Chaos, Solitons and Fractals 42 (2009) 256–264

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

Invariant and semi-invariant probabilistic normed spaces

M.B. Ghaemi [a,*], B. Lafuerza-Guillén [b], S. Saiedinezhad [a]

[a] School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran
[b] Departamento de Estadística y Matemática Aplicada, Universidad de Almería, Almería E-04120, Spain

ARTICLE INFO

Article history:
Accepted 23 November 2008

ABSTRACT

Probabilistic metric spaces were introduced by Karl Menger. Alsina, Schweizer and Sklar gave a general definition of probabilistic normed space based on the definition of Menger [1]. We introduce the concept of *semi-invariance* among the PN spaces. In this paper we will find a sufficient condition for some PN spaces to be *semi-invariant*. We will show that PN spaces are *normal* spaces. Urysohn's lemma, and Tietze extension theorem for them are proved.

1. Introduction and preliminaries

Menger proposed transferring the probabilistic notions of quantum mechanics from physics to the underlying geometry. The theory of probabilistic normed spaces (briefly, PN spaces) is important as a generalization of deterministic results of linear normed spaces and also in the study of random operator equations. The PN spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important applications in quantum particle physics particularly in strings' theory and in ε^∞ theory which were studied by El Naschie [2,3].

PN spaces were first defined by Šerstnev in 1962 (see [12]). Their definition was generalized in [1]. We recall the definition of probabilistic space briefly as given in [1], together with the notation that will be needed (see [11]). We shall consider the space of all distance probability distribution functions (briefly, d.f.'s), namely the set of all left-continuous and non-decreasing functions from $\overline{\mathbb{R}}$ into $[0,1]$ such that $F(0) = 0$ and $F(+\infty) = 1$; here as usual, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. The spaces of these functions will be denoted by Δ^+, while the subset $D^+ \subseteq \Delta^+$ will denote the set of all proper distance d.f.'s, namely those for which $\ell^- F(+\infty) = 1$. Here $\ell^- f(x)$ denotes the left limit of the function f at the point x, $\ell^- f(x) := \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, i.e., $F \leqslant G$ if and only if $F(x) \leqslant G(x)$ for all x in $\mathbb{R}$ [10]. For any $a \geqslant 0$, ε_a is the d.f. given by

$$\varepsilon_a = \begin{cases} 0 & \text{if } x \leqslant a, \\ 1 & \text{if } x > a. \end{cases}$$

The space Δ^+ can be metrized in several ways [11], but we shall here adopt the Sibley metric d_S which is the metric denoted by d_L in [11], viz. the Lvy metric as a modified by Sibley [13]. If F, G are d.f.'s and h is in $]0,1[$, let $[F, G; h]$ denote the condition:

$$G(x) \leqslant F(x+h) + h \quad \text{for all } x \in \left]0, \frac{1}{h}\right[.$$

* Corresponding author.
E-mail addresses: mghaemi@iust.ac.ir (M.B. Ghaemi), blafuerz@ual.es (B. Lafuerza-Guillén), ssaiedinezhad@yahoo.com (S. Saiedinezhad).

0960-0779/$ - see front matter
doi:10.1016/j.chaos.2008.11.017

Then the Sibley metric d_S is defined by

$$d_S(F,G) := \inf\{h \in]0,1] : \text{both } [F,G;h] \text{ and } [G,F;h] \text{ hold}\}. \tag{1.1}$$

Lemma 1.1. see [11], Lemma 4.2.2If $d_S(F,G) = h > 0$, then both $[F,G;h]$ and $[G,F;h]$ hold.

Definition 1.2. A triangular norm or, briefly, a t-norm is a function $T : [0,1]^2 \to [0,1]$ that satisfies the following conditions:

(T1) it is commutative, i.e., $T(s,t) = T(t,s)$ for all s and t in $[0,1]$;
(T2) it is associative, i.e., $T(T(s,t),u) = T(s,T(t,u))$ for all s,t and u in $[0,1]$;
(T3) it is increasing in each place, i.e., $T(s,t) \leqslant T(s',t)$ for all $t \in [0,1]$ whenever $s \leqslant s'$;
(T4) it satisfies the boundary condition $T(1,t) = t$ for every $t \in [0,1]$.

The most important t-norms are the minimum M, the product Π, the Lukasiewicz t-norm W and the drastic product D given by

$$M(x,y) := \min\{x,y\},$$
$$\Pi(x,y) := xy,$$
$$W(x,y) := \max\{0, x+y-1\},$$
$$D(x,y) := \begin{cases} \min(x,y) & \text{if } \max\{x,y\} = 1, \\ 0 & \text{otherwise}. \end{cases}$$

A triangle function is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, non-decreasing in each place and has ε_0 as identity, that is, for all F, G and H in Δ^+:

(TF1) $\tau(\tau(F,G),H) = \tau(F,\tau(G,H))$,
(TF2) $\tau(F,G) = \tau(G,F)$,
(TF3) $F \leqslant G \Rightarrow \tau(F,H) \leqslant \tau(G,H)$,
(TF4) $\tau(F,\varepsilon_0) = \tau(\varepsilon_0,F) = F$.

In particular, under the usual pointwise ordering of functions, ε_0 is the maximal element of Δ^+. Moreover, a triangle function is *continuous* if it is continuous in the metric space (Δ^+, d_S).

