,1]$ and the curves $\Sigma_1$ and $\Sigma_3$; the inner lower domain, bounded by $[-1,1]$ and the curves $\Sigma_2$ and $\Sigma_4$, and finally the outer domain, bounded by curves $\Sigma_i$ and containing the infinity. Denote $\Sigma \stackrel{\text{def}}{=} [-1,1] \cup \bigcup_{k=1}^{4} \Sigma_k$.

Using the matrix $\mathbf{T}$ from (9) we define

$$\mathbf{S}(z) \stackrel{\text{def}}{=} \begin{cases} \mathbf{T}(z), & \text{for } z \text{ in the outer domain,} \\[2mm] \mathbf{T}(z) \begin{pmatrix} 1 & 0 \\ -\frac{1}{w_{c,\gamma}(z)}\varphi^{-2n}(z) & 1 \end{pmatrix}, & \text{for } z \text{ in the inner upper domain,} \\[4mm] \mathbf{T}(z) \begin{pmatrix} 1 & 0 \\ \frac{1}{w_{c,\gamma}(z)}\varphi^{-2n}(z) & 1 \end{pmatrix}, & \text{for } z \text{ in the inner lower domain.} \end{cases} \tag{13}$$

Then $\mathbf{S}$ is a solution of a new RH problem, now with jumps on $\Sigma$, that are easy to compute explicitly. The uniqueness is guaranteed if we impose the additional local requirement: as $z \to x_0$, $z \in \mathbb{C}\backslash\Sigma$,

- for $-1 < \gamma < 0$, $\mathbf{S}(z) = \mathcal{O}\begin{pmatrix} 1 & |z-x_0|^\gamma \\ 1 & |z-x_0|^\gamma \end{pmatrix}$, as $z \to x_0$,

- for $\gamma = 0$:

$$\mathbf{S}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & \log|z - x_0| \\ 1 & \log|z - x_0| \end{pmatrix}, & \text{as } z \to x_0 \text{ from the outer domain,} \\ \mathcal{O}\begin{pmatrix} \log|z - x_0| & \log|z - x_0| \\ \log|z - x_0| & \log|z - x_0| \end{pmatrix}, & \text{as } z \to x_0 \text{ from the inner domains,} \end{cases}$$

- for $\gamma > 0$:

$$\mathbf{S}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to x_0, \text{ from the outer domain,} \\ \mathcal{O}\begin{pmatrix} |z - x_0|^{-\gamma} & 1 \\ |z - x_0|^{-\gamma} & 1 \end{pmatrix}, & \text{as } z \to x_0, \text{ from the inner domain.} \end{cases}$$

### 2.2. Outer parametrix

In the next step, which is also standard, we build the so-called outer parametrix for the RH problem for $\mathbf{S}$ in terms of the Szegő function $D(\cdot, w_{c,\gamma})$ corresponding to the weight $w_{c,\gamma}$. Namely, we construct the $2 \times 2$ matrix $N$ that satisfies

(N1) $\mathbf{N}$ is analytic in $\mathbb{C} \backslash [-1, 1]$;

(N2) $\mathbf{N}_+(x) = \mathbf{N}_-(x) \begin{pmatrix} 0 & w_{c,\gamma}(x) \\ -w_{c,\gamma}(x)^{-1} & 0 \end{pmatrix}$, $x \in (-1, x_0) \cup (x_0, 1)$;

(N3) $\mathbf{N}(z) = \mathbf{I} + \mathcal{O}(1/z)$, as $z \to \infty$.

The solution is given by

$$\mathbf{N}(z) \stackrel{\text{def}}{=} D_\infty^{\sigma_3} \mathbf{A}(z) D(z, w_{c,\gamma})^{-\sigma_3}, \tag{14}$$

and we will describe the three factor appearing in the r.h.s. of (14). Matrix $\mathbf{A}$ is

$$\mathbf{A}(z) \stackrel{\text{def}}{=} \begin{pmatrix} A_{11} & A_{12} \\ -A_{12} & A_{11} \end{pmatrix}, \quad A_{11}(z) = \frac{\varphi(z)^{1/2}}{\sqrt{2}(z^2 - 1)^{1/4}},$$

$$A_{12}(z) = \frac{i\varphi(z)^{-1/2}}{\sqrt{2}(z^2 - 1)^{1/4}} = \frac{i}{\varphi(z)} A_{11}(z), \tag{15}$$

with the main branches of the roots, in such a way that $A_{11}$ is analytic in $\mathbb{C} \backslash [-1, 1]$ with $A_{11}(z) \to 1$, and $A_{12}(z) \to 0$, as $z \to \infty$. The Szegő function $D(\cdot, w_{c,\gamma})$ for $w_{c,\gamma}$ is given by

$$D(z, w_{c,\gamma}) = D(z, h)D(z, w_{1,\gamma})D(z, \Xi_c), \tag{16}$$

8

where

$$D(z,h) = \exp\left(\sqrt{1-z^2}\,\mathcal{C}\left(\frac{\log h(t)}{\sqrt{1-t^2}}\right)(z)\right), \quad D(z,w_{1,\gamma}) = \frac{(z-1)^{\alpha/2}\,(z+1)^{\beta/2}\,(z-x_0)^{\gamma/2}}{\varphi^{(\alpha+\beta+\gamma)/2}(z)},$$

(17)

and

$$D(z,\Xi_c) = c\,\exp\left(-\lambda\log\left(\frac{1-zx_0-i\sqrt{z^2-1}}{z-x_0}\right)\right),$$

(18)

with

$$\lambda \stackrel{\text{def}}{=} i\frac{\log c}{\pi},$$

(19)

where we take the main branches of $(z-1)^{\alpha/2}$, $(z+1)^{\beta/2}$, $(z-x_0)^{\gamma/2}$ and $\sqrt{z^2-1}$ that are positive for $z>1$, as well as the main branch of the logarithm (see [11, section 2.3] for a detailed computation). Finally,

$$D_\infty \stackrel{\text{def}}{=} D(\infty, w_{c,\gamma}) = \sqrt{c}\,D\left(\infty, h\right) 2^{-(\alpha+\beta+\gamma)/2}\,e^{i\lambda\arcsin x_0} > 0.$$

(20)

Some of the properties of this function are summarized in the following lemma:

**Lemma 2.** *The Szegő function $D(\cdot, w)$ for the weight $w$ defined in (11) exhibits the following boundary behavior:*

$$\lim_{\substack{z\to x\in(-1,1),\\ \pm\operatorname{Im}z>0}} D(z,w) = \sqrt{w(x)}\,\exp\left(\pm i\widehat{\Phi}(x)\right),$$

(21)

*with*

$$\Phi(x) = \frac{\pi\alpha}{2} - \frac{\alpha+\beta+\gamma}{2}\arccos x - \frac{\sqrt{1-x^2}}{2\pi}\int_{-1}^{1}\frac{\log h(t)}{\sqrt{1-t^2}}\frac{dt}{t-x},$$

(22)

$$\widehat{\Phi}(x) = \begin{cases} \Phi(x) + \frac{\pi\gamma}{2}, & -1 < x < x_0 \\ \Phi(x), & x_0 < x < 1 \end{cases}$$

(23)

*Furthermore, for the step function $\Xi_c$,*

$$\lim_{\substack{z\to x\in(-1,x_0)\cup(x_0,1),\\ \pm\operatorname{Im}z>0}} D(z,\Xi_c) = \sqrt{\Xi_c(x)}\,\exp\left(\mp i\,\frac{\log c}{\pi}\log\left|\frac{1-x_0 x+\sqrt{(1-x^2)(1-x_0^2)}}{x-x_0}\right|\right),$$

*and*

$$D(z,\Xi_c) = c^{1\pm\frac{i}{\pi}\log\left(\frac{z-x_0}{2(1-x_0^2)}\right)}\,[1+o(1)], \quad as\ z\to x_0,\ \pm\operatorname{Im}z>0.$$

(24)

9

The proof of this lemma is similar to [11, Lemma 7], up to the difference that our jump here takes place at a generic point $x_0$, and that $w(z)$ (see (11)) has an extra factor which makes $w(z)$ non-analytic across $\Sigma_5 \cup \Sigma_6$.

The main purpose for constructing $\mathbf{N}$ is that it solves the "stripped" RH problem, obtained from the RH problem for $\mathbf{S}$ by ignoring all jumps asymptotically close to identity. Unfortunately, this property of $\mathbf{N}$ is not uniform on the whole plane: the jumps of $\mathbf{N}$ and $\mathbf{S}$ are no longer close in the neighborhoods of $\pm 1$ and $x_0$. The analysis at these points requires a separate treatment, called local analysis, that we perform next. The outline of this analysis at each point is the following: take a small disc centered at the point and build a matrix-valued function (local parametrix) that:

(i) matches exactly the jumps of $\mathbf{S}$ within the disc, and

(ii) coincides with $\mathbf{N}$ on the boundary of the disc, at least to an order $o(1)$, $n \to \infty$.

*2.3. Local parametrix*

We fix a $\delta > 0$ small enough such that discs $U_\zeta \overset{\text{def}}{=} \{z \in \mathbb{C} : |x - \zeta| < \delta\}$, $\zeta \in \{-1, x_0, 1\}$ are mutually disjoint and lie in the domain of analyticity of the function $h$. We skip the details of construction of the local parametrices $\mathbf{P}_{\pm 1}$ at $z = \pm 1$ and refer the reader to [14].

For the local parametrix at the jump we need to build a $2 \times 2$ matrix-valued function $\mathbf{P}_{x_0} \overset{\text{def}}{=} \mathbf{P}_0$ in $U_{x_0} \setminus \Sigma$ that satisfies the following conditions:

($\mathbf{P}_0$1) $\mathbf{P}_0$ is holomorphic in $U_{x_0} \setminus \Sigma$ and continuous up to the boundary.

