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A common fixed point for operators in probabilistic normed spaces

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Abstract

Probabilistic Metric spaces was introduced by Karl Menger. Alsina, Schweizer and Sklar gave a general definition of probabilistic normed space based on the definition of Menger [Alsina C, Schweizer B, Sklar A. On the definition of a probabilistic normed spaces. Aequationes Math 1993;46:91–8]. Here, we consider the equicontinuity of a class of linear operators in probabilistic normed spaces and finally, a common fixed point theorem is proved. Application to quantum Mechanic is considered.

1. Introduction

The theory of probabilistic metric spaces, introduced in 1942 by Meneger [10], was developed by numerous authors, for instance [4], as well as those in [12,13]. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of quantum physics as shown by El Naschie. The notion of a probabilistic normed space was introduced by Šerstnev [14]. Alsina, Schweizer and Sklar gave a general definition of probabilistic normed space based on the definition of Menger [13] for probabilistic metric spaces in [3,2]. Linear operator in probabilistic normed spaces has been studied by Guillen, Lallena and Sempi in [8,9]. In this paper, we consider the equicontinuity of a class of linear operators on probabilistic normed space and by applying these results, a common fixed point theorem is proved. In order to do this, we recall some definitions from [1–3,6].

Definition 1.1. A distribution function is a function $F:\overline{\mathbb{R}}\to[0,1]$ that is nondecreasing and left continuous on $\mathbb{R}$; moreover, $F(-\infty)=0$ and $F(\infty)=1$.

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The set of all the distribution functions is denoted by Δ and the set of those distribution functions such that $F(0) = 0$ is denoted by Δ^+. The distance distribution functions are denoted by D^+ and $D^+ = \{F \in \Delta^+ : \lim_{x\to\infty} F(x) = 1\}$. A natural ordering in Δ and Δ^+ is defined by $F \leqslant G$ whenever $F(x) \leqslant G(x)$ for every $x \in \mathbb{R}$. The maximal element in this order for Δ^+ is ϵ_0, where for $a \leqslant \infty$ the distribution function ϵ_a is defined by

$$\epsilon_a = \begin{cases} 0, & \text{if } t \leqslant a, \\ 1, & \text{if } t > a. \end{cases}$$

Definition 1.2. A *triangle function* is a binary operation on Δ^+ that is commutative, associative, nondecreasing in each place, and has ϵ_0 as identity.

Note that the continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+.

Example 1.3. Let T be a continuous t-norm, i.e. a continuous binary operation in $[0,1]$ that is associative, nondecreasing and has 1 as identity; T^* is a continuous t-conorm, namely a continuous binary operation on $[0,1]$ that is related to a continuous t-norm through

$$T^*(x,y) = 1 - T(1-x, 1-y).$$

Typical continuous triangle functions are convolution, the operations τ_T and τ_{T^*}, which are given by

$$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s), G(t))$$

and

$$\tau_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s), G(t))$$

for all F, G in Δ^+ and all x in $\mathbb{R}$ [13, Sections 7.2 and 7.3], respectively.

It follows without difficulty from the above that:

$$\tau_T(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b} = \tau_{T^*}(\varepsilon_a, \varepsilon_b)$$

for any continuous t-norm T, any continuous t-conorm T^* and any $a, b \geqslant 0$.

Definition 1.4. A *probabilistic normed space* (briefly, PN-space) is a quadruple $(X, \Theta, \tau, \tau^*)$, where X is a real vector space, τ and τ^* are continuous triangle functions, and Θ is a mapping from X into Δ^+ such that, for $p, q \in X$, the following conditions hold:

(PN1) $\Theta_p = \epsilon_0$ iff $p = 0$ where 0 is the null vector in X,
(PN2) $\Theta_{-p} = \Theta_p$,
(PN3) $\Theta_{p+q} \geqslant \tau(\Theta_p, \Theta_q)$,
(PN4) $\Theta_p \leqslant \tau^*(\Theta_{\alpha p}, \Theta_{(1-\alpha)p})$, for all $\alpha \in I = [0,1]$.