Typical continuous triangle functions are

(a) $\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s),G(t))$,
(b) $\tau_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s),G(t))$,
(c) $\Pi_T(F,G)(x) = T(F(x),G(x))$,

and here T is a continuous t-norm.

The definition below is more general; it has been proposed in [1].

Definition 1.3. A *Probabilistic normed space* (briefly, PN space) is a quadruple (V,ν,τ,τ^*), where V is a real vector space, τ and τ^* are continuous triangle functions with $\tau \leqslant \tau^*$ and ν is a mapping (the *probabilistic norm*) from V into Δ^+, such that for every choice of p and q in V the following hold:

(N1) $\nu_p = \varepsilon_0$ if and only if, $p = \theta$(θ is the null vector in V);
(N2) $\nu_{-p} = \nu_p$;
(N3) $\nu_{p+q} \geqslant \tau(\nu_p, \nu_q)$;
(N4) $\nu_p \leqslant \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0,1]$.

There is a natural topology in a PN space (V,ν,τ,τ^*), called *strong topology*; it is defined for $p \in V$ and $t > 0$, by the neighborhoods

$$N_p(t) := \{q \in V; \nu_{q-p}(t) > 1-t\} = \{q \in V; d_S(\nu_{q-p}, \varepsilon_0) < t\}$$

We recall that a set A in a PN space (V,ν,τ,τ^*), is said to be *D-bonded* if its probabilistic radius R_A belong to D^+, where

$$R_A(x) = \begin{cases} l^- \inf\{\nu_p(x); p \in A\}, & \text{if } x \in [0,+\infty), \\ 1, & \text{if } x = +\infty. \end{cases}$$

There are special PN spaces, only some of which we list below; for the others we refer to [5].

When there is a continuous t-norm T (see [11]) such that $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, where $T^*(x,y) := 1 - T(1-x, 1-y)$,

$$\tau_T(F,G)(x) := \sup_{s+t=x} T(F(s), G(t)) \quad \text{and} \quad \tau_{T^*}(F,G)(x) := \inf_{s+t=x} T^*(F(s), G(t))$$

the PN space $(V, \nu, \tau_T, \tau_{T^*})$ is called a *Menger* PN space, and is denoted by (V, ν, T).

A PN space is called a *Šerstnev space* if it satisfies (N1), (N3) and the following condition, which implies both (N2) and (N4)

$$\forall\, p \in V \;\; \forall\, \alpha \in \mathbb{R}\setminus\{0\} \;\; \forall\, x > 0 \quad \nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right).$$

One speaks of an *equilateral* PN space when there is $F \in \Delta^+$ different from both ε_0 and ε_∞ such that, for every $p \neq \theta$, $\nu_p = F$, and when $\tau = \tau^* = \mathbf{M}$, which is the triangle function defined for G and H in Δ^+ by $\mathbf{M}(G,H)(x) := \min\{G(x), H(x)\}$.

Let $G \in \Delta^+$ be different from ε_0 and from ε_∞ and let $(V, \|\cdot\|)$ be a normed space; then define, for $p \neq \theta$,

$$\nu_p(x) := G\left(\frac{x}{\|p\|}\right).$$

Then (V, ν, M) is a Menger space denoted by $(V, \|\cdot\|, G, M)$ $(M(x,y) := \min\{x,y\})$. In the same conditions, if ν is defined by

$$\nu_p(x) := G\left(\frac{x}{\|p\|^\alpha}\right),$$

with $\alpha \geqslant 0$, then the pair (V, ν) is a PSN space called *α-simple* and it is denoted by $(V, \|\cdot\|, G; \alpha)$. Let $(\Omega, \mathscr{A}, P)$ be a probability space, $(V, \|\cdot\|)$ a normed space and S a vector space of V-valued random variables (possibly, the entire space). For every $p \in S$ and for every $x \in \overline{\mathbb{R}}_+$, let $\nu : S \to \Delta^+$ be defined by $\nu_p(x) := P\{\omega \in \Omega : \|p(\omega)\| < x\}$; then (S, ν) is called an *E-normed space* (briefly, EN space) with *base* $(\Omega, \mathscr{A}, P)$ and *target* $(V, \|\cdot\|)$. Every EN space (S, ν) is a PPN space under τ_W and τ_M. It is said to be *canonical* if it is a PN space under the same two triangle functions. In this latter case, it is a Šerstnev space. See [5,8] for properties of PN spaces.

If $\nu_p(x)$ is viewed as the probability $P(\|p\| < x)$ that the usual norm of p is less than x, then, the fact that, for some $p \in V$, ν_p does not belong to $\mathscr{D}^+$ means that $P(\|p\| < +\infty) < 1$; this is to be regarded as being "odd". Therefore we shall call *strict* any PN space (V, ν, τ, τ^*) such that $\nu(V) \subseteq \mathscr{D}^+$, or, equivalently, such that ν_p belongs to $\mathscr{D}^+$ for every $p \in V$.

Definition 1.4. A copula is a function $C : [0,1]^2 \to [0,1]$ that satisfies the following conditions:

(C1) for every $t \in [0,1]$, $C(0,t) = C(t,0) = 0$ and $C(1,t) = C(t,1) = t$;
(C2) C is 2-increasing, i.e., for all s, s', t and t' in $[0,1]$, with $s \leqslant s'$ and $t \leqslant t'$,

$$C(s',t') - C(s',t) - C(s,t') + C(s,t) \geqslant 0. \tag{1.2}$$

It follows from the definition that every copula C is increasing in each place.