($\mathbf{P}_0$2) $\mathbf{P}_0$ satisfies the following jump relations:

$$\mathbf{P}_{0+}(z) = \mathbf{P}_{0-}(z) \begin{pmatrix} 1 & 0 \\ \frac{1}{w_{c,\gamma}(z)} \varphi(z)^{-2n} & 1 \end{pmatrix}, \quad \text{for } z \in U_{x_0} \cap \left( \bigcup_{i=1}^{4} \Sigma_i \right) \setminus \{x_0\};$$

$$\mathbf{P}_{0+}(x) = \mathbf{P}_{0-}(x) \begin{pmatrix} 0 & w_{c,\gamma}(z) \\ -\frac{1}{w_{c,\gamma}(z)} & 0 \end{pmatrix}, \quad \text{for } z \in U_{x_0} \cap ((-1, x_0) \cup (x_0, 1)).$$

($\mathbf{P}_0$3) $\mathbf{P}_0(z)\mathbf{N}^{-1}(z) = \mathbf{I} + \mathcal{O}(1/n)$, as $n \to \infty$, uniformly for $z \in \partial U_{x_0} \setminus \Sigma$.

($\mathbf{P}_0$4) $\mathbf{P}_0$ has the same behavior than $\mathbf{S}$ as $z \to x_0$, $z \in U_{x_0} \setminus \Sigma$.

Following a standard procedure, we obtain the solution of this problem in two steps, getting first a matrix $\mathbf{P}^{(1)}$ that satisfies conditions ($\mathbf{P}_0$1, $\mathbf{P}_0$2, $\mathbf{P}_0$4). After that, using an additional freedom in the construction, we take care of the matching condition ($\mathbf{P}_0$3).

We define an auxiliary function $W$, holomorphic in $\widetilde{U} \setminus \mathbb{R}$. In the next formula we understand by $\left( h(z)(1-z)^\alpha (1+z)^\beta (z - x_0)^\gamma c \right)^{1/2}$ the holomorphic branch of this function in $U \setminus ((-\infty, x_0] \cup [1, +\infty))$, positive on $(x_0, 1)$. Analogously, $\left( h(z)(1-z)^\alpha (1+z)^\beta (x_0 - z)^\gamma c \right)^{1/2}$

10

stands for the holomorphic branch in $U \setminus ((-\infty, -1] \cup [x_0, +\infty))$, positive on $(-1, x_0)$. With this convention we set

$$W(z) = \begin{cases} \left( h(z)(1-z)^\alpha (1+z)^\beta (z-x_0)^\gamma c \right)^{1/2}, & z \in Q_+^L \cup Q_-^L, \\ \left( h(z)(1-z)^\alpha (1+z)^\beta (x_0-z)^\gamma c \right)^{1/2}, & z \in Q_+^R \cup Q_-^R. \end{cases} \tag{25}$$

It is easy to see from (11)–(12) that

$$W^2(z) = \begin{cases} w_{c,\gamma}(z) e^{-\gamma \pi i} c^{-1}, & z \in Q_+^R, \\ w_{c,\gamma}(z) e^{\gamma \pi i} c, & z \in Q_+^L, \\ w_{c,\gamma}(z) e^{-\gamma \pi i} c, & z \in Q_{-,}^L, \\ w_{c,\gamma}(z) e^{\gamma \pi i} c^{-1}, & z \in Q_-^R. \end{cases}$$

This shows that $W$ satisfies the following jump relations:

$$W_+(x) = \begin{cases} W_-(x) e^{i\gamma \pi}, & -1 < x < x_0, \\ W_-(x) e^{-i\gamma \pi}, & x_0 < x < 1, \end{cases}$$

and

$$W_+(z) = e^{i\gamma \pi/2} W_-(z), \quad z \in \Sigma_5 \cup \Sigma_6. \tag{26}$$

Moreover,

$$W_+(x) = \begin{cases} \sqrt{w_{c,\gamma}(x) c} \, e^{i\frac{\gamma \pi}{2}} = \sqrt{w_{1,\gamma}(x) c} \, e^{i\frac{\gamma \pi}{2}}, & -1 < x < x_0, \\ \sqrt{w_{c,\gamma}(x) c^{-1}} \, e^{-i\frac{\gamma \pi}{2}} = \sqrt{w_{1,\gamma}(x) c} \, e^{-i\frac{\gamma \pi}{2}}, & x_0 < x < 1, \end{cases} \tag{27}$$

and

$$W_+(x) W_-(x) = \begin{cases} w_{c,\gamma}(x) c, & -1 < x < x_0, \\ w_{c,\gamma}(x) c^{-1}, & x_0 < x < 1. \end{cases} \tag{28}$$

We construct the matrix function $\mathbf{P}_0$ in the following form:

$$\mathbf{P}_0(z) = \mathbf{E}_n(z) \mathbf{P}^{(1)}(z) W(z)^{-\sigma_3} \varphi(z)^{-n\sigma_3}, \tag{29}$$

where $\mathbf{E}_n$ is an analytic matrix-valued function in $U_{x_0}$ (to be determined). Matrix $\mathbf{P}^{(1)}$ is analytic in $U_{x_0} \setminus \Sigma$. Denote by

$$J_1 = \begin{pmatrix} 0 & c \\ -1/c & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 1 & 0 \\ e^{-\gamma \pi i} c^{-1} & 1 \end{pmatrix}, \quad J_3 = J_7 = \begin{pmatrix} e^{i\gamma \pi/2} & 0 \\ 0 & e^{-i\gamma \pi/2} \end{pmatrix}, \tag{30}$$

$$J_4 = \begin{pmatrix} 1 & 0 \\ e^{\gamma \pi i} c & 1 \end{pmatrix}, \quad J_5 = \begin{pmatrix} 0 & 1/c \\ -c & 0 \end{pmatrix}, \quad J_6 = \begin{pmatrix} 1 & 0 \\ e^{-\gamma \pi i} c & 1 \end{pmatrix}. \tag{31}$$

Using the properties of $W$ and $\varphi$ it is easy to show that

$$\mathbf{P}_+^{(1)}(x) = \mathbf{P}_-^{(1)}(x) \begin{cases} J_5, & x \in (x_0 - \delta, x_0), \\ J_1, & x \in (x_0, x_0 + \delta), \end{cases} \tag{32}$$

and

$$\mathbf{P}_+^{(1)}(z) = \mathbf{P}_-^{(1)}(z) \begin{cases} J_4, & z \in \Sigma_1 \cap U_{x_0} \setminus \{x_0\}, \\ J_6, & z \in \Sigma_2 \cap U_{x_0} \setminus \{x_0\}, \\ J_2, & z \in \Sigma_3 \cap U_{x_0} \setminus \{x_0\}, \\ J_8, & z \in \Sigma_4 \cap U_{x_0} \setminus \{x_0\}, \end{cases} \tag{33}$$

and, as $W$ has a jump on $\Sigma_5 \cup \Sigma_6$, by (26), we have two additional jumps on $\Sigma_5 \cup \Sigma_6$:

$$\mathbf{P}_+^{(1)}(z) = \mathbf{P}_-^{(1)}(z) \begin{cases} J_3, & z \in \Sigma_5 \cap U_{x_0} \setminus \{x_0\}, \\ J_7, & z \in \Sigma_6 \cap U_{x_0} \setminus \{x_0\}. \end{cases} \tag{34}$$

Taking into account that $W(z) = \mathcal{O}\left(|z - x_0|^{\gamma/2}\right)$ and $\varphi(z) = \mathcal{O}(1)$ as $z \to x_0$, we conclude also from $(\mathrm{P}_04)$ that $\mathbf{P}^{(1)}$ has the following behavior at $x_0$: as $z \to x_0$, $z \in \mathbb{C} \setminus (\Sigma \cup \Sigma_5 \cup \Sigma_6)$,

- for $\gamma < 0$:
$$\mathbf{P}^{(1)}(z) = \mathcal{O}\left( \begin{array}{cc} |z - x_0|^{\gamma/2} & |z - x_0|^{\gamma/2} \\ |z - x_0|^{\gamma/2} & |z - x_0|^{\gamma/2} \end{array} \right), \tag{35}$$

- for $\gamma = 0$
$$\mathbf{P}^{(1)}(z) = \begin{cases} \mathcal{O}\left( \begin{array}{cc} \log|z - x_0| & \log|z - x_0| \\ \log|z - x_0| & \log|z - x_0| \end{array} \right), & \text{from inside the lens,} \\ \mathcal{O}\left( \begin{array}{cc} 1 & \log|z - x_0| \\ 1 & \log|z - x_0| \end{array} \right), & \text{from outside the lens,} \end{cases} \tag{36}$$

- for $\gamma > 0$:
$$\mathbf{P}^{(1)}(z) = \begin{cases} \mathcal{O}\left( \begin{array}{cc} |z - x_0|^{\gamma/2} & |z - x_0|^{-\gamma/2} \\ |z - x_0|^{\gamma/2} & |z - x_0|^{-\gamma/2} \end{array} \right), & \text{from outside the lens,} \\ \mathcal{O}\left( \begin{array}{cc} |z - x_0|^{-\gamma/2} & |z - x_0|^{-\gamma/2} \\ |z - x_0|^{-\gamma/2} & |z - x_0|^{-\gamma/2} \end{array} \right), & \text{from inside the lens.} \end{cases} \tag{37}$$
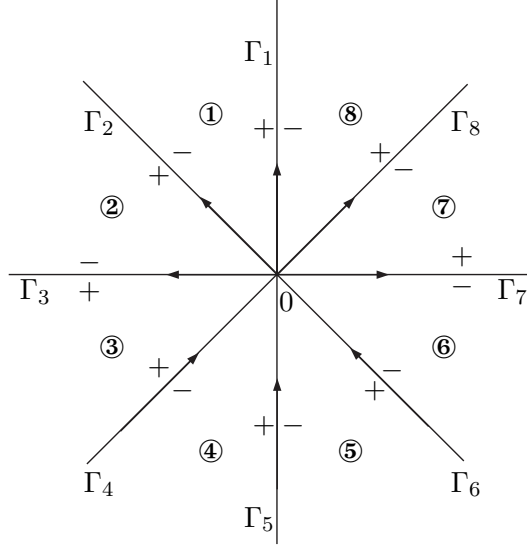
Figure 3: Auxiliary contours $\Gamma$.

