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Probabilistic Total Paranorms, F-norms and PN Spaces

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ABSTRACT

The concept of paranorm given by A. Wilansky in (Wilansky, 1964) suggests to us the construction of a more restrictive type of probabilistic spaces: one introduces the notion of Probabilistic Total Paranormed spaces (briefly PTPN spaces) and we characterize a class of probabilistic normed spaces(briefly PN spaces) which are also Probabilistic Total Paranormed spaces. In section 3 we use the F-norms and the F-normed spaces in connection with the PN spaces. Finally we show the relationship between F-spaces and PTPN spaces.

Keywords: Probabilistic total paranorms, Probabilistic total paranormed space,F-normed space.

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1 Introduction and Preliminaries

Probabilistic normed spaces were first defined by Šerstnev in 1962 (see (Šerstnev, 1963)). Their definition was generalized in (Alsina, Schweizer and Sklar, 1993). We recall the definition of probabilistic space briefly as given in (Alsina et al., 1993), together with the notation that will be needed (see (Schweizer and Sklar, 2005)). We shall consider the space of all distance probability distribution functions (briefly, d.f.'s), namely the set of all left–continuous and non–decreasing functions from $\overline{\mathbb{R}}$ into $[0, 1]$ such that $F(0) = 0$ and $F(+\infty) = 1$; here as usual, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. The spaces of these functions will be denoted by Δ^+, while the subset $D^+ \subseteq \Delta^+$ will denote the set of all proper distance d.f.'s, namely those for which $\ell^- F(+\infty) = 1$. Here $\ell^- f(x)$ denotes the left limit of the function f at the point x, $\ell^- f(x) := \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions i.e., $F \leq G$ if and only if $F(x) \leq G(x)$ for all x in $\mathbb{R}$. For any $a \geq 0$, ε_a is the d.f. given by

$$\varepsilon_a = \begin{cases} 0, & x \leq a \\ 1, & x > a \end{cases} \qquad \varepsilon_{+\infty} \begin{cases} 0, & x < +\infty \\ 1, & x = +\infty \end{cases}$$

The space Δ^+ can be metrized in several ways (Schweizer, 1975; Seibley, 1971; Taylor, 1985), but we shall here adopt the Sibley metric d_S. If F, G are d.f.'s and h is in $]0,1[$, let $(F, G; h)$ denote the condition:

$$G(x) \leq F(x+h) + h \text{ for all } x \in \left]-\frac{1}{h}, \frac{1}{h}\right[.$$

Then the Sibley metric d_S is defined by

$$d_S(F, G) := \inf\{h \in]0,1[: \text{both } (F, G; h) \text{ and } (G, F; h) \text{ hold}\}.$$

For any F in Δ^+,

$$d_S(F, \varepsilon_0) = \inf\{h \in]0,1[: (F, \varepsilon_0; h) \text{ holds}\} = \inf\{h \in]0,1[: F(h^+) > 1 - h\}$$

In particular, under the usual pointwise ordering of functions, ε_0 is the maximal element of Δ^+. A triangle function is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ that is associative, commutative, nondecreasing in each place and has ε_0 as identity, that is, for all F, G and H in Δ^+:

(TF1) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$,

(TF2) $\tau(F, G) = \tau(G, F)$,

(TF3) $F \leq G \Longrightarrow \tau(F, H) \leq \tau(G, H)$,

(TF4) $\tau(F, \varepsilon_0) = \tau(\varepsilon_0, F) = F$.