Moreover for any copula C one has $W \leqslant C \leqslant M$.

2. Invariant and semi-invariant PN spaces

In any PN space (V, ν, τ, τ^*) with regard to the distance d_S, one has, in general, the following:

$$d_S(\nu_{p-q}, \varepsilon_0) \neq d_S(\nu_p, \nu_q).$$

But it would be interesting to know in which cases and under which conditions the inequality $d_S(\nu_{p-q}, \varepsilon_0) \geqslant d_S(\nu_p, \nu_q)$ holds. In this case, the corresponding relationship in functional analysis is

$$\|p - q\| \geqslant |\|p\| - \|q\||.$$

Definition 2.1. The probabilistic normed space (V, ν, τ, τ^*) is said to be *invariant*, if for every $p, q \in V$ we have $d_S(\nu_{p-q}, \varepsilon_0) = d_S(\nu_p, \nu_q)$.

Definition 2.2. The probabilistic normed space (V, ν, τ, τ^*) is said to be *semi-invariant*, if for every $p, q \in V$ we have $d_S(\nu_{p-q}, \varepsilon_0) \geqslant d_S(\nu_p, \nu_q)$.

Example 2.3. The quadruple $(\mathbb{R}, \nu, \tau_\pi, \tau_{\pi^*})$ where $\nu : \mathbb{R} \to \Delta^+$ is defined by

$$\nu_p(x) = \begin{cases} 0, & \text{if } x = 0, \\ \exp\left(-\sqrt{\|p\|}\right), & \text{if } 0 < x < +\infty, \\ 1, & \text{if } x = +\infty. \end{cases}$$

and $\nu_0 = \varepsilon_0$ is a semi-invariant PN space strictly.

Proof. $(\mathbb{R}, \nu, \tau_\pi, \tau_{\pi^*})$ is a PN space of Menger that is not a Sertnev space [1]. By theorem 4.3.3 of [11] one has $d_S(\nu_{p-q}, \varepsilon_0) = \inf\{h \in]0,1[: \nu_{p-q}(h^+) > 1 - h\} = \inf\{h \in]0,1[: exp(-\|p-q\|^{1/2}) > 1 - h\} = 1 - exp(-\|p-q\|^{1/2})$.

Now let us see $d_S(\nu_p, \nu_q)$. Suppose, without lost of generality, $\|p\| \geqslant \|q\|$. Then for every $h \in]0,1[$ and for all $t > 0$, particularly for $t \in]0, 1/h[$, one has

$$\nu_p(t) \leqslant \nu_q(t+h) + h,$$

so that, in our example, the condition $[\nu_q, \nu_p; h]$ clearly holds. The other condition $[\nu_p, \nu_q; h]$ says that

$$h \geqslant exp(-\|q\|^{1/2}) - exp(-\|p\|^{1/2}).$$

Consequently, if $p \neq q$,

$$d_S(\nu_p, \nu_q) = exp(-(\min\{\|q\|, \|p\|\})^{1/2}) - exp(-(\max\{\|q\|, \|p\|\})^{1/2}).$$

holds. And taking into account the following relations

$$\begin{aligned}
\sqrt{\|p\|} &\leqslant \sqrt{\|p-q\| + \|q\|} \leqslant \sqrt{\|p-q\|} + \sqrt{\|q\|} \\
&\Rightarrow exp(-\sqrt{\|p\|}) \geqslant exp(-\sqrt{\|p-q\|}) \cdot exp(-\sqrt{\|q\|}) \\
&\Rightarrow exp(-\sqrt{\|q\|}) - exp(-\sqrt{\|p\|}) \leqslant exp(-\sqrt{\|q\|}) - exp(-\sqrt{\|p-q\|}) \cdot exp(-\sqrt{\|q\|}) \\
&= exp(-\sqrt{\|q\|}) \cdot [1 - exp(-\sqrt{\|p-q\|})] \leqslant 1 - exp(-\sqrt{\|p-q\|}).
\end{aligned}$$

one has finally the strict inequality

$$d_S(\nu_{p-q}, \varepsilon_0) > d_S(\nu_p, \nu_q). \quad \square$$

Lemma 2.4. *If $a, b \in [0, +\infty]$, then the statement*

$$d_S(\varepsilon_a, \varepsilon_b) = min\left\{1, \frac{1}{min\{a,b\}} \cdot |a-b|\right\}$$

holds. Particularly, the cases $b = 0$ and $b = +\infty$ are

$$\begin{aligned}
d_S(\varepsilon_a, \varepsilon_0) &= min\{1, a\} \\
d_S(\varepsilon_a, \varepsilon_\infty) &= min\{1, 1/a\}
\end{aligned}$$

Proof. Based on the definition (1.1) one has

$$d_S(\varepsilon_a, \varepsilon_b) = \inf\{h \in]0,1] : \text{both} \quad [\varepsilon_a, \varepsilon_b, h] \quad \text{and} \quad [\varepsilon_b, \varepsilon_a, h], \quad \text{hold}\}.$$

Let us recall that for every $x \in]0, \frac{1}{h}[$ the relation $[\varepsilon_a, \varepsilon_b, h] \iff \varepsilon_b(x) \leqslant \varepsilon_a(x+h) + h$ holds.