If $(X, \|\cdot\|)$ is a real normed space, τ a triangle function such that $\tau(\epsilon_a, \epsilon_b) \leqslant \epsilon_{a+b}$ for all $a, b \geqslant 0$, and if $\Theta : X \to \Delta^+$ is defined via $\Theta_p = \epsilon_{\|p\|}$, then $(X, \Theta, \tau, \tau^*)$ is a PN-space.

Note that if $\tau^* = \tau_M$ (where τ_M is the t-norm defined as $\tau_M(x,y) = \mathrm{Min}\{x,y\}$) and equality holds in (PN4), then $(S, \Theta, \tau, \tau_M)$ is a Šerstnev PN-space [1]. In this case, as shown in [3], the conditions

$$\Theta_p = \tau_M(\Theta_{\alpha p}, \Theta_{(1-\alpha)p}) \text{ for all } p \text{ in } X \text{ and } \alpha \text{ in } I$$

and (PN2), taken together, are equivalent to Šerstnev's condition

$$\Theta_{\lambda p}(x) = \Theta_p\left(\frac{x}{|\lambda|}\right) \text{ for all } \lambda \in \mathbb{R} - \{0\} \text{ and } x \text{ in } \mathbb{R}.$$

Every PN-space $(V, \Theta, \tau, \tau^*)$ can be endowed with the strong topology; this topology is generated by the strong neighborhood, which are defined as follows: for every $t > 0$, the neighborhood $N_p(t)$ at a point p of V is defined by

$$N_p(t) := \{q \in V : d_S(\Theta_{p-q}, \varepsilon_0) < t\} = \{q \in V : \Theta_{p-q}(t) > 1 - t\}.$$

Definition 1.5. A topological vector space (TVS) is a vector space V together with a topology such that with respect to this topology:

(i) The map of $V \times V \to V$ defined by $(x, Y) \mapsto x + y$ is continuous.
(ii) The map of $\mathbb{F} \times X \to X$ defined by $(\alpha, x) \mapsto \alpha x$ is continuous. Where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.6. A topological vector space X is called locally convex if the neighborhood filter around 0 has a basis of convex sets.

Note that every PN-space $(V, \Theta, \tau, \tau^*)$, when it is endowed with the strong topology induced by the probabilistic norm Θ is a topological vector space if, and only if, for every $p \in V$ the map from $\mathbb{R}$ into V defined by $\alpha \to \alpha p$ is continuous (see [2]) for more details). Henceforth a PN-space $(V, \Theta, \tau, \tau^*)$ which is a topological vector space is denoted by TV PN-space.

Remark 1.7. It was proved [3, Theorem 4] that if the triangle function τ^* is Archimedean, i.e. if τ^* admits no idempotents other than ε_0 and ε_∞ [13] then the mapping $\alpha \mapsto \alpha p$ is continuous and as a consequence of this, the PN-space (V, v, τ, τ^*) is a TV PN-space.

Definition 1.8. A PN-space (V, v, τ, τ^*) is characteristic if $v(V) \subseteq \mathcal{D}^+$, or equivalently $v_p \in \mathcal{D}^+$ for every $p \in V$.

Theorem 1.9. *A characteristic Šerstnev space (V, v, τ) with $\tau = \tau_M$ is locally convex.*

Proof. ([11]). We prove it here because the notation in Prochaska's thesis is different from the one that has become usual after the publication of [12].