Moreover, a triangle function is *continuous* if it is continuous in the metric space (Δ^+, d_S). Typical continuous triangle functions are $\tau_T(F, G)(x) = \sup_{s+t=x} T(F(s), G(t))$, and $\tau_{T^*}(F, G) = \inf_{s+t=x} T^*(F(s), G(t))$. Here T is a continuous t-norm, i.e. a continuous binary operation on $[0,1]$ that is commutative, associative, nondecreasing in each variable and has 1 as identity; T^* is a continuous t–conorm , namely a continuous binary operation on $[0,1]$ which is related to the continuous t-norm T through $T^*(x, y) = 1 - T(1-x, 1-y)$. Let us recall that among the triangle function one has the function defined via $\mathbf{T}(F, G)(x) = T(F(x), G(x))$ and $\mathbf{T}^*(F, G)(x) = T^*(F(x), G(x))$. For example $T = \min$. Recall that the maximum and minimum continuous t–norm are respectively given by $M(x, y) := min\{x, y\}$ and $W(x, y) := max\{x + y - 1, 0\}$; and another important continuous t–norm is $\Pi(x, y) := xy$.
The definition below has been proposed in (Alsina et al., 1993).

Definition 1.1. A *Probabilistic Normed space* (briefly, PN space) is a quadruple (V, ν, τ, τ^*), where V is a real vector space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$ and ν is a mapping (the *probabilistic norm*) from V into Δ^+, such that for every choice of a and b in V the following hold:

(N1) $\nu_a = \varepsilon_0$ if, and only if, $a = \theta$(θ is the null vector in V);

(N2) $\nu_{-a} = \nu_a$;

(N3) $\nu_{a+b} \geq \tau(\nu_a, \nu_b)$;

(N4) $\nu_a \leq \tau^*(\nu_{\lambda a}, \nu_{(1-\lambda)a})$ for every $\lambda \in [0, 1]$.

If ν satisfies (N2), (N3), (N4) and $\nu_\theta = \varepsilon_0$(but not necessarily (N1)), then (V, ν, τ, τ^*) is said to be a *probabilistic pseudonormed* space (briefly, a PPN space). The pair (V, ν) is said to be a Probabilistic Seminormed Space(briefly PSN space) if $\nu : V \to \Delta^+$ satisfies (N1) and (N2).
If a PN space (V, ν, τ, τ^*) satisfies the following condition

(Š) $\quad \forall a \in V \quad \forall \lambda \in \mathbb{R} \setminus 0 \quad \forall x > 0 \quad \nu_{\lambda a}(x) = \nu_a\left(\frac{x}{|\lambda|}\right)$,

then it is called a *Šerstnev* PN space; the condition (Š) implies that the best-possible selection for τ^* is $\tau^* = \tau_M$, which satisfies a stricter version of $(N4)$, namely

$$\forall \lambda \in [0, 1] \quad \nu_a = \tau_M(\nu_{\lambda a}, \nu_{(1-\lambda)a}).$$

Therefore, condition (N4) is satisfied for every τ^* such that $\tau_M \leq \tau^*$. In this sense, if (V, ν, τ, τ^*) is a *Šerstnev* PN space then (V, ν, τ, τ_M) is a better structure than (V, ν, τ, τ^*). This motivates the following partial order relation:

Definition 1.2. A PN space $(V, \nu, \tau_1, \tau_1^*)$ is *better* than another PN space $(V, \nu, \tau_2, \tau_2^*)$ with the same V and ν, if for all $a, b \in V$ and $\lambda \in [0, 1]$, the following statements hold:

- $\tau_1(\nu_a, \nu_b) \geq \tau_2(\nu_a, \nu_b)$,
- $\tau_1^*(\nu_{\lambda a}, \nu_{(1-\lambda)a}) \leq \tau_2^*(\nu_{\lambda a}, \nu_{(1-\lambda)a})$.