In order to construct $\mathbf{P}^{(1)}$ we solve first an auxiliary RH problem on a set $\Gamma \stackrel{\mathrm{def}}{=} \bigcup_{j=1}^{8} \Gamma_j$ of unbounded oriented straight lines converging at the origin, like in Fig. 3. More precisely,

$$\Gamma_1 = \left\{ t\,e^{i\pi/2} : t > 0 \right\}, \ \Gamma_2 = \left\{ t\,e^{3i\pi/4} : t > 0 \right\}, \ \Gamma_3 = \left\{ -t : t > 0 \right\}, \ \Gamma_4 = \left\{ t\,e^{5i\pi/4} : t > 0 \right\},$$

$$\Gamma_5 = \left\{ t\,e^{3i\pi/2} : t > 0 \right\}, \ \Gamma_6 = \left\{ t\,e^{-i\pi t/4} : t > 0 \right\}, \ \Gamma_7 = \left\{ t : t > 0 \right\}, \ \Gamma_8 = \left\{ e^{i\pi t/4} : t > 0 \right\}.$$

These lines split the plane into 8 sectors, enumerated anti-clockwise from ① to ⑧ as in Fig. 3.

We look for a $2 \times 2$ matrix valued function $\mathbf{\Psi}(z)$, satisfying the following conditions:

($\Psi$1) $\mathbf{\Psi}$ is analytic in $\mathbb{C}\backslash\Gamma$.

($\Psi$2) for $k = 1, \ldots, 8$, $\mathbf{\Psi}$ satisfies the jump relation $\mathbf{\Psi}_+(\zeta) = \mathbf{\Psi}_-(\zeta)J_k$ on $\Gamma_k$, with $J_k$ given by (30) and (31).

($\Psi$3) the behavior of $\mathbf{\Psi}$ as $\zeta \to 0$ is obtained from that of $\mathbf{P}^{(1)}$ at $x_0$ by replacing $(z - x_0)$ with $\zeta$. Now the region "inside lens" correspond to ① $\cup$ ④ $\cup$ ⑤ $\cup$ ⑧ and the region "outside lens" corresponds to ② $\cup$ ③ $\cup$ ⑥ $\cup$ ⑦.

We construct $\mathbf{\Psi}$ explicitly using the confluent hypergeometric functions

$$\phi\left(a, \gamma + 1; \zeta\right) \stackrel{\mathrm{def}}{=} {}_1F_1\left(a; \gamma + 1; \zeta\right) \quad \text{and} \quad \psi\left(a, \gamma + 1; \zeta\right) \stackrel{\mathrm{def}}{=} z^{-a}\,{}_2F_0\left(a, a - \gamma; -; -1/\zeta\right),$$

that are solutions of the confluent hypergeometric equation $\zeta w'' + (\gamma + 1 - \zeta)\,w' - aw = 0$, see [1, formula (13.1.1)]. Namely, let

$$G\left(a, \gamma; \zeta\right) \stackrel{\mathrm{def}}{=} \zeta^{\gamma/2}\phi\left(a, \gamma + 1; \zeta\right)e^{-\zeta/2}, \quad H\left(a, \gamma; \zeta\right) \stackrel{\mathrm{def}}{=} \zeta^{\gamma/2}\psi\left(a, \gamma + 1; \zeta\right)e^{-\zeta/2}, \tag{38}$$

13

they form a basis of solutions of the confluent equation (see e.g. [1, formula (13.1.35)])

$$4\zeta^2 w'' + 4\zeta w' + \left[-\gamma^2 + 2\zeta\left(\gamma + 1 - 2a\right) - \zeta^2\right] w = 0. \tag{39}$$

We can relate $G$ and $H$ with the Whittaker functions: $G(a,\gamma;z) = M_{\kappa,\mu}(z)/\sqrt{z}$ and $H(a,\gamma;z) = W_{\kappa,\mu}(z)/\sqrt{z}$ with $\mu = \gamma/2$ and $\kappa = 1/2 + \mu - a$ (see [1, formula (13.1.32)]).

In general, $G\left(a,\gamma;\zeta\right)$ and $H\left(a,\gamma;\zeta\right)$ from (39) are multi-valued, and we take its principal branch in $-\frac{\pi}{2} < \arg\left(\zeta\right) < \frac{3\pi}{2}$. For these values of $\zeta$ we define

$$\widehat{\boldsymbol{\Psi}}\left(\zeta\right) \overset{\text{def}}{=} \begin{pmatrix} \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & -H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \end{pmatrix} e^{\frac{\gamma\pi i}{4}\sigma_3}.$$

By $(\Psi_2)$, if we set

$$\boldsymbol{\Psi}\left(\zeta\right) \overset{\text{def}}{=} \begin{cases} \widehat{\boldsymbol{\Psi}}\left(\zeta\right) J_8 J_1, & \text{for } \zeta \in \text{①}; \\ \widehat{\boldsymbol{\Psi}}\left(\zeta\right) J_8 J_1 J_2, & \text{for } \zeta \in \text{②}; \\ \widehat{\boldsymbol{\Psi}}\left(\zeta\right) J_8 J_1 J_2 J_3, & \text{for } \zeta \in \text{③}; \\ \widehat{\boldsymbol{\Psi}}\left(\zeta\right) J_8 J_1 J_2 J_3 J_4^{-1}, & \text{for } \zeta \in \text{④}; \\ \widehat{\boldsymbol{\Psi}}\left(\zeta\right) J_7^{-1} J_6, & \text{for } \zeta \in \text{⑤}; \\ \widehat{\boldsymbol{\Psi}}\left(\zeta\right) J_7^{-1}, & \text{for } \zeta \in \text{⑥}; \\ \widehat{\boldsymbol{\Psi}}\left(\zeta\right), & \text{for } \zeta \in \text{⑦}; \\ \widehat{\boldsymbol{\Psi}}\left(\zeta\right) J_8, & \text{for } \zeta \in \text{⑧}. \end{cases} \tag{40}$$

then $\boldsymbol{\Psi}$ has the jumps across $\Gamma$ specified in $(\Psi_2)$. Explicitly, $\boldsymbol{\Psi}\left(\zeta\right) =$

$$\begin{pmatrix} c^{-1} H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)} H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-\pi i}\right) \\ -c^{-1}\frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{-\pi i}\right) \end{pmatrix} e^{-\frac{\gamma\pi i}{4}\sigma_3}, \ \zeta \in \text{①},$$

$$\tag{41}$$

$$\begin{pmatrix} \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) e^{-\frac{\gamma\pi i}{2}} & -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)} H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-\pi i}\right) \\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) e^{-\frac{\gamma\pi i}{2}} & H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{-\pi i}\right) \end{pmatrix} e^{-\frac{\gamma\pi i}{4}\sigma_3}, \ \zeta \in \text{②},$$

$$\tag{42}$$

$$\begin{pmatrix} \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)} H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-\pi i}\right) e^{-\frac{\gamma\pi i}{2}} \\ \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{-\pi i}\right) e^{-\frac{\gamma\pi i}{2}} \end{pmatrix} e^{-\frac{\gamma\pi i}{4}\sigma_3}, \ \zeta \in \text{③},$$

$$\tag{43}$$

$$\begin{pmatrix} c\, H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-2\pi i}\right) & -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)} H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-\pi i}\right) \\ -c\frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-2\pi i}\right) & H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{-\pi i}\right) \end{pmatrix} e^{\frac{\gamma\pi i}{4}\sigma_3}, \ \zeta \in \text{④},$$

$$\tag{44}$$

14

$$
\begin{pmatrix}
-\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\lambda+\frac{\gamma}{2}\right)} H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{\pi i}\right) e^{-\lambda\pi i} & -H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \\
H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{\pi i}\right) e^{-\lambda\pi i} & \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right)
\end{pmatrix} e^{-\frac{\gamma\pi i}{4}\sigma_3}, \quad \zeta \in \text{⑤},
$$
(45)

$$
\begin{pmatrix}
\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & -H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \\
\frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right)
\end{pmatrix} e^{-\frac{\gamma\pi i}{4}\sigma_3}, \quad \zeta \in \text{⑥},
$$
(46)

$$
\begin{pmatrix}
\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & -H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \\
\frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)} G\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right) & \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right)
\end{pmatrix} e^{\frac{\gamma\pi i}{4}\sigma_3}, \quad \zeta \in \text{⑦},
$$
(47)

$$
\begin{pmatrix}
-c^{-1}\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)} H\left(1-\lambda+\frac{\gamma}{2},\gamma;\zeta e^{-\pi i}\right) & -H\left(\lambda+\frac{\gamma}{2},\gamma;\zeta\right) \\
c^{-1} H\left(\frac{\gamma}{2}-\lambda,\gamma;\zeta e^{-\pi i}\right) & \frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} H\left(1+\lambda+\frac{\gamma}{2},\gamma;\zeta\right)
\end{pmatrix} e^{\frac{\gamma\pi i}{4}\sigma_3}, \quad \zeta \in \text{⑧}.
$$
(48)

**Proposition 3.** *The solution of the RH problem* $(\Psi 1)$, $(\Psi 2)$, $(\Psi 3)$ *is given by* (40) *and* $\det \boldsymbol{\Psi}(z) = 1$, *for* $z \in \mathbb{C} \setminus \Gamma$.