It suffices to consider the family of neighborhoods of the origin θ, $N_\theta(t)$, with $t > 0$. Let $t > 0$, $p, q \in N_\theta(t)$ and $\alpha \in [0, 1]$. Then

$$\begin{aligned} v_{\alpha p+(1-\alpha)q}(t) &\geqslant \tau_M(v_{\alpha p}, v_{(1-\alpha)q})(t) = \sup M(v_{\alpha p}(\beta t), v_{(1-\alpha)q}((1-\beta)t) : \beta \in [0,1]) \geqslant M(v_{\alpha p}(\alpha t), v_{(1-\alpha)q}((1-\alpha)t)) \\ &= M(v_p(t), v_q(t)) > 1 - t. \end{aligned}$$

Thus $\alpha p + (1-\alpha)q$ belongs to $N_\theta(t)$ for every $\alpha \in [0, 1]$. □

Theorem 1.10. *Let (V, v, τ, τ^*) be a TV PN-space. If $A : X \to X$ is linear and continuous at 0, then A is continuous.*

A linear operator $T : V_1 \to V_2$ where $(V_1, v_1, \tau_1, \tau_1^*)$ and $(V_2, v_2, \tau_2, \tau_2^*)$ are TV PN-spaces, is bounded if it transform bounded subset of V_1 into bounded subset of V_2. Note that continuous linear operators are bounded.

Remark 1.11. A linear operator between two locally convex TV PN-spaces is continuous if and only if it is bounded, see [5, p. 477].

2. Common fixed point

In this section, we prove some theorems and as a result of these theorems, one can prove the existence of a common fixed point theorem. Due to this, the next result is a uniform boundedness theorem for TV-PN spaces.

Theorem 2.1. *If Γ is a collection of continuous linear maps between two TV PN-spaces $(V_1, v_1, \tau_1, \tau_1^*)$ and $(V_2, v_2, \tau_2, \tau_2^*)$ and if the set*

$$\Gamma(x) = \{Ax; A \in \Gamma\}$$

is a bounded subset of V_2 for every $x \in V_1$, then Γ is equicontinuous.

Proof. Let $N_0(t) = \{p \in V_1 : v_p(t) > 1 - t\}$ be a neighborhood of 0, then

$$\overline{N_0\left(\frac{1}{3}t\right)} = \overline{\left\{p : v_p(t) > 1 - \frac{t}{3}\right\}} = \left\{p \in V_1 : d_s(v_p, \varepsilon_0) \leqslant \frac{1}{3}\right\}$$

and we have

$$\overline{N_0\left(\frac{1}{3}t\right)}+\overline{N_0\left(\frac{1}{3}t\right)}\subset\{p\in V_1:d_s(v_p,\varepsilon_0)\leqslant 1\}.$$

Put

$$E=\bigcap\nolimits_{A\in\Gamma}A^{-1}\left(N_0\left(\frac{1}{3}t\right)\right).$$

V_1 is a complete metric space and $V_1\subset\bigcup_{n=1}^{\infty}nE$. Therefore E is a closed subset of V_1 also the interior of E is not empty. Hence $x-E$ contains a neighborhood $N_0(s)$ of 0 such that

$$A(N_0(s))\subset A(x)-A(E)\subset\overline{N_0\left(\frac{1}{3}t\right)}+\overline{N_0\left(\frac{1}{3}t\right)}$$

for every $A\in\Gamma$. This proves Γ is equicontinuous. □

Corollary 2.2. *If Γ is a collection of continuous linear maps from PN-space $(V_1,v_1,\tau_1,\tau_1^*)$ onto PN-space $(V_2,v_2,\tau_2,\tau_2^*)$, where τ_1^* and τ_2^* are Archimedean and $\Gamma(x)=\{Ax;\ A\in\Gamma\}$ is a bounded subset of V_2 for every $x\in V_1$, then Γ is equicontinuous.*

Proof. By Remark 1.7 the PN-spaces $(V_1,v_1,\tau_1,\tau_1^*)$ and $(V_2,v_2,\tau_2,\tau_2^*)$ are TV PN-spaces. Now the result follows from Theorem 2.1. □

Corollary 2.3. *If $\Gamma=\{A:V_1\to V_2\}$ is a collection of continuous linear maps, where $(V_1,v_1,\tau_1,\tau_1^*)$ and $(V_2,v_2,\tau_2,\tau_2^*)$ are characteristic Šerstnev spaces with $\tau_1=\tau_2=\tau_M$ and if $\Gamma(x)=\{Ax;A\in\Gamma\}$ is a bounded subset of V_2 for every $x\in V_1$, then Γ is equicontinuous.*