Consequently

$$d_S(\varepsilon_a, \varepsilon_b) = \inf\{h \in]0,1] : \varepsilon_{min\{a,b\}}(x) \leqslant \varepsilon_{max\{a,b\}}(x+h) + h, x \in]0, 1/h[\}. \tag{2.1}$$

The inequality on the right hand of (2.1) occurs in the following cases:

(a) $h \geqslant 1$,
(b) $\frac{1}{h} \leqslant min\{a,b\}$, or equivalently $h \geqslant \frac{1}{min\{a,b\}}$
(c) For every $x \in]min\{a,b\}, 1/h[$ one has

$$\begin{aligned}
&1 \leqslant \varepsilon_{max\{a,b\}}(x+h) + h \quad \text{and} \quad h < \tfrac{1}{min\{a,b\}} \\
&\iff h < min\left\{1, \tfrac{1}{min\{a,b\}}\right\} \quad \text{with} \quad x + h > max\{a,b\} \quad \text{for all} \quad x \in]min\{a,b\}, 1/h[\\
&\iff h < min\left\{1, \tfrac{1}{min\{a,b\}}\right\} \quad \text{with} \quad h \geqslant max\{a,b\} - min\{a,b\} = |a-b|
\end{aligned}$$

for all $x \in]min\{a,b\}, 1/h[$. $\square$

Theorem 2.5. *Let (V, ν, τ, τ^*) be a PN space where V is a linear space and $\nu_p = \varepsilon_{\varphi(p)}$ and $\varphi : V \to \mathbb{R}$ is a positive function such that for every $p, q \in V$: $\varphi(q) - \varphi(p) \leqslant \varphi(q-p)$. Then (V, ν, τ, τ^*) is a semi-invariant PN space.*

Proof. It is enough we prove the below relation

$$d_S(\varepsilon_{\varphi(q)}, \varepsilon_{\varphi(p)}) \leqslant d_S(\varepsilon_{\varphi(q-p)}, \varepsilon_0). \tag{2.2}$$

Let p, q be in V. Then, from Lemma 2.4, one has:

$$\begin{aligned} d_S(\nu_{p-q}, \varepsilon_0) &= d_S(\varepsilon_{\varphi(p-q)}, \varepsilon_0) = min\{1, \varphi(p-q)\} \geqslant min\{1, |\varphi(p) - \varphi(q)|, 1/min\{\varphi(p), \varphi(q)\}\} = d_S(\varepsilon_{\varphi(p)}, \varepsilon_{\varphi(q)}) \\ &= d_S(\nu_p, \nu_q). \quad \square \end{aligned}$$

Example 2.6. The quadruple (V, ν, τ_M, Π_M) where $(V, \|\cdot\|)$ is a classical normed space and $\nu_p = \varepsilon_{\varphi(p)}$ with $\varphi(p) = \frac{\|p\|}{1-\|p\|}$, is a semi-invariant PN space.

Proof. $(V, \nu, \tau_M, \mathbf{M})$ is a PN space (see [6]) which is not a Šerstnev space. From Lemma 2.4 $d_S(\varepsilon_{\varphi(p-q)}, \varepsilon_0) = \frac{\|p-q\|}{1+\|p-q\|}$. On the other hand the function $\varphi(p) = \frac{\|p\|}{1+\|p\|}$ satisfies the condition given in Theorem 2.5

$$\begin{aligned} \varphi(q) - \varphi(p) &= \frac{\|q\|}{1+\|q\|} - \frac{\|p\|}{1+\|p\|} \\ &= \frac{\|q\| - \|p\|}{1+\|q\|+\|p\|+\|p\|\|q\|} \leqslant \frac{\|q\| - \|p\|}{1+\|q\|+\|p\|} \leqslant \frac{\|q-p\|}{1+\|q-p\|} = \varphi(q-p). \end{aligned}$$

and the proof is complete. □

It is possible to give sufficient conditions under which the inequality

$$d_S(\nu_{p-q}, \varepsilon_0) \geqslant d_S(\nu_p, \nu_q)$$

holds, as in the following theorem but we need before a lemma:

Lemma 2.7. *According with some considerations in [9] one has:*
Let F_i, G_i (i=1,2) be distance distribution functions in Δ^+. Let A_i, $i = 1, 2$ be the set defined via

$$A_i := \{h \in]0, 1] : both \quad [F_i, G_i; h] \quad and \quad [G_i, F_i; h] \quad hold\}.$$

Then,

$$\begin{aligned} &d_S(F_1, G_1) < d_S(F_2, G_2) \quad \text{if and only if} \quad A_1 \supset A_2 \\ &d_S(F_1, G_1) = d_S(F_2, G_2) \quad \text{if and only if} \quad A_1 = A_2. \end{aligned}$$

Proof. Let us recall that $d_S(F_i, G_i) := \inf A_i$, $(i = 1, 2)$. By the definition of A_i, if $h_0 \in A_i$, then $]h_0, 1] \subset A_i \subset]0, 1]$; from Lemma 1.1, if $h = \inf A_i > 0$, then

$$[F_i, G_i; h] \quad \text{and} \quad [G_i, F_i; h].$$

hold.