For every PN space (V, ν, τ, τ^*), if $a \in V$ and $x \geq 0$, then $\nu_a(x)$ may be thought of as the probability $P(\| a \| < x)$, where $\| \cdot \|$ is a norm for V. So the fact that ν_a does not belong to $\mathcal{D}^+$ means that $P(\| a \| < +\infty) < 1$; this is to be regarded as being "odd". Therefore we shall call *strict* any PN space (V, ν, τ, τ^*) such that $\nu(V) \subseteq \mathcal{D}^+$, i.e., such that ν_a belongs to $\mathcal{D}^+$ for every $a \in V$. When there is a continuous t-norm T (see (Schweizer and Sklar, 2005),(Klement, Mesiar and Pap, 2000)) such that $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, the PN space (V, ν, τ, τ^*) is called a Menger PN space, and is denoted by (V, ν, τ). One speaks of an *equilateral* PN space when there is $F \in \Delta^+$ different from both ε_0 and ε_∞ such that, for every $a \neq \theta$, $\nu_a = F$, and when $\tau = \tau^* = \Pi_M$, which is the triangle function defined for G and H in Δ^+ by $\Pi_M(G, H)(x) := M(G(x), H(x))$.
Let $G \in \Delta^+$ be different from ε_0 and from ε_∞ and let $(V, \| \cdot \|)$ be a normed space; then, define, for $a \neq \theta$,

$$\nu_a(x) := G\left(\frac{x}{\| a \|}\right).$$

Then (V, ν, M) is a Menger PN space denoted by $(V, \| \cdot \|, G, M)$. This type of Menger PN spaces are known as *simple* PN spaces. Observe that simple PN spaces belongs to the class of Šerstnev spaces. In the same conditions, if ν is defined by

$$\nu_a(x) := G\left(\frac{x}{\| a \|^\alpha}\right),$$

with $\alpha \geq 0$, then the pair (V, ν) is a PSN space called *α–simple* and it is denoted by $(V, \| \cdot \|, G; \alpha)$. The α– simple spaces can be endowed with a structure of PN space in a very general setting(G should be a continuous and strictly increasing function in $\mathcal{D}^+$, see (Lafuerza-Guillén, Rodríguez Lallena and Sempi, 1997)).

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(V, \| \cdot \|)$ a normed space and S a vector space of V–valued random variables (possibly, the entire space). For every $p \in S$ and for every $x \in \overline{\mathbb{R}}_+$, let $\nu : S \to \Delta^+$ be defined by $\nu_p(x) := P\{\omega \in \Omega : \| p(\omega) \| < x\}$; then (S, ν) is called an E- *normed space* (briefly, EN space) with *base* $(\Omega, \mathcal{A}, P)$ and *target* $(V, \| \cdot \|)$. Every EN space (S, ν) is a PPN space under τ_W and τ_M. It is said to be *canonical* if it is a PN space under the same triangle functions. In th later case, it is a Šerstnev space. See (Lafuerza-Guillén et al., 1997), (Lafuerza-Guillén, Rodríguez Lallena and Sempi, 1995), (Sempi, 1982) for properties of PN spaces.

Definition 1.3. Let (V, ν, τ, τ^*) a PN space, then:

(i) The strong neighbourhoods are defined as follows: for every $t > 0$, the neighbourhood $N_p(t)$ at a point p of V is defined by

$$N_p(t) := \{q \in V : d_S(\nu_{p-q}, \varepsilon_0) < t\} = \{q \in V : \nu_{p-q}(t) > 1 - t\}.$$

The *strong topology* is the topology generated by strong neighbourhoods.

(ii) A sequence (p_n) in V is said to be strongly convergent to p in V if for each $\lambda > 0$, there exists a positive integer N such that $p_n \in N_p(\lambda)$, for $n \geq N$.

Recall that a vector space endowed with a topology, is a *topological vector space* (briefly a TV space) if both the addition $+ : V \times V \to V$ and multiplication by scalars $\eta : \mathbb{R} \times V \to V$ are continuous. If only the addition is assumed to be continuous then V is a *topological group*; if furthermore η is continuous at the first place, then it is called a *topological vector group* (briefly, a TV group). In (Alsina, Schweizer and Sklar, 1997) the authors showed that PN with τ continuous ere topological vector groups

Theorem 1.1. ***(Alsina, Schweizer, Sklar*** *(Alsina et al., 1997)). Every PN space (V, ν, τ, τ^*), when it is endowed with the strong topology induced by the probabilistic norm ν, is a topological vector space if, and only if, for every $a \in V$ the map from $\mathbb{R}$ into V defined by*

$$\lambda \mapsto \lambda a \tag{1.1}$$

is continuous.