PROOF. If we take the branch cut across $\arg\zeta = -\pi/2$ oriented towards the origin (we consider $-\pi/2 < \arg\zeta < 3\pi/4$), we have that the matrix $\boldsymbol{\Psi}$ has on this cut the following jump (using (19)):

$$
\boldsymbol{\Psi}_+(\zeta) = \boldsymbol{\Psi}_-(\zeta) J_5, \quad \zeta \in \Gamma_5, \tag{49}
$$

$$
\widehat{\boldsymbol{\Psi}}_+(\zeta) = \widehat{\boldsymbol{\Psi}}_-(\zeta)
\begin{pmatrix}
e^{i\pi\gamma} & -e^{-i\pi\lambda} + e^{i\pi\lambda}e^{-i\pi\gamma} \\
0 & e^{-i\pi\gamma}
\end{pmatrix}, \quad \zeta \in \Gamma_5. \tag{50}
$$

Using the following relations (see [13, appendix: formulas (7.18), (7.30), (7.27)]),

$$
\phi\left(a,b;e^{\pm 2\pi i}z\right) = \phi\left(a,b;z\right), \tag{51}
$$

$$
\psi\left(a,b;e^{2\pi i}z\right) = e^{-2i\pi a}\psi\left(a,b;z\right) + e^{-i\pi a}\frac{2\pi i}{\Gamma(a)\Gamma(1+a-b)}\psi\left(b-a,1;e^{i\pi}z\right)e^{z},
$$

$$
\psi\left(b-a,b;e^{i\pi}z\right)e^{z} = \frac{-\Gamma(a)}{\Gamma(b-a)}e^{-i\pi b}\psi\left(a,b;z\right) + \frac{\Gamma(a)}{\Gamma(b)}e^{-i\pi(b-a)}\phi\left(a,b;z\right),
$$

$$
\Gamma(s)\Gamma(1-s) = \frac{2\pi i}{e^{i\pi s} - e^{-i\pi s}},
$$

and, combining the last three formulas we obtain:

$$
\psi\left(a,b;e^{2\pi i}z\right) = \psi\left(a,b;z\right)e^{-2\pi ib} + \phi\left(a,b;z\right)\frac{2\pi i}{\Gamma(1+a-b)\Gamma(b)}e^{-\pi ib}. \tag{52}
$$

15

Set

$$\widehat{\Psi}_{11}(\zeta) = \frac{\Gamma\left(1 - \lambda + \frac{\gamma}{2}\right)}{\Gamma(\gamma + 1)} \zeta^{\gamma/2} \phi\left(\lambda + \frac{\gamma}{2}, \gamma + 1; \zeta\right) e^{-\zeta/2} e^{i\pi\gamma/4},$$

$$\widehat{\Psi}_{12} = -\zeta^{\gamma/2} \psi\left(\lambda + \frac{\gamma}{2}, \gamma + 1; \zeta\right) e^{-\zeta/2} e^{-i\pi\gamma/4}.$$

Then from (51) and (52) if follows that for $\zeta \in \Gamma_5$,

$$\left(\widehat{\Psi}_{11}\right)_+ (\zeta) = \left(e^{2\pi i}\zeta\right)^{\gamma/2} \phi\left(\lambda + \frac{\gamma}{2}, \gamma + 1; e^{2\pi i}\zeta\right) e^{-\zeta/2} e^{i\pi\gamma/4} \frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma(\gamma+1)}$$

$$= e^{i\pi\gamma} \left(\widehat{\Psi}_{11}\right)_- (\zeta),$$

and

$$\left(\widehat{\Psi}_{12}\right)_+ (\zeta) = -\left(e^{2\pi i}\zeta\right)^{\gamma/2} \psi\left(\lambda + \frac{\gamma}{2}, \gamma + 1; e^{2\pi i}\zeta\right) e^{-\zeta/2} e^{-i\pi\gamma/4}$$

$$= \frac{2\pi i e^{-\pi i} e^{-i\pi\gamma/4}\Gamma(\gamma+1)}{\Gamma\left(\lambda-\frac{\gamma}{2}\right)\Gamma(\gamma+1)\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)e^{i\pi\gamma/4}} \left(\widehat{\Psi}_{11}\right)_- (\zeta) + e^{\pi i(\gamma-2\gamma-2)} \left(\widehat{\Psi}_{12}\right)_- (\zeta)$$

$$= \left[-e^{-i\pi\lambda} + e^{i\pi\lambda} e^{-i\pi\gamma}\right] \left(\widehat{\Psi}_{11}\right)_- (\zeta) + e^{-i\pi\gamma} \left(\widehat{\Psi}_{12}\right)_- (\zeta),$$

in accordance with (49). Analogously, we can satisfy the second row of (49) if we take

$$\widehat{\Psi}_{21} = \frac{\Gamma\left(1 + \lambda + \frac{\gamma}{2}\right)}{\Gamma(\gamma + 1)} \zeta^{\gamma/2} \phi\left(1 + \lambda + \frac{\gamma}{2}, \gamma + 1; \zeta\right) e^{-\zeta/2} e^{i\pi\frac{\gamma}{4}},$$

$$\widehat{\Psi}_{22} = \frac{\Gamma\left(1 + \lambda + \frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2} - \lambda\right)} \zeta^{\gamma/2} \psi\left(1 + \lambda + \frac{\gamma}{2}, \gamma + 1; \zeta\right) e^{-\zeta/2} e^{-i\pi\frac{\gamma}{4}}.$$

By construction, $\Psi$ satisfies the jumps relations in ($\Psi$2). Using formulas (7.26), (7.27) and (7.29) from [13, appendix], we can write explicitly the matrix $\Psi$ in all regions. Since the local behavior of $\psi(a, b; z)$ depends only on the value of the parameter $b$, by construction, all rows of $\widehat{\Psi}$ have the same asymptotics as $\zeta \to 0$. Hence, it is sufficient to analyze the first row.

From formulas (13.5.5) and (13.5.12) from [1] if follows that for $\zeta \in \textcircled{7}$, $\widehat{\Psi}$ has the behavior described in ($\Psi$3), as $\zeta \to 0$. Indeed, for $\gamma > 0$,

$$\widehat{\Psi}_{11} = \mathcal{O}\left(\zeta^{\gamma/2}\right), \quad \widehat{\Psi}_{12} = \mathcal{O}\left(\zeta^{-\gamma/2}\right);$$

for $\gamma = 0$,

$$\widehat{\Psi}_{11} = \mathcal{O}(1), \quad \widehat{\Psi}_{12} = \mathcal{O}(\ln\zeta);$$

and for $-1 < \gamma < 0$,

$$\widehat{\Psi}_{11} = \mathcal{O}\left(\zeta^{\gamma/2}\right), \quad \widehat{\Psi}_{12} = \mathcal{O}\left(\zeta^{\gamma/2}\right).$$

16

Analogously we can check that $\boldsymbol{\Psi}$ satisfies ($\Psi3$) in all regions of the plane.

Finally, using formula (13.1.22) from [1],

$$\begin{vmatrix} \phi(a,b;\zeta) & \psi(a,b;\zeta) \\ \phi'(a,b;\zeta) & \psi'(a,b;\zeta) \end{vmatrix} = -\frac{\Gamma(b)\,e^{\zeta}}{\zeta^{b}\Gamma(a)},$$

as well as the differential relations (13.4.23) and (13.4.10) from [1], we easily get that

$$\begin{vmatrix} \frac{\Gamma(b-a)}{\Gamma(b)}\zeta^{\frac{b-1}{2}}\phi(a,b;\zeta)\,e^{-\zeta/2}e^{\frac{i\pi\gamma}{4}} & -\zeta^{\frac{b-1}{2}}\psi(a,b;\zeta)\,e^{-\zeta/2}e^{-\frac{i\pi\gamma}{4}} \\ \frac{\Gamma(1+a)}{\Gamma(b)}\zeta^{\frac{b-1}{2}}\phi(a+1,b;\zeta)\,e^{-\zeta/2}e^{\frac{i\pi\gamma}{4}} & \frac{\Gamma(1+a)}{\Gamma(-(1+a-b))}\zeta^{\frac{b-1}{2}}\psi(a+1,b;\zeta)\,e^{-\zeta/2}e^{-\frac{i\pi\gamma}{4}} \end{vmatrix} = 1.$$

This implies tha $\det\ \widehat{\boldsymbol{\Psi}} = 1$, and, by construction, $\det \boldsymbol{\Psi} = 1$, which concludes the proof.