Moreover

$$A_i = \begin{cases}]0, 1] & \text{if } d_S(F_i, G_i) = 0, \\ [a_i, 1] & \text{if } d_S(F_i, G_i) = a_i. \end{cases} \quad \square$$

Theorem 2.8. *Let C be an associative copula and (V, ν, τ_C, τ^*) a PN space. Then (V, ν, τ_C, τ^*) is semi-invariant.*

Proof. Since C is an associative copula, then C is a continuous t-norm and as a consequence one can say that τ_C is a continuous triangle function.

When calculating $d_S(\nu_{p-q}, \varepsilon_0)$ notice that the condition $[\varepsilon_0, \nu_{p-q}; h]$ is always right. The other one, $[\nu_{p-q}, \varepsilon_0; h]$, tell us that given $h \in]0, 1[$ and for all $x \in]0, 1/h[$, one has the relation $1 \leqslant \nu_{p-q}(x+h) + h$ holds. Following Lemma 2.7 assume that $F_1 = \nu_p$, $G_1 = \nu_q$ and $F_2 = \nu_{p-q}$, $G_2 = \varepsilon_0$. We have to prove that given $h \in]0, 1[$ and for all $x \in]0, 1/h[$, if

$$1 - h \leqslant \nu_{p-q}(x+h), \tag{2.3}$$

then

$$\nu_p(x) \leqslant \nu_q(x+h) + h \quad \text{and} \quad \nu_q(x) \leqslant \nu_p(x+h) + h.$$

hold. We only prove the second inequality because the other one is symmetrical: it suffices to interchange p, q.

For every $x \in]0, 1/h[$ and applying (2.3) one has

$$\nu_p(x+h)+h \geqslant \tau_C(\nu_{p-q},\nu_q)(x+h)+h = \sup\{C(\nu_{p-q}(u),\nu_q(v)) : u+v = x+h\}+h \geqslant \sup\{C(\nu_{p-q}(u),\nu_q(v)) : u+v = x+h, v\langle x,u\rangle h\}+h \geqslant \sup\{C(1-h,\nu_q(v)) : v<x\}+h = C(1-h,\nu_q(x))+h.$$

Moreover, since C is a copula, it follows from (1.2) that

$$C(1,1)-C(1,\nu_q(x))-C(1-h,1)+C(1-h,\nu_q(x)) = 1-\nu_q(x)-(1-h)+C(1-h,\nu_q(x)) \geqslant 0,$$

so that

$$C(1-h,\nu_q(x))+h \geqslant \nu_q(x). \quad \square$$

As a consequence of this theorem, if (V,ν,τ_T,τ^*) is a PN space such that $\tau_T \geqslant \tau_C$, with C an associative copula, then (V,ν,τ_T,τ^*) is semi-invariant.

Corollary 2.9.

(a) Every equilateral space (V,F,Π_M) is semi-invariant.
(b) Every simple PN space $(V,\|\cdot\|,G,M)$ is semi-invariant.
(c) Every EN space (S,ν) is semi-invariant.

Before introducing a new class of PN spaces we need the following technical lemma from [7].

Lemma 2.10. *Let $f:[0,+\infty]\to[0,1]$ be a right-continuous non-increasing function. Let we define $f^{[-1]}(1):=0$ and $f^{[-1]}(y):=\sup\{x:f(x)>y\}$ for all $y\in[0,1[$ ($f^{[-1]}(y)$ might be infinite). If $x_0\in[0,+\infty]$ and $y_0\in[0,1]$, then the following facts are equivalent: (a) $f(x_0)>y_0$; (b) $x_0<f^{[-1]}(y_0)$.*

Proof. If $f(x_0)>y_0$ then $f^{[-1]}(y_0)=\sup\{x:f(x)>y_0\}\geqslant x_0$. If we suppose that $\sup\{x:f(x)>y_0\}=x_0$, then $f(x)\leqslant y_0$ for every $x>x_0$. Thus $f(x_0)=f(x_0+)\leqslant y_0$, against the assumption; whence (a) $\Rightarrow$ (b). The converse result is an immediate consequence of the monotonicity of f. $\square$

The following theorem (see [7]) introduces a new class of PN spaces – which generalizes an example in [9] – and also provides some properties of the spaces in that class. As has been said above, such properties are interesting in order for the purposes of this paper.

Theorem 2.11. *Let $(V,\|\cdot\|)$ a normed space and let T be a continuous t-norm. Let f be a function as in Lemma 2.9, and satisfying the following two properties:*

(a) $f(x)=1$ if and only if $x=0$;
(b) $f(\|p+q\|)\geqslant T(f(\|p\|),f(\|q\|))$ for every $p,q\in V$.

If $\nu:V\to\Delta^+$ is given by

$$\nu_p(x)=\begin{cases}0, & x\leqslant 0,\\ f(\|p\|), & x\in]0,+\infty[,\\ 1, & x=+\infty,\end{cases} \tag{2.4}$$

for every $p\in V$, then (V,ν,τ_T,τ_M) is a Menger PN space satisfying the following properties:

(F1) (V,ν,τ_T,τ_M) is a TV space;
(F2) (V,ν,τ_T,τ_M) is normable;
(F3) If $p\in V$ and $t>0$, then the strong neighborhood $N_p(t)$ in (V,ν,τ_T,τ_M) is not $\mathcal{D}$-bounded, but $N_p(t)$ is topologically bounded whenever $N_p(t)\neq V$;
(F4) (V,ν,τ_T,τ_M) is not a Šerstev space;
(F5) (V,ν,τ_T,τ_M) is not a characteristic PN space.