2 Main Results-I: Probabilistic Total Paranormed Spaces

The concept of paranorm is a generalization of that of absolute value. The paranorm of x may be thought of as the distance from x to 0.

Definition 2.1. A paranorm is a real function $p : V \to \mathbb{R}$ where V is a vector space, and satisfying conditions (i) through (v) for all vectors a, b in V.

(i) $p(\theta) = 0$

(ii) $p(a) \geq 0$

(iii) $p(-a) = p(a)$

(iv) $p(a + b) \leq p(a) + p(b)$

(v) If t_n is a sequence of scalars with $t_n \to t$ and u_n is a sequence of vectors with $u_n \to u$, then $p(t_n u_n - tu) \to 0$ (continuity of multiplication).

A paranorm p for which $p(a) = 0$ implies $a = \theta$ will be called *total*.

Definition 2.2. A *Probabilistic Total Paranormed space* (briefly, PTPN space) is a triple (V, ν, τ), where V is a real vector space, τ is a continuous triangle functions and ν is a mapping (the *probabilistic total paranorm*) from V into Δ^+, such that for every choice of a and b in V the following hold:

(P1) $\nu_a = \varepsilon_0$ if, and only if, $a = \theta(\theta$ is the null vector in $V)$;

(P2) $\nu_{-a} = \nu_a$;

(P3) $\nu_{a+b} \geq \tau(\nu_a, \nu_b)$;

(P4) If (u_n), (α_n) are two sequences of vectors and scalars respectively with $(u_n) \to u$ as regards the strong topology, and $(\alpha_n) \to \alpha$, then $\nu_{\alpha_n u_n - \alpha u} \to \varepsilon_0$.

The next Example shows a PTPN space which is not Šerstnev .

Example 2.1. Suppose p is a total paranorm on the vector space V and let τ be a triangle function such that $\tau(\varepsilon_a, \varepsilon_b) \leq \varepsilon_{a+b}$ and $\nu_a = \varepsilon_{p(a)}$. Then (V, ν, τ) is a PTPN space.

Proof. (P1) $\nu_a = \varepsilon_0 \Leftrightarrow \varepsilon_{p(a)} = \varepsilon_0 \Leftrightarrow p(a) = 0 \Leftrightarrow a = \theta$.
(P2) $\nu_a = \varepsilon_{p(a)} = \varepsilon_{p(-a)} = \nu_{-a}$.
(P3) $\nu_{a+b} = \varepsilon_{p(a+b)} \geq \varepsilon_{p(a)+p(b)} \geq \tau(\varepsilon_{p(a)}, \varepsilon_{p(b)}) = \tau(\nu_a, \nu_b)$.
(P4) Suppose u_n and α_n be two sequences of vectors and scalars respectively and $u_n \to u$, $\alpha_n \to \alpha$. Then $\nu_{\alpha_n u_n - \alpha u} = \varepsilon_{p(\alpha_n u_n - \alpha u)} \to \varepsilon_0$. □

Example 2.2. Let V be a normed vector space. Define $p(a) = \|a\|/(1 + \|a\|)$ and $\nu_a = \varepsilon_{p(a)}$. Then (V, ν, τ) is a PTPN space for every triangle function τ such that $\tau(\varepsilon_a, \varepsilon_b) \leq \varepsilon_{a+b}$.

Proof. In this case notice that condition (iv) of the Definition 2.1 holds for any complex numbers a, b (see pp. 52 in (Wilansky, 1964)). □

Example 2.3. Let p be a total paranorm. The quadruple (V, ν, τ, τ^*) with $\nu_a(x) := e^{-p(a)}$, $\tau = \tau_\Pi$ and $\tau^* = \tau_{\Pi^*}$ is a PN space which is not strict, and the triple (V, ν, τ_Π) is a PTPN space.