In order to construct the analytic function $\mathbf{E}_n$ in (29) we need to study also the asymptotic behavior of $\boldsymbol{\Psi}$ at infinity. Let us introduce the notation

$$\upsilon_n \stackrel{\text{def}}{=} \upsilon_n(\lambda) = \frac{\left(\lambda+\frac{\gamma}{2}\right)_n \left(\lambda-\frac{\gamma}{2}\right)_n}{n!}, \tag{53}$$

$$\tau_\lambda \stackrel{\text{def}}{=} \frac{\Gamma\left(-\lambda+\frac{\gamma}{2}\right)}{\left(-\frac{\gamma}{2}-\lambda\right)\Gamma\left(\frac{\gamma}{2}+\lambda\right)} = -\frac{\Gamma\left(\frac{\gamma}{2}-\lambda\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda+1\right)}; \tag{54}$$

observe that

$$\tau_{-\lambda} = \overline{\tau_\lambda}, \quad \upsilon_n(-\lambda) = \overline{\upsilon_n(\lambda)}, \quad \text{and} \quad \upsilon_1 = \left(\lambda^2 - \frac{\gamma^2}{4}\right) \in \mathbb{R}. \tag{55}$$

**Lemma 4.** *As $\zeta \to \infty$, $\zeta \in \mathbb{C}\backslash\Gamma$,*

$$\boldsymbol{\Psi}(\zeta) = \left[\mathbf{I} + \sum_{n=1}^{R-1} \frac{1}{\zeta^n} \begin{pmatrix} (-1)^n\,\upsilon_n & n\tau_\lambda\overline{\upsilon_n} \\ (-1)^n\,n\overline{\tau_\lambda}\upsilon_n & \overline{\upsilon_n} \end{pmatrix} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] \zeta^{-\lambda\sigma_3}e^{\frac{-\zeta\sigma_3}{2}}M^{-1}(\zeta) \tag{56}$$

*with $\upsilon_n$ defined by (53), $\tau_\lambda$ defined by (54), $\lambda = i\log(c)/\pi$, and*

$$M(\zeta) \stackrel{\text{def}}{=} \begin{cases} e^{\frac{\gamma}{4}\pi i\sigma_3}e^{-\lambda\pi i\sigma_3}, & \frac{\pi}{2} < \arg\zeta < \pi, \\ e^{-\frac{\gamma}{4}\pi i\sigma_3}e^{-\lambda\pi i\sigma_3}, & \pi < \arg\zeta < \frac{3\pi}{2}, \\ e^{\frac{\gamma}{4}\pi i\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & -\frac{\pi}{2} < \arg\zeta < 0, \\ e^{-\frac{\gamma}{4}\pi i\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & 0 < \arg\zeta < \frac{\pi}{2}, \end{cases}$$

*where we use the main branch of $\zeta^{-\lambda} = e^{-\lambda\log\zeta}$ with the cut along $i\mathbb{R}_-$.*

17

PROOF. We use the classical formulas (13.5.1) and (13.5.2) from [1] for the confluent hyper-geometric functions. If we take $b = \gamma + 1$, and multiply $\phi$ and $\psi$ by $(z^{\gamma/2} e^{-z/2})$, using (38), we have that, as $|z| \to \infty$,

$$G(a, \gamma; z) = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(a)} z^{a-\gamma/2} \left[ \frac{1}{z} \left( 1 + \sum_{n=1}^{R-1} \frac{(\gamma+1-a)_n (1-a)_n}{n! z^n} + \mathcal{O}\left( |z|^{-R} \right) \right) \right] e^{z/2}, & \operatorname{Re} z > 0, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-a)} e^{a\pi i} z^{\gamma/2-a} \left[ 1 + \sum_{n=1}^{R-1} \frac{(a)_n (a-\gamma)_n}{(-1)^n n! z^n} + \mathcal{O}\left( |z|^{-R} \right) \right] e^{-z/2}, & \operatorname{Re} z < 0, \end{cases}$$

(57)

$$H(a, \gamma; z) = z^{\gamma/2-a} \left[ 1 + \sum_{n=1}^{R-1} \frac{(a)_n (a-\gamma)_n}{(-1)^n n! z^n} + \mathcal{O}\left( |z|^{-R} \right) \right] e^{-z/2}.$$

(58)

Replacing these expansions in the expression for $\boldsymbol{\Psi}$ for $\zeta \in \textcircled{1}$, $\frac{\pi}{2} < \arg \zeta < \frac{3\pi}{4}$ and $\frac{-\pi}{2} < \arg\left( e^{-\pi i} \zeta \right) < \frac{-\pi}{4}$, we get for $|\zeta| \to \infty$,

$$\boldsymbol{\Psi}(\zeta) = \begin{pmatrix} \zeta^{-\lambda} \left[ 1 + \sum_{n=1}^{R-1} \frac{\left(\lambda+\frac{\gamma}{2}\right)_n \left(\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n n! \zeta^n} + \mathcal{O}\left( |\zeta|^{-R} \right) \right] e^{-\zeta/2} e^{\lambda\pi i} \\ \frac{-\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} \zeta^{-1-\lambda} \left[ 1 + \sum_{n=1}^{R-1} \frac{\left(1+\lambda+\frac{\gamma}{2}\right)_n \left(1+\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n n! \zeta^n} + \mathcal{O}\left( |\zeta|^{-R} \right) \right] e^{-\zeta/2} e^{\lambda\pi i} \end{pmatrix}$$
$$\begin{pmatrix} -\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)} \left( e^{-\pi i} \zeta \right)^{-1+\lambda} \left[ 1 + \sum_{n=1}^{R-1} \frac{\left(1-\lambda+\frac{\gamma}{2}\right)_n \left(1-\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n (-1)^n \zeta^n n!} + \mathcal{O}\left( |\zeta|^{-R} \right) \right] e^{\zeta/2} \\ \left( e^{-\pi i} \zeta \right)^{\lambda} \left[ 1 + \sum_{n=1}^{R-1} \frac{\left(\frac{\gamma}{2}-\lambda\right)_n \left(-\frac{\gamma}{2}-\lambda\right)_n}{(-1)^n (-1)^n \zeta^n n!} + \mathcal{O}\left( |\zeta|^{-R} \right) \right] e^{\zeta/2} \end{pmatrix} e^{-\frac{\gamma\pi i}{4} \sigma_3},$$

which can be rewritten using notation (53)–(54) as

$$= \left[ \mathbf{I} + \begin{pmatrix} \left[ \sum_{n=1}^{R-1} (-1)^n \frac{v_n}{\zeta^n} \right] \\ \overline{\tau_\lambda} \left[ \sum_{n=1}^{R-1} (-1)^n \frac{n v_n}{\zeta^n} \right] \end{pmatrix} \right.$$
$$\left. \begin{pmatrix} \tau_\lambda \left[ \sum_{n=1}^{R-1} \frac{n \overline{v_n}}{\zeta^n} \right] \\ \left[ \sum_{n=1}^{R-1} \frac{\overline{v_n}}{\zeta^n} \right] \end{pmatrix} + \mathcal{O}\left( |\zeta|^{-R} \right) \right] \left( \zeta^{-\lambda} e^{-\zeta/2} e^{\lambda\pi i} e^{-\frac{\gamma\pi i}{4}} \right)^{\sigma_3}.$$

This yields (56) for $\pi/2 < \zeta < 3\pi/4$; this expansion is also valid for $\zeta \in \textcircled{2}$. A comparison of (42) with (43) shows that the behavior for $\zeta \in \textcircled{3}$, $\pi < \arg \zeta < \frac{5\pi}{4}$, can be obtained from the expansion in $\textcircled{2}$ by multiplying by $e^{i\frac{\gamma}{2}\pi\sigma_3}$, which again yields (56) for $\pi < \zeta < 5\pi/4$. It is easy to see that asymptotics in $\textcircled{3}$ is also valid in $\textcircled{4}$.

Using (45), (57), (58) and comparing the expression for $\boldsymbol{\Psi}$ in $\textcircled{1}$ and $\textcircled{5}$, we conclude that

18

for $\zeta \in \text{⑤}$, $-\frac{\pi}{2} < \arg \zeta < -\frac{\pi}{4}$ ($\frac{\pi}{2} < \arg(\zeta) e^{\pi i} < \frac{3\pi}{4}$ and $\operatorname{Re} \zeta > 0$), as $|\zeta| \to \infty$,

$$
\Psi(\zeta) = \begin{pmatrix}
-\frac{\Gamma\left(1-\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}+\lambda\right)} \left(e^{\pi i}\zeta\right)^{-1+\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(1-\lambda+\frac{\gamma}{2}\right)_n \left(1-\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n (-1)^n \zeta^n n!} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{-\lambda\pi i} e^{\zeta/2} \\
\left(e^{\pi i}\zeta\right)^{\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(\frac{\gamma}{2}-\lambda\right)_n \left(-\frac{\gamma}{2}-\lambda\right)_n}{(-1)^n (-1)^n \zeta^n n!} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{-\lambda\pi i} e^{\zeta/2} \\
-\zeta^{-\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(\lambda+\frac{\gamma}{2}\right)_n \left(\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n n! \zeta^n} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{-\zeta/2} \\
-\frac{\Gamma\left(1+\lambda+\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma}{2}-\lambda\right)} \zeta^{-1-\lambda} \left[1 + \sum_{n=1}^{R-1} \frac{\left(1+\lambda+\frac{\gamma}{2}\right)_n \left(1+\lambda-\frac{\gamma}{2}\right)_n}{(-1)^n n! \zeta^n} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] e^{-\zeta/2}
\end{pmatrix} e^{-\frac{\gamma\pi i}{4}\sigma_3}
$$

$$
= \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \sum_{n=1}^{R-1} \frac{1}{\zeta^n} \begin{pmatrix} n\tau_\lambda \overline{\upsilon_n} & -(-1)^n \upsilon_n \\ \overline{\upsilon_n} & -(-1)^n n\overline{\tau_\lambda}\upsilon_n \end{pmatrix} + \mathcal{O}\left(|\zeta|^{-R}\right)\right]
$$

$$
\times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \zeta^{\lambda\sigma_3} e^{-\frac{\gamma}{4}\pi i \sigma_3} e^{\frac{\zeta}{2}\sigma_3}
$$

$$
= \left[\mathbf{I} + \sum_{n=1}^{R-1} \frac{1}{\zeta^n} \begin{pmatrix} (-1)^n \upsilon_n & n\tau_\lambda \overline{\upsilon_n} \\ (-1)^n n\overline{\tau_\lambda}\upsilon_n & \overline{\upsilon_n} \end{pmatrix} + \mathcal{O}\left(|\zeta|^{-R}\right)\right] \zeta^{-\lambda\sigma_3} e^{\frac{\gamma}{4}\pi i \sigma_3} e^{-\frac{\zeta}{2}\sigma_3} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

This expression is valid in $\text{⑥}$ as well. Finally, comparing (46) with (47) we see that the behavior for $\zeta \in \text{⑦}$, $0 < \arg \zeta < \frac{\pi}{4}$, corresponds to that in $\text{⑥}$ times the constant factor $e^{i\frac{\gamma}{2}\pi\sigma_3}$, which yields (56). Since the asymptotics for $\zeta \in \text{⑧}$ is the same than in $\text{⑦}$, this concludes the proof of Lemma.