Now we consider some special cases and use the preceding theorem in order to give some examples.

Example 2.12. Suppose that, in Theorem 2.11, $T=\Pi$. Then, property (b) reads $f(\|p+q\|)\geqslant f(\|p\|)f(\|q\|)$ for all $p,q\in V$. It is not difficult to prove that, under the given assumptions on f, property (b) is equivalent to the following one:

$$f(x+y)\geqslant f(x)f(y), \quad \text{for all } x,y\in]0,\infty[. \tag{2.5}$$

The following are examples of functions f satisfying the assumptions of Theorem 2.11:

$$f_{\alpha,\beta}(x) := 1 - \frac{\beta}{\alpha} + \frac{\beta}{x+\alpha}, \quad 0 \leqslant \beta \leqslant \alpha,$$
$$g_{\alpha,\beta}(x) := 1 - \alpha + \alpha\exp(-x^{\beta}), \quad 0 < \alpha \leqslant 1, \ \beta > 0.$$

Example 2.13. Take $T = W$ in Theorem. In this case property (b) reads

$$f(\|p+q\|) \geqslant f(\|p\|) + f(\|q\|) - 1 \qquad \text{for all } p, q \in V.$$

Since W is the smallest continuous t-norm, all the functions f satisfying the assumptions of Theorem 2.11 with respect to any t-norm T also satisfy such assumptions with respect to W. It is not hard to prove that, under those assumptions, property (b) is equivalent to the following one:

$$1 + f(x+y) \geqslant f(x) + f(y) \qquad \text{for all } x, y \in]0, \infty[.$$

For instance, the following functions satisfy this property but not that considered in Example 2.12, since they do not satisfy (2.5):

$$h_{\alpha,\beta}(x) := \begin{cases} 1 - \alpha x, & 0 \leqslant x \leqslant \beta, \\ 1 - \alpha\beta, & x > \beta, \end{cases} \quad 0 < \beta \leqslant 1/\alpha.$$

Theorem 2.14. *Every PN space belonging to the class considered in Theorem 2.11 is semi-invariant.*

Proof. Since $d_S(\nu_{p-q}, \varepsilon_0) = \inf\{h \in]0,1[: \nu_{p-q}(h^+) > 1-h\}$ one has $f(\|p-q\|) > 1 - h \Rightarrow h > 1 - f(\|p-q\|)$, and it follows:

$$d_S(\nu_{p-q}, \varepsilon_0) = 1 - f(\|p-q\|).$$

By the other hand, according with (1.1),

$$d_S(\nu_p, \nu_q) = \inf\{h \in]0,1[: \text{ both } \ [\nu_q, \nu_p; h] \ \text{ and } \ [\nu_p, \nu_q; h] \text{ hold}\}.$$

Suppose, without lost of generality, $\|p\| \geqslant \|q\|$, then $[\nu_q, \nu_p; h]$ is equivalent to $f(\|p\|) \leqslant f(\|q\|) + h$. In fact this inequality is strict. Moreover, from $[\nu_p, \nu_q; h]$ one has $h \geqslant f(\|q\|) - f(\|p\|)$, hence

$$d_S(\nu_p, \nu_q) = f(\|q\|) - f(\|p\|).$$

Now we need to investigate under which particular conditions one has for the PN spaces considered in Theorem 2.11 the inequality

$$1 - f(\|p-q\|) \geqslant f(\|q\|) - f(\|p\|). \tag{2.6}$$

If one chooses f among the type's functions $f_{\alpha,\beta}$, then it is not difficult to check that

$$\frac{\|p\| - \|q\|}{\|p\|\|q\| + (\|p\| + \|q\|)\alpha + \alpha^2} \leqslant \frac{\|p-q\|}{\alpha^2 + \|p-q\|\alpha}$$

and the inequality $d_S(\nu_{p-q}, \varepsilon_0) \geqslant d_S(\nu_p, \nu_q)$ holds. And if one chooses f among the type's functions $g_{\alpha,\beta}$, then the PN spaces of Theorem 2.11 are always semi-invariant strictly: since

$$d_S(\nu_{p-q}, \varepsilon_0) = 1 - g_{\alpha,\beta}(\|p-q\|) = \alpha - \alpha^{-\|p-q\|^{\beta}},$$

and

$$d_S(\nu_p, \nu_q) = g_{\alpha,\beta}(\|q\|) - g_{\alpha,\beta}(\|p\|) = \alpha e^{-\|q\|^{\beta}} - \alpha e^{-\|p\|^{\beta}},$$

one has only to check the inequality $1 - e^{-\|p-q\|^{\beta}-1} \geqslant e^{-\|q\|^{\beta}}(1 - e^{-(\|p\|^{\beta} - \|q\|^{\beta})})$, which is equivalent to check on the inequality $1 - e^{-\|p-q\|^{\beta}} \geqslant 1 - e^{-(\|p\|^{\beta} - \|q\|^{\beta})}$, and this is true because of the well known inequality $(1-s)^{\beta} \geqslant 1 - s^{\beta}$. In fact, the inequality is strict and the verification is complete. Finally, if one chooses f among the type's functions $h_{\alpha,\beta}$ the PN spaces considered in Theorem 2.11 are also semi-invariant. □

Definition 2.15 [14]. A topological space is called *normal space*, if any two disjoint closed subset of it can be separated by open sets.