Corollary 2.1. *A PN-space (V, ν, τ, τ^*) is a PTPN space if, and only if, (V, ν, τ) is a PTPN space.*

Proof. Suppose (V, ν, τ) is a PTPN space. By (p4) we have $\nu_{\alpha_n u_n - \alpha u} \to \varepsilon_0$ when $u_n \to u$ and $\alpha_n \to \alpha$. Therefore $d_S(\nu_{\alpha_n p_n - \alpha p}, \varepsilon_0) \to 0$ as $n \to \infty$. This shows that the product by a scalar of the Theorem 1.3 is a continuous operation in the strong topology. If (V, ν, τ) is a topological vector space, then the product by a scalar is continuous in the strong topology and hence $d_S(\nu_{\alpha_n u_n - \alpha u}, \varepsilon_0) \to 0$ $n \to \infty$. Therefore $\nu_{\alpha_n u_n - \alpha u} \to \varepsilon_0$ when $u_n \to u$ and $\alpha_n \to \alpha$. □

Theorem 2.2. *(i) No equilateral space (V, F, M) is a PTPN space.*

(ii) A Šerstnev space (V, ν, τ) is a PTPN space if, and only if, the probabilistic total paranorm ν maps V into $\mathcal{D}^+$, viz. $\nu(V) \subseteq \mathcal{D}^+$.

(iii) A simple space $(V, \|\cdot\|, G, M)$ is a PTPN space if, and only if, G belongs to $\mathcal{D}^+$.

(iv) If G is a distribution function different from ε_0 and ε_∞, then the α–simple space $(V, \|\cdot\|, G; \alpha)$ is a PTPN space if, and only if, G belongs to $\mathcal{D}^+$.

(v) An EN space (S, ν) is a PTPN space if, and only if, ν_a belongs to $\mathcal{D}^+$ for every $a \in S$.

Proof. Let θ denote the null vector of the linear space V. Since any PN space can be metrized, one can limit oneself to investigating the behaviour of sequences. Moreover, because of the linear structure of V, one can take $a \neq \theta$ and an arbitrary sequence (λ_n) with $\lambda_n \neq 0 (n \in \mathbb{N})$ such that $\lambda_n \to 0$ as n tends to ∞.

(i) For every $n \in \mathbb{N}$, one has $\nu_{\lambda_n a} = F$ while $\nu_\theta = \varepsilon_0$. Therefore the map (1.1) is not continuous.

(ii) If ν maps V into $\mathcal{D}^+$, then, for every $t > 0$, one has

$$\lim_{n\to\infty} \nu_{\lambda_n a}(t) = \lim_{n\to\infty} \nu_a\left(\frac{t}{|\lambda_n|}\right) = 1,$$

whence $\lim_{n\to\infty} \nu_{\lambda_n a}(t) = \varepsilon_0(t)$, and $\lim_{n\to\infty} \nu_{\lambda_n a} = \varepsilon_0$. Conversely, if there exists at least one $a \in V$ such that $\nu_a \in \Delta^+ \setminus \mathcal{D}^+$, namely such that $\lim_{x\to\infty} \nu_a(x) = \gamma < 1$, then, for $x > 0$,

$$\lim_{n\to\infty} \nu_{\lambda_n a}(t) = \nu_a\left(\frac{t}{|\lambda_n|}\right) = \gamma < 1,$$

so that the mapping (1.1) is not continuous.

(iii) Is a trivial consequence of part (ii), since every simple space is a Šerstnev space.

(iv) Let (λ_n) be a sequence of real numbers that tends to 0, when n goes to ∞. Then, for all $a \in V$ and $x > 0$, one has, for every $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \nu_{\lambda_n a}(t) = \lim_{n\to\infty} G\left(\frac{t}{\|\lambda_n a\|^\alpha}\right) = \lim_{n\to\infty} G\left(\frac{t}{|\lambda_n|^\alpha \|a\|^\alpha}\right) = 1.$$

(v) The proof is analogous to that of part (ii).