Proof. Note that V_1 and V_2 are locally convex spaces by [8, Theorem 7]. Now the result is immediate consequence of Theorem 2.1. □

Theorem 2.4. *If $\{A_n\}$ is a sequence of continuous linear mapping from a TV PN-space $(V_1,v_1,\tau_1,\tau_1^*)$ into a TV PN-space $(V_2,v_2,\tau_2,\tau_2^*)$ and if $Ax=\lim_{n\to\infty}A_nx$, exist for every $x\in V_1$, then A is continuous.*

Proof. Theorem 2.1 implies that $\{A_n\}$ is equicontinuous. Suppose U_2 is a neighborhood of 0 in V_2, then $A_n(U_1)=U_2$ for all $n\in\mathbb{N}$ and some neighborhood U_1 of 0 in V_1. It follows that $A(U_1)\subseteq\overline{U_2}$; hence A is continuous. □

Definition 2.5. A linear operator T of V_1 into V_2 is called bounded if it transform bounded subset of V_1 into bounded subset of V_2 [5, p. 63].

Corollary 2.6. *If $\{A_n\}$ is a sequence of bounded linear mapping from a characteristic Šerstnev $(V_1,v_1,\tau_1,\tau_1^*)$ into a characteristic Šerstnev$(V_2,v_2,\tau_2,\tau_2^*)$ with $\tau_1=\tau_2=\tau_M$ and if $Ax=\lim_{n\to\infty}A_nx$, exist for every $x\in V_1$, then A is bounded.*

Proof. By Remark 1.11, A_n is continuous for all n. Now the result is a immediate consequence of Theorem 2.4. □

Lemma 2.7. *In a characteristic Šerstnev (V,v,τ,τ^*) the following statement are equivalent for a subset A of V:*

A is $\mathscr{D}$ – bounded (a)

A is topologicaly bounded (b)

Proof. (a) ⇒ (b) Let A any $\mathscr{D}$-bounded subset of V and let p_n be any sequence of elements of A and α_n any sequence of real numbers that converges to 0; there is no loss of generality in assuming $\alpha_n=0$ for every $n\in\mathbb{N}$

$$v_{\alpha_np_n}(x)=v_{p_n}\left(\frac{x}{|\alpha_n|}\right)\geqslant R_A\left(\frac{x}{|\alpha_n|}\right)\to 1$$

as $n\to+\infty$.

Thus $\alpha_n p_n \to \theta$ in the strong topology and A is topologically bounded. (b) $\Rightarrow$ (a) Let A be a subset of V which is not $\mathcal{D}$-bounded. Then

$$R_A(x) \to \gamma < 1$$

as $x \to +\infty$.

By definition of R_A for every $n \in \mathbb{N}$ ther is $p_n \in A$ such that

$$v_{p_n}(n^2) < \frac{1+\gamma}{2} < 1.$$

If $\alpha_n = 1/n$, then

$$v_{\alpha_n p_n}(1/2) \leqslant v_{\alpha_n p_n}(n) = v_{p_n}(n^2) < \frac{1+\gamma}{2} < 1,$$

which shows that $v_{\alpha_n p_n}$ does not tend to ε_0, even if it has a weak limit, viz. $\alpha_n p_n$ does not tend to θ in the strong topology; in other words, A is not topologically bounded.

The following corollary is immediate from Corollary 2.6 and Lemma 2.7. □

Corollary 2.8. *If $\Gamma = \{A: V_1 \to V_2\}$ is a collection of continuous linear maps, where $(V_1, v_1, \tau_1, \tau_1^*)$ and $(V_2, v_2, \tau_2, \tau_2^*)$ are characteristic Šerstnev spaces with $\tau_1 = \tau_2 = \tau_M$ and if $\Gamma(x) = \{Ax;\ A \in \Gamma\}$ is a $\mathcal{D}$-bounded subset of V_2 for every $x \in V_1$, then Γ is equicontinuous.*

In order to give the final result of this article, the following definition is given:

Definition 2.9. A set $A \subset V$ is balanced, if $\alpha x \in A$ whenever $x \in V$ and $|\alpha| \leqslant 1$. A set A is absorbing, if for each $x \in V$ there is an $\varepsilon > 0$, such that $tx \in A$ for $0 < t < \varepsilon$.