Theorem 2.16. *Every semi-invariant PN space (V, ν, τ, τ^*) is normal.*

Proof. If A, B are closed subset of V we should construct two disjoint open subset U, W of V such that $A \subset U$ and $B \subset W$.

Let $U := \{p \in V; \inf_{a \in A} d_S(\nu_{p-a}, \varepsilon_0) < \inf_{b \in B} d_S(\nu_{p-b}, \varepsilon_0)\}$. If $p \in A$ then $\inf_{a \in A} d_S(\nu_{p-a}, \varepsilon_0) = 0$ and hence $A \subset U$. Now we prove that U is open.

Suppose $p \in U$, $\alpha_p = \inf d_S(\nu_{p-a}, \varepsilon_0)$ and $\beta_p = \inf d_S(\nu_{p-b}, \varepsilon_0)$. Therefore $\alpha_p < \beta_p$.

For every $\epsilon > 0$ there exist $a_0 \in A$ such that $d_S(\nu_{p-a_0}, \varepsilon_0) < \alpha_p + \epsilon$.

For every $p \in U$ we will show that $N_p\left(\frac{\beta_p-\alpha_p}{2}\right) \subset U$. Let $q \in N_p\left(\frac{\beta_p-\alpha_p}{2}\right)$ then $d_S(v_{q-p}, \varepsilon_0) < \frac{\beta_p-\alpha_p}{2}$ and

$$\inf d_S(v_{q-a}, \varepsilon_0) \leqslant \inf d_S(v_{q-a}, v_{q-p}) + d_S(v_{q-p}, \varepsilon_0) \leqslant \inf d_S(v_{q-a}, v_{q-p}) + \frac{\beta_p - \alpha_p}{2}.$$

The space (V, v, τ, τ^*) is a semi-invariant and hence

$$\inf d_S(v_{q-a}, v_{q-p}) \leqslant \inf d_S(v_{p-a}, \varepsilon_0) = \alpha_p.$$

Therefore $d_S(v_{q-a}, \varepsilon_0) \leqslant \alpha_p + \frac{\beta_p-\alpha_p}{2} = \frac{\beta_p+\alpha_p}{2} < \beta_p$ and $q \in U$.

Similarly if $W := \{p \in V : \inf d_S(v_{p-a}, \varepsilon_0) > \inf d_S(v_{p-b}, \varepsilon_0)\}$ then W is open subset of V such that $B \subset W$ and the construction of U and W shows that $U \cap W = \emptyset$. □

Immediately the next corollaries come from Theorem 2.8.

Corollary 2.17. Urysohn's Lemma for PN space*Any two disjoint subset of every semi-invariant PN space can be separated by a continuous function.*

Corollary 2.18 (*The Tietze extension Theorem for PN space*). *If A is any closed subset of the semi-invariant PN space (V, v, τ, τ^*) and $f \in C(A, [a, b])$ then there exists $F \in C(V, [a, b])$ such that $F \mid A = f$.*

Corollary 2.19. *If A is any closed subset of the semi-invariant PN space (V, v, τ, τ^*) and $f \in C(A)$ then there exists $F \in C(V)$ such that $F \mid A = f$.*

Open problems. One needs several previous results.

Definition 2.20 [11]. Let $\mathscr{L}$ be the set of all binary operations, L, on $\mathbb{R}^+$ and $Ran(L) = \mathbb{R}^+$ satisfying the following conditions:

(i) L is non-decreasing in each place,
(ii) L is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$ except at the most in $(0, \infty)$ and $(\infty, 0)$.

Theorem 2.21 [11]. *Let T be a left-continuous t-norm and let $L \in \mathscr{L}$ satisfying the following conditions:*

(i) L is commutative.
(ii) L is associative.
(iii) $u_1 < u_2$ and $v_1 < v_2$ imply $L(u_1, v_1) < L(u_2, v_2)$.
(iv) $L(x, 0) = x$, then the function $\tau_{T,L}$ is a triangle function.

The condition (*iv*) of the Theorem 2.21 implies that $L \geqslant Max$, and then $\tau_{T,L} \leqslant \prod_T$.

Corollary 2.22. *Let $L \in \mathscr{L}$ satisfying the hypothesis of Theorem 2.21. Then the following statements hold*

(i) $L \geqslant Max$ (trivial and known)
(ii) (see [9]) $L = Max \iff L(a, a) = a$ for every $a \in (0, \infty)$.

Theorem 2.23. *Let T be an Archimedean t-norm and assume that $L \in \mathscr{L}$ satisfies the hypothesis of the Theorem 2.21. Then the following statements are equivalent:*

(i) $\tau_{T,L}$ is a triangle function no having non-trivial idempotents in Δ^+.
(ii) L satisfies the condition $L(x, x) > x$ for every $x \in (0, \infty)$.

Example 2.24. This theorem provides us with many examples of Archimedean triangle functions: it suffices to have $L = K_\alpha(x, y) = (x^\alpha + y^\alpha)^{1/\alpha}$ defined for every $x, y \in \mathbb{R}^+$ and for $\alpha \geqslant 1$.