Now we are ready to build $\mathbf{P}^{(1)}$ in (29). Using the properties of $\varphi$ we define an analytic function $f$ in a neighborhood of $x_0$,

$$
f(z) \overset{\text{def}}{=} \begin{cases} 2i\arccos x_0 - 2\log\varphi(z), & \text{for } \operatorname{Im} z > 0, \\ 2i\arccos x_0 + 2\log\varphi(z), & \text{for } \operatorname{Im} z < 0, \end{cases} \tag{59}
$$

where we take the main branch of the logarithm. Using that $\varphi_+(x)\varphi_-(x) = 1$ on $(-1,1)$ we conclude that $f$ can be extended to a holomorphic function in $\mathbb{C} \setminus ((-\infty,-1] \cup [1,+\infty))$. For $|z| < 1$ we have

$$
f(z) = \frac{2i}{\sqrt{1-x_0^2}} (z - x_0) + \mathcal{O}\left((z-x_0)^2\right), \quad \text{as } z \to x_0. \tag{60}
$$

Hence, for $\delta > 0$ sufficiently small, $f$ is a conformal mapping of $U_{x_0}$. Moreover, since

$$
\varphi_+(x) = x + i\sqrt{1-x^2} = e^{i\arccos x}, \quad x \in (-1,1), \tag{61}
$$

then

$$
f(x) = 2i(\arccos x_0 - \arccos x), \quad x \in (-1,1), \tag{62}
$$

19

so that $f$ maps the real interval $(-1, x_0)$ one-to-one onto the purely imaginary interval $(2i(\arccos x_0 - \pi), 0)$, as well as $(x_0, 1)$ one-to-one onto the purely imaginary interval $(0, 2i \arccos x_0)$.

We can always deform our contours $\Sigma_k$ close to $z = x_0$ in such a way that

$$f(\Sigma_1 \cap U_{x_0}) \subset \Gamma_4, \quad f(\Sigma_2 \cap U_{x_0}) \subset \Gamma_6, \quad f(\Sigma_3 \cap U_{x_0}) \subset \Gamma_2, \quad f(\Sigma_4 \cap U_{x_0}) \subset \Gamma_8$$
$$f(\Sigma_5 \cap U_{x_0}) \subset \Gamma_3, \quad f(\Sigma_6 \cap U_{x_0}) \subset \Gamma_7.$$

With this convention, set

$$\zeta \stackrel{\text{def}}{=} nf(z), \quad z \in U_{x_0}, \tag{63}$$

and, we define

$$\mathbf{P}^{(1)}(z) \stackrel{\text{def}}{=} \mathbf{\Psi}(nf(z)), \quad z \in U_{x_0}. \tag{64}$$

By $(\Psi 1)$–$(\Psi 3)$ and (60), this matrix-valued function has the jumps and the local behavior at $z = x_0$ specified in (32)–(35). Taking into account the definition (59) we get that

$$e^{nf(z)} = \varphi_+^{2n}(x_0) \varphi^{\mp 2n}(z), \quad \text{for } \pm \operatorname{Im} z > 0,$$

and for $[nf(z)]^\lambda$ we take the cut along $(-\infty, x_0]$. Since

$$[f(z)]^\lambda = |f(z)|^\lambda \exp\left(-\frac{\log c}{\pi} \arg(f(z))\right),$$

straightforward computations show that

$$[f(x)]_\pm^\lambda = \begin{cases} |f(x)|^\lambda c^{-1/2}, & \text{for } x_0 < x < 1, \\ |f(x)|^\lambda c^{-1/2 \mp 1}, & \text{for } -1 < x < x_0, \end{cases} \tag{65}$$

where we assume the natural orientation of the interval.

In order to satisfy $(\text{P}_0 3)$ above, we define

$$\mathbf{E}_n(z) \stackrel{\text{def}}{=} \mathbf{N}(z) W(z)^{\sigma_3} \begin{cases} (nf(z))^{\lambda \sigma_3} \varphi_+^{n\sigma_3}(x_0) e^{i\frac{\gamma\pi}{4}\sigma_3} c^{\sigma_3}, & \text{if } z \in Q_+^R, \\[2mm] \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (nf(z))^{\lambda \sigma_3} \varphi_+^{n\sigma_3}(x_0) e^{i\frac{\gamma\pi}{4}\sigma_3}, & \text{if } z \in Q_-^R, \\[2mm] (nf(z))^{\lambda \sigma_3} \varphi_+^{n\sigma_3}(x_0) e^{-i\frac{\gamma\pi}{4}\sigma_3} c^{\sigma_3}, & \text{if } z \in Q_+^L, \\[2mm] \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (nf(z))^{\lambda \sigma_3} \varphi_+^{n\sigma_3}(x_0) e^{-i\frac{\gamma\pi}{4}\sigma_3}, & \text{if } z \in Q_-^L. \end{cases} \tag{66}$$

By construction, $\mathbf{E}_n$ is analytic in $U_{x_0} \setminus (\mathbb{R} \cup \Sigma_5 \cup \Sigma_6)$. Furthermore, by (N2) and (28), for $x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$,

$$W_-(x)^{-\sigma_3} \mathbf{N}_-^{-1}(x) \mathbf{N}_+(x) W_+(x)^{\sigma_3} = \begin{pmatrix} 0 & \frac{w_{c,\gamma}(x)}{W_-(x)W_+(x)} \\ -\frac{W_-(x)W_+(x)}{w_{c,\gamma}(x)} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & c^{\pm 1} \\ -c^{\mp 1} & 0 \end{pmatrix}, \quad \text{for } \pm x > x_0;$$

and, by (26), for $z \in \Sigma_6 \cap U_{x_0}$ (oriented from above to bellow) and for $z \in \Sigma_5 \cap U_{x_0}$ (oriented from bellow to above) we have,

$$W_-(z)^{-\sigma_3} \mathbf{N}_-^{-1}(z)\mathbf{N}_+(z)W_+(z)^{\sigma_3} = \begin{pmatrix} W_+(z)/W_-(z) & 0 \\ 0 & W_-(z)/W_+(z) \end{pmatrix} = e^{i\frac{\gamma\pi}{2}\sigma_3}.$$

From (65) and (66) it follows that

$$\mathbf{E}_{n-}^{-1}(z) \mathbf{E}_{n+}(z) = \mathbf{I}, \quad \text{for } z \in U_{x_0} \setminus \{x_0\}.$$

In this form, $x_0$ is the only possible isolated singularity of $\mathbf{E}_n$ in $U_{x_0}$. The following proposition shows that this is in fact a removable singularity of $\mathbf{E}_n$:

**Proposition 5.**

$$\lim_{z \to x_0} \mathbf{E}_n(z) = \frac{\sqrt{2}}{2} D_\infty^{\sigma_3} \begin{pmatrix} e^{-i \arcsin(x_0)/2} & e^{i \arcsin(x_0)/2} \\ -e^{i \arcsin(x_0)/2} & e^{-i \arcsin(x_0)/2} \end{pmatrix} e^{i\eta_n \sigma_3},$$

*with $\eta_n$ defined by*

$$\eta_n \overset{def}{=} \frac{\log c}{\pi} \log\left(4n\sqrt{1-x_0^2}\right) + n \arccos(x_0) - \frac{\gamma\pi}{4} - \Phi(x_0) \tag{67}$$

*and $\Phi$ given by (22). In particular, $\mathbf{E}_n$ is analytic in $U_{x_0}$.*

PROOF. Since $\mathbf{E}_n$ is analytic in a neighborhood of $x_0$, it is sufficient to analyze its limit as $z \to x_0$ from the first quarter of the plane, $z \in Q_+^R$. By (24) and (60),

$$\lim_{\substack{z \to x_0 \\ z \in Q_+^R}} D(z, \Xi_c) f(z)^{-\lambda} = \lim_{\substack{z \to x_0 \\ z \in Q_+^R}} c^{1+\frac{i}{\pi}\log(z/2)-\frac{i}{\pi}\log(f(z))} = c^{3/2}\left(4\sqrt{1-x_0^2}\right)^{-\lambda}.$$

On the other hand, by (21) and (27) (notice that $w_{1,\gamma}$ defined in (12) coincides with $w$ defined in (11)),

$$\lim_{\substack{z \to x_0 \\ z \in Q_+^R}} D(z, w) W(z)^{-1} = c^{-1/2} e^{i\Phi(x_0)} e^{i\gamma\pi/2}.$$

Summarizing,

$$\lim_{\substack{z \to x_0 \\ z \in Q_+^R}} D(z, w_{c,\gamma})^{-1} W(z) f(z)^\lambda = \frac{\left(4\sqrt{1-x_0^2}\right)^\lambda}{c} e^{-i\Phi(x_0)-i\gamma\pi/2}.$$

21

By (14) and (66), if $z \in Q_+^R$ $(\operatorname{Im} z > 0)$,

$$\mathbf{E}_n(z) = D_\infty^{\sigma_3} \, \mathbf{A}(z) \, m_n(z)^{\sigma_3}, \tag{68}$$

with

$$m_n(z) \stackrel{\text{def}}{=} \frac{W\left(z\right) f\left(z\right)^\lambda}{D\left(z, w_{c,\gamma}\right)} \, \varphi_+^n(x_0) n^\lambda e^{i\gamma\pi/4} c = e^{\eta_n}, \tag{69}$$

where $\eta_n$ is defined in (67).