Theorem 2.10. *Suppose K is a nonempty convex compact subset of locally convex TV PN-space (V, v, τ, τ^*). Let G be a group of linear mapping such that*

$$\Gamma(x) = \{Ax : A \in G\}$$

is bounded in V for every $x \in V$. Also $A(K) \subseteq K$ for every $A \in G$. Then G has a common fixed point in K; that is there exist $p \in K$ such that $Ap = p$ for every $A \in G$.

Proof. The group G is equicontinuous by Theorem 2.1 and V_1 has a local base consisting of balanced convex set U which satisfies $A(U) \subseteq U$ for every $A \in G$. Let Ω be the collection of all nonempty convex sets $H \subseteq K$ such that $A(H) = H$ for every $A \in G$. Now, we partially ordered the set Ω by set inclusion. Then Hausdorff's maximality theorem shows that Ω contains a maximal totally ordered subcollection Ω_0. The intersection H_0 of all members of Ω_0 is a minimal member of Ω. Now, it is enough to show that H_0 have exactly one point. Otherwise, if $H \in \Omega$ contains more than one point. Then $H - H \neq \{0\}$ and there is a convex balanced member of the above local base which does not cover $H - H$. Since $H - H$ is compact, there exists some $s > 0$ such that $H - H \subset sU$. Let t be the greatest lower bound of these member s. Set $W = tU$, then W is a convex balanced open set such that

$$(1 - r)\overline{W} \text{ does not cover } H - H \text{ if } 0 < r < 1. \tag{1}$$

H is compact and hence, there exist $x_1, x_2, \ldots, x_n \in H$ such that $H \subset \bigcup_{i=1}^{n}(x_i + 1/2W)$. Let $r = 1/4n$ and define $H_1 = H \bigcap_{y \in H}(y + (1 - r)\overline{W})$. H_1 is compact, convex and $AH_1 \subset H_1$ for every $A \in G$. By (1) there are points $x \in H$ and $y \in H$ such that $x - y$ does not lie in $(1 - r)\overline{W}$. Any such x is not in H_1. Thus $H_1 \neq H$. The point $x_0 = 1/n$; $(x_1 + x_2 + \cdots + x_n) \in H_1$ and therefore $H_1 \neq \Phi$. This shows that H_0 contains only one point. □

The following corollaries are immediate consequence from Theorems 2.10 and 1.9.

Corollary 2.11. *Suppose K is a nonempty convex subset of characteristic Šerstnev space (V, v, τ, τ^*) with $\tau = \tau_M$. Let G be a group of linear mapping such that $\Gamma(x) = \{Ax: A \in G\}$ is bounded in V for every $x \in V$. Moreover, $A(K) \subseteq K$ for every $A \in G$. Then G has a common fixed point in K.*

Corollary 2.12. *Suppose K is a nonempty convex subset of characteristic Šerstnev space (V, v, τ, τ^*) with $\tau = \tau_M$ and A is an invertible operator on V such that the set $\{Ax, A^{-1}x\}$ is bounded for every $x \in V$, then A and A^{-1} has a common fixed point.*

3. Application in physics

Menger sponge is a random space which could be used for instance to predict the Background micro wave radiation (see El Naschie and also He's Book [7]).

4. Conclusions

In this worked we have analyzed in some detail the problem of equicontinuity of a class of linear operators on probabilistic normed spaces. We have shown for the class of characteristic Šerstnev spaces if $\Gamma(x) = \{Ax; A \in \Gamma\}$ is bounded then Γ is equicontinuous. A detailed study of how we can have a common fixed point for a group of linear operators is given.

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