Let us recall the definition of τ_{T,K_α}:

$$\tau_{T,K_\alpha} = \sup\{T(F(u), G(v)) : u^\alpha + v^\alpha = x^\alpha\}$$

Applying the previous example, and by Theorem 7.2.12 in [11], one has: $K_2 < K_1$ and $T_1 \leqslant T_2$ implies that $\tau_{T_1,K_1} \leqslant \tau_{T_2,K_2}$.

In analogous way, as $K_4 \leqslant K_2$, the inequality $\tau_{T_1,K_2} \leqslant \tau_{T_2,K_4}$.

We are ready to propose the following open problems all of which need from the computation sciences.

(1) Is any α-simple space invariant or semi-invariant? If not, are there any conditions under which it is? Also, if not, why not?

(2) In the following examples and for the moment, we do not know the answer to the same questions mentioned in (1).

(a) The quadruple $(V, \nu, \star, \tau^*)$ where V is a normed linear space $\tau^* = \Pi_M$, and ν, the probabilistic norm, is a map $\nu : V \to \Delta^+$ defined via $\nu_p(x) := \frac{x}{x+1} e^{-\|p\|}$, and $\star$ the triangle function convolution, is a PN space; it is neither a Šerstnev space nor a topological vector space.

(b) The quadruple (V, ν, Π_Π, Π_M) where V is a normed linear space and ν, the probabilistic norm, is a map $\nu : V \to \Delta^+$ defined via $\nu_p(x) := e^{-\frac{\|p\|}{x}}$, is a PN space and a topological vector (in short, TV) space.

(c) Let (V, ν, τ, τ^*) a PN space and let τ^* be non-Archimedean triangle function. Let f, g be functions satisfying the following conditions:

(i) f is a continuous non-increasing from $[0, +\infty]$ into $[0, 1]$ and $f(0) = 1$.

(ii) g is a function from $\mathbb{R}^+ \times \mathbb{R}^+$ into $[0, 1]$, continuous in the first and in the second places, non-decreasing in the first place and non-increasing in the second place with $g(x, 0) = 1$. Then the quadruple (V, ν, τ, τ^*) where the probabilistic norm ν is defined via

$$\nu_p(x) = f(\|p\|) \cdot g(x, \|p\|),$$

is a topological vector space.

(d) With the same assumptions of the previous example one has that the quadruple (V, ν, Π_Π, Π_M) where

$$\nu_p(x) = \frac{1}{1 + \|p\|} \cdot \frac{x}{x + \|p\|}$$

is a PN space that is a TV space and it is not a PN space of Šerstnev. (V, ν, Π_Π, Π_M) is not a strict PN space, and it is normable.

3. Application in physics

Menger sponge is a random space which could be used for instance to predict the Background microwave radiation (see El Naschie and also He's Book [4]).

4. Conclusions

In this work, we have analyzed some detail of semi-invariance for some class of PN spaces. We have shown that PN spaces are normal spaces. A detailed study of how we can have the Urysohn's Lemma and Tietze extension Theorem for PN spaces is given.

Acknowledgement

The Second author was supported by grants from Ministerio de Ciencia y Tecnologa (MTM2006-12218) of Spain.

References

[1] Alsina C, Schweizer B, Sklar A. On the definition of a probabilistic normed space. Aequationes Math 1993;46:91–8.
[2] El Naschie MS. On the uncertainty of Cantorian geometry and the two slit experiment. Chaos, Solitons & Fractals 1998;9:517–29.
[3] El Naschie MS. On the unification heterotic strings theory and ε^{∞} theory. Chaos, Solitons & Fractals 2000;11:397–407.
[4] Ji-Huan He, El Naschie MS. Transfinite physics. Beijing: China Education and Culture publishing Co.; 2006.
[5] Lafuerza-Guillén B, Rodríguez Lallena JA, Sempi C. Some classes of probabilistic normed spaces. Rend Mat 1997;17:237–52.
[6] Lafuerza-Guillén B. Latest results on probabilistic normed spaces. Int J Math Anal 2006;3:269–309.
[7] Lafuerza-Guillén B, Rodríguez Lallena JA, Sempi C. Normability of probabilistic normed spaces, Note di Matematica, in press.
[8] Lafuerza-Guillén B, Rodríguez Lallena JA, Sempi C. A study of boundedness in probabilistic normed spaces. J Math Anal Appl 1999;232:183–96.
[9] Lafuerza-Guillén B. Primeros resultados en el estudio de los espacios normados probabilísticos con nuevos conceptos de acotación, Ph.D. Thesis, Universidad de Almería, Spain; 1996.
[10] Saminger S, Sempi C. A primer on triangle functions I. Aequationes Math 2008;76:201–40.
[11] Schweizer B, Sklar A. Probabilistic metric spaces. New York: Dover Publication, Inc., Mineola; 2005.
[12] Šerstnev AN. On the motion of a random normed space. Dokl Akad Nauk SSSR 1963;149:280–3 [English translation in Soviet Math Dokl 1963;4:388–90].
[13] Sibley DA. A metric for weak convergence of distribution functions. Rocky Mountain J Math 1971;1:427–30.
[14] Wilansky A. Functional analysis. New York, Toronto, London: Blaisdell Publishing Company; 1964.