Gathering the limits computed above and using that

$$\lim_{\substack{z \to x_0 \\ z \in Q_+^R}} A_{11}(z) = e^{-i\arcsin(x_0)/2} = \lim_{\substack{z \to x_0 \\ z \in Q_+^R}} \overline{A_{12}(z)}$$

and by definition of $\eta_n$, the statement follows.

Therefore, by construction the matrix-valued function $\mathbf{P}_{x_0}$ given by (29) satisfies conditions $(P_0 1)$–$(P_0 4)$. Moreover, it is easy to check that

$$\det \mathbf{P}_{x_0}\left(z\right) = 1 \quad \text{for every } z \in U_{x_0} \backslash \Sigma.$$

At this point all the ingredients are ready to define the final transformation. We take

$$\mathbf{R}\left(z\right) \stackrel{\text{def}}{=} \begin{cases} \mathbf{S}\left(z\right) \mathbf{N}^{-1}\left(z\right), & z \in \mathbb{C} \backslash \left\{\Sigma \cup U_{-1} \cup U_{x_0} \cup U_1\right\}; \\ \mathbf{S}\left(z\right) \mathbf{P}_\zeta^{-1}\left(z\right), & z \in U_j \backslash \Sigma, \ j \in \{-1, x_0, 1\}. \end{cases} \tag{70}$$

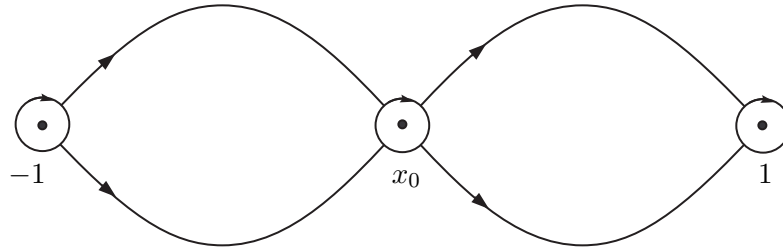$\mathbf{R}$ is analytic in the complement to the contours $\Sigma_R$ depicted in Fig. 4.



Figure 4: Contours $\Sigma_R$.

Let

$$\Sigma_R^{out} \stackrel{\text{def}}{=} \Sigma_R \backslash \left(\partial U_{-1} \cup \partial U_{x_0} \cup \partial U_1\right).$$

**R** satisfies the jump relation

$$\mathbf{R}_+(z) = \mathbf{R}_-(z)(\mathbf{I} + \mathbf{\Delta}(z)), \quad z \in \Sigma_R,$$

with

$$\mathbf{\Delta}(s) \stackrel{\text{def}}{=} \begin{cases} \mathbf{N}(s) \begin{pmatrix} 1 & 0 \\ w_{c,\gamma}(s)^{-1} \varphi(s)^{-2n} & 1 \end{pmatrix} \mathbf{N}^{-1}(s) - \mathbf{I}, & \text{for } s \in \Sigma_R^{out}; \\ \mathbf{P}_\zeta(s) \mathbf{N}^{-1}(s) - \mathbf{I}, & \text{for } s \in \partial U_\zeta, \ j \in \{-1, x_0, 1\}. \end{cases}$$

Standard arguments show that $\mathbf{\Delta}$ has an asymptotic expansion in powers of $1/n$ of the form

$$\mathbf{\Delta}(s) \sim \sum_{k=1}^{\infty} \frac{\mathbf{\Delta}_k(s, n)}{n^k}, \quad \text{as } n \to \infty, \text{ uniformly for } s \in \Sigma_R, \tag{71}$$

and, for $k \in \mathbb{N}$,

$$\mathbf{\Delta}_k(s) = 0, \quad \text{for } s \in \Sigma_R^{out}. \tag{72}$$

So, it remains to determine $\mathbf{\Delta}_k$ on $\partial U_{x_0}$. Here we find explicitly only the first term, $\mathbf{\Delta}_1$. Using (14), (25), (29), (59), (56), (63), (66) and (55), we obtain

$$\mathbf{\Delta}(s) = \mathbf{E}_n(s) \left[ \frac{(\lambda^2 - \gamma^2/4)}{nf(s)} \begin{pmatrix} -1 & \tau_\lambda \\ -\overline{\tau_\lambda} & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{n^2}\right) \right] \mathbf{E}_n^{-1}(s), \quad s \in \partial U_{x_0}, \quad n \to \infty.$$

Let us define

$$\mathbf{\Delta}_1(s) \stackrel{\text{def}}{=} \frac{(\lambda^2 - \gamma^2/4)}{f(s)} \mathbf{E}_n(s) \begin{pmatrix} -1 & \tau_\lambda \\ -\overline{\tau_\lambda} & 1 \end{pmatrix} \mathbf{E}_n^{-1}(s), \quad s \in \partial U_{x_0}. \tag{73}$$

Using that by (66),

$$\mathbf{E}_n(s) = \mathbf{F}(s) \left( \varphi_+(x_0)^n n^\lambda \right)^{\sigma_3} = \mathbf{F}(s) \left( e^{in \arccos(x_0)} c^{\frac{i}{\pi} \log n} \right)^{\sigma_3},$$

with

$$\mathbf{F}(s) \stackrel{\text{def}}{=} \begin{cases} \mathbf{N}(s) W(s)^{\sigma_3} c^{\sigma_3} e^{\pm \frac{\gamma\pi}{4}\sigma_3} f(s)^{\lambda\sigma_3}, & \text{if } \operatorname{Im} s > x_0, \\ \mathbf{N}(s) W(s)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{\pm \frac{\gamma\pi}{4}\sigma_3} f(s)^{\lambda\sigma_3}, & \text{if } \operatorname{Im} s < x_0, \end{cases}$$

where we take $\pm$ for $\pm \operatorname{Re} s > x_0$, we conclude that, for $s \in \partial U_{x_0}$, $\Delta_1(z, n)$ is uniformly bounded in $n$; indeed, $\mathbf{F}$ does not depend on $n$ and

$$\left| e^{in \arccos x_0} c^{\frac{i \log n}{2\pi}} \right| = 1, \quad \forall n \in \mathbb{N}.$$

So $\boldsymbol{\Delta}_1$ in (73) is genuinely the first coefficient in the expansion (71).

Similar analysis can be performed for $\boldsymbol{\Delta}_k(\cdot, n)$, $k \geq 2$, taking higher order terms in the expansion of $\boldsymbol{\Psi}$ in (56).

The explicit expression (56) and the local behavior of $f$ show that $\boldsymbol{\Delta}_1(s, n)$ has an analytic continuation to $U_{x_0}$ except for $x_0$, where it has a simple pole. Again, similar conclusion is valid for other $\boldsymbol{\Delta}_k(s, n)$, except that now the pole has order $k$.

Like in [5, Theorem 7.10] we obtain from (71) that

$$\mathbf{R}(z) \sim \mathbf{I} + \sum_{j=1}^{\infty} \frac{\mathbf{R}^{(j)}(z, n)}{n^j}, \quad \text{as } n \to \infty, \tag{74}$$

uniformly for $z \in \mathbb{C} \backslash \{\partial U_{-1} \cup \partial U_{x_0} \cup \partial U_1\}$, where each $\mathbf{R}^{(j)}(z)$ is analytic, uniformly bounded in $n$, and

$$\mathbf{R}^{(j)}(z, n) = \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \to \infty.$$

Since $\mathbf{R}^{(1)}$ is analytic in the complement of $\partial U_{-1} \cup \partial U_{x_0} \cup \partial U_1$ (see (72)) and vanishes at infinity, by Sokhotskii-Plemelj formulas,

$$\mathbf{R}^{(1)}(z, n) = \frac{1}{2\pi i} \int_{\partial U_{-1} \cup \partial U_{x_0} \cup \partial U_1} \frac{\boldsymbol{\Delta}_1(s, n)}{s - z} \, ds.$$

Recall that $\boldsymbol{\Delta}_1$ can be extended analytically inside $U_j$'s with simple poles at $\pm 1$ and $x_0$; let us denote by $A^{(1)}(n)$, $B^{(1)}(n)$ and $C^{(1)}(n)$ the residues of $\boldsymbol{\Delta}_1(\cdot, n)$ at $1$, $-1$ and $x_0$, respectively. Then the residue calculus gives

$$\mathbf{R}^{(1)}(z, n) = \begin{cases} \dfrac{A^{(1)}(n)}{z - 1} + \dfrac{B^{(1)}(n)}{z + 1} + \dfrac{C^{(1)}(n)}{z - x_0}, & \text{for } z \in \mathbb{C} \backslash \{U_{-1} \cup U_{x_0} \cup U_1\}; \\ \dfrac{A^{(1)}(n)}{z - 1} + \dfrac{B^{(1)}(n)}{z + 1} + \dfrac{C^{(1)}(n)}{z - x_0} - \boldsymbol{\Delta}_1(z, n), & \text{for } z \in U_{-1} \cup U_{x_0} \cup U_1. \end{cases} \tag{75}$$

Residues $A^{(1)}(n)$ and $B^{(1)}(n)$ are in fact independent of $n$; they have been determined in [14, Section 8]:

$$A^{(1)}(n) = A^{(1)} = \frac{4\alpha^2 - 1}{16} D_\infty^{\sigma_3} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} D_\infty^{-\sigma_3},$$

$$B^{(1)}(n) = B^{(1)} = \frac{4\beta^2 - 1}{16} D_\infty^{\sigma_3} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} D_\infty^{-\sigma_3} \tag{76}$$

(notice however that the constant $D_\infty$ is different with respect to [14]). The value of the remaining residue $C^{(1)}(n)$ is given in the following

**Proposition 6.** *If we denote*

$$
C^{(1)}(n) = \begin{pmatrix} C_{11}^{(1)}(n) & C_{12}^{(1)}(n) \\ C_{21}^{(1)}(n) & C_{11}^{(1)}(n) \end{pmatrix}
$$

*then the entries are given explicitly by:*

$$
C_{11}^{(1)}(n) = -\left(\frac{\log^2 c}{2\pi^2} + \frac{\gamma^2}{8}\right) x_0 + \sqrt{\frac{\log^2 c}{4\pi^2} + \frac{\gamma^2}{16}} \sin\theta_n \tag{77}
$$

$$
C_{12}^{(1)}(n) = iD_\infty^2 \left(\frac{\log^2 c}{2\pi^2} + \frac{\gamma^2}{8} - \sqrt{\frac{\log^2 c}{4\pi^2} + \frac{\gamma^2}{16}} \cos\left(\arcsin(x_0) - \theta_n\right)\right) \tag{78}
$$

$$
C_{21}^{(1)}(n) = \frac{i}{D_\infty^2} \left(\frac{\log^2 c}{2\pi^2} + \frac{\gamma^2}{8} + \sqrt{\frac{\log^2 c}{4\pi^2} + \frac{\gamma^2}{16}} \cos\left(\arcsin(x_0) + \theta_n\right)\right) \tag{79}
$$

$$
C_{22}^{(1)}(n) = \left(\frac{\log^2 c}{2\pi^2} + \frac{\gamma^2}{8}\right) x_0 - \sqrt{\frac{\log^2 c}{4\pi^2} + \frac{\gamma^2}{16}} \sin\theta_n \tag{80}
$$

*where*

$$
\theta_n = 2\eta_n + \varsigma, \tag{81}
$$

*with $\eta_n$ defined by (67) and $\varsigma = -2\arg\Gamma\left(\frac{\gamma}{2} + \lambda\right) - \arg\left(\frac{\gamma}{2} + \lambda\right)$.*

PROOF. Taking into account (60) and (73) we conclude that the residue $C^{(1)}(n)$ of $\boldsymbol{\Delta}_1(z,n)$ at $z = x_0$ is given by

$$
C^{(1)}(n) = \frac{(\lambda^2 - \gamma^2/4)\sqrt{1 - x_0^2}}{2i} \mathbf{E}_n(x_0) \begin{pmatrix} -1 & \tau_\lambda \\ -\overline{\tau_\lambda} & 1 \end{pmatrix} \mathbf{E}_n^{-1}(x_0). \tag{82}
$$

Since $\mathbf{E}_n$ is analytic in a neighborhood of $x_0$ (see Proposition 5),

$$
\mathbf{E}_n(x_0) = \lim_{\substack{z \to x_0 \\ z \in Q_+^R}} \mathbf{E}_n(z) = \frac{1}{\sqrt{2}\sqrt[4]{1 - x_0^2}} D_\infty^{\sigma_3} \begin{pmatrix} e^{-i\arcsin(x_0)/2} & e^{i\arcsin(x_0)/2} \\ -e^{i\arcsin(x_0)/2} & e^{-i\arcsin(x_0)/2} \end{pmatrix} e^{i\eta_n\sigma_3}, \tag{83}
$$

so that

$$
\mathbf{E}_n^{-1}(x_0) = \frac{1}{\sqrt{2}\sqrt[4]{1 - x_0^2}} e^{-i\eta_n\sigma_3} \begin{pmatrix} e^{-i\arcsin(x_0)/2} & -e^{i\arcsin(x_0)/2} \\ e^{i\arcsin(x_0)/2} & e^{-i\arcsin(x_0)/2} \end{pmatrix} D_\infty^{-\sigma_3}. \tag{84}
$$

25

From (83) we obtain

$$C^{(1)}(n) = \frac{\left(\lambda^2 - \gamma^2/4\right)}{4i} D_\infty^{\sigma_3} \begin{pmatrix} e^{-i\arcsin(x_0)/2} & e^{i\arcsin(x_0)/2} \\ -e^{i\arcsin(x_0)/2} & e^{-i\arcsin(x_0)/2} \end{pmatrix} e^{i\eta_n \sigma_3}$$

$$\times \begin{pmatrix} -1 & \tau_\lambda \\ -\overline{\tau_\lambda} & 1 \end{pmatrix} e^{-i\eta_n \sigma_3} \begin{pmatrix} e^{-i\arcsin(x_0)/2} & -e^{i\arcsin(x_0)/2} \\ e^{i\arcsin(x_0)/2} & e^{-i\arcsin(x_0)/2} \end{pmatrix} D_\infty^{-\sigma_3}.$$

Using formulas (6.1.28), (6.1.23) and (4.3.2) from [1] and (54) we can rewrite

$$\tau_\lambda = -\frac{\overline{\Gamma\left(\frac{\gamma}{2} + \lambda\right)}}{\left(\frac{\gamma}{2} + \lambda\right)\Gamma\left(\frac{\gamma}{2} + \lambda\right)} = -\left| \frac{\overline{\Gamma\left(\frac{\gamma}{2} + \lambda\right)}}{\left(\frac{\gamma}{2} + \lambda\right)\Gamma\left(\frac{\gamma}{2} + \lambda\right)} \right| e^{i\varsigma} = -\frac{e^{i\varsigma}}{\sqrt{\gamma^2/4 + |\lambda|^2}},$$

where

$$\varsigma = \arg\left( \frac{\overline{\Gamma\left(\frac{\gamma}{2} + \lambda\right)}}{\left(\frac{\gamma}{2} + \lambda\right)\Gamma\left(\frac{\gamma}{2} + \lambda\right)} \right).$$

Then,

$$C^{(1)}(n) = \frac{\left(\lambda^2 - \gamma^2/4\right)}{2} D_\infty^{\sigma_3} \begin{pmatrix} x_0 - \frac{\sin\theta_n}{\sqrt{\gamma^2/4 + |\lambda|^2}} & -i + i\frac{\cos(\arcsin(x_0) - \theta_n)}{\sqrt{\gamma^2/4 + |\lambda|^2}} \\ -i - i\frac{\cos(\arcsin(x_0) + \theta_n)}{\sqrt{\gamma^2/4 + |\lambda|^2}} & -x_0 - \frac{\sin(-\theta_n)}{\sqrt{\gamma^2/4 + |\lambda|^2}} \end{pmatrix} D_\infty^{-\sigma_3}.$$

We can simplify this expression using that $\left(\lambda^2 - \frac{\gamma^2}{4}\right) = -\left(\frac{\log^2 c}{\pi^2} + \frac{\gamma^2}{4}\right) = -\left(\sqrt{\frac{\gamma^2}{4} + |\lambda|^2}\right)^2$,
and this settles the proof.

## 3. Proof of Theorem 1

Unraveling the transformations $\mathbf{Y} \to \mathbf{T} \to \mathbf{S} \to \mathbf{R}$ we can obtain an expression for $\mathbf{Y}$. Repeating the arguments in [8] (see also [14], [11, Section 3] and [17]) we see that the recurrence coefficients (3) are given by

$$a_n^2 = \lim_{z \to \infty} \left( -\frac{D_\infty^2}{2i} + z\mathbf{R}_{12}(z, n) \right) \left( z\mathbf{R}_{21}(z, n) + \frac{1}{2iD_\infty^2} \right), \tag{85}$$

$$b_n = \lim_{z \to \infty} z\left(1 - \mathbf{R}_{11}(z, n+1)\mathbf{R}_{22}(z, n)\right). \tag{86}$$

Taking into account the expression for $\mathbf{R}^{(1)}$ in (75), as well as (76) and Proposition 6, we obtain for $a_n$:

$$a_n^2 = \frac{1}{4} - \frac{1}{n}\sqrt{\frac{\gamma^2}{16} + \frac{\log^2 c}{4\pi^2}} \cos(\arcsin x_0)\cos(\theta_n) + O\left(\frac{1}{n^2}\right), \quad n \to \infty,$$

26

where $\theta_n$ is given by (81). It also can be rewritten in the form

$$\theta_n = \frac{2\log c}{\pi}\log\left(4n\sqrt{1-x_0^2}\right) + 2n\arccos x_0 - \Theta,$$

with $\Theta$ defined in (6). This proves (4).

Analogously,

$$b_n = -\frac{\sqrt{\frac{\log^2 c}{4\pi^2} + \frac{\gamma^2}{16}}\left(\sin\theta_{n+1} - \sin\theta_n\right)}{n} + O\left(\frac{1}{n^2}\right), \quad n \to \infty.$$

By (81),

$$\theta_{n+1} - \theta_n = 2\arccos(x_0) + 2\frac{\log c}{\pi}\log\left(1 + \frac{1}{n}\right),$$

and

$$\sin\theta_{n+1} - \sin\theta_n = \sin\left(\theta_n + 2\arccos x_0\right) - \sin\theta_n + \mathcal{O}\left(\frac{1}{n}\right)$$

$$= 2\cos\left(\theta_n + \arccos x_0\right)\sin\left(\arccos x_0\right) + \mathcal{O}\left(\frac{1}{n}\right), \quad n \to \infty.$$

Thus we obtain

$$b_n = -\frac{1}{n}\sqrt{\frac{\log^2 c}{\pi^2} + \frac{\gamma^2}{4}}\sin\left(\arccos x_0\right)\cos\left(\theta_n + \arccos x_0\right) + O\left(\frac{1}{n^2}\right),$$

which proves (5).

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