□

However, in general PN spaces, the condition $\nu(V) \subseteq \mathcal{D}^+$ is not necessary to obtain a PTPN (see Example 2.3).
Example 2.1, Corolario 2.1 and Theorem 2.2 shows that the class of PTPN spaces contains the class of strict Šerstnev spaces and the class of α-simple spaces with $G \in \mathcal{D}^+$: the class of PTPN spaces is larger than any other of these two classes of PN spaces.

3 Main Results-II: F-normed spaces and PN spaces

Definition 3.1. An F-norm on a vector space V is a map $g : V \to \mathbb{R}_+$ satisfying for all $a \in V$ the following conditions:

(i) $g(a) = 0$ if, and only if, $a = \theta$,

(ii) $g(\lambda a) \leq g(a)$ if $|\lambda| \leq 1$,

(iii) $g(a + b) \leq g(a) + g(b)$.

The pair (V, g) is called an F-*normed space*.

It is a TV group with respect to the metric $d(a, b) = g(a - b)$, but in general it is not a TV space. In [(Schaefer and Wolff, 1999), Exercise 12(b), p. 35] one can find such an example where V is the vector space of all continuous functions $p : \mathbb{R} \to \mathbb{R}$, and $g(p) := \sup_{t\in\mathbb{R}} \frac{|p(x)|}{\alpha+|p(x)|}$, with $\alpha > 0$. (The problem arises with the unbounded functions the F-norm of which is 1).
Clearly different F-norms may induce the same metric-topology. For instance, if $(V, \|\cdot\|)$ is a normed space, then $g(p) = \| p \|^\alpha$, or $g(p) = \frac{\|p\|}{\alpha+\|p\|}$, where $\alpha > 0$, are F-norms which induce the same topology as $\|\cdot\|$.
The above condition (ii) implies $\| -p \| = \| p \|$. This remark and the fact that for every a, b in $\mathbb{R}$ is $\tau_M(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b}$, yield easily the following result that is a correspondence between F-norms and some specific PN spaces.

Theorem 3.1. *Let $G : V \to \mathbb{R}_+$ be any map and define ν by $\nu_a := \varepsilon_{g(a)}$. Then, (V, g) is a F-*normed space *if, and only if, $(V, \nu, \tau_M, \mathbf{M})$ is a PN space, where **M** is defined via $\mathbf{M}(F, G)(x) = M(F(x), G(x))$.*

Note that **M** is the maximal triangle function, so that $(V, \varepsilon_g, \tau_M, \mathbf{M})$ copuld not be the best PN structure for a given F-norm g (indeed, if g is a norm then $(V, \varepsilon_g, \tau_M, \mathbf{M})$ is better). So in this sense PN structures stratify all possible F-norms on a given vector space, depending on the triangle functions we use.

Theorem 3.2. *Let $g : V \to \mathbb{R}_+$ be any map and define ν by $\nu_a := \varepsilon_{g(a)}$. Let τ and τ^* be two (non necessarily associative) triangle functions. Then one has the following statements:*

(i) If $\tau(\varepsilon_a, \varepsilon_b) \geq \varepsilon_{a+b}$, for all a, b in $\mathbb{R}_+$, and (V, ν, τ, τ^*) is a PN space, then g is a F-norm.

(ii) If $\tau(\varepsilon_a, \varepsilon_b) \leq \varepsilon_{a+b}$, for all a, b in $\mathbb{R}_+$, and g is a F-norm, then (V, ν, τ, τ^*) is a PN space if, and only if, (N4) holds.

(iii) *If* $\tau(\varepsilon_a, \varepsilon_b) \leq \varepsilon_{a+b}$, *for all a, b in* $\mathbb{R}_+$, *then* g *is a norm if, and only if,* (V, ν, τ, τ^*) *is a Šerstnev PN space.*

4 Main Results-III: F-spaces and PTPN spaces

In this section the relation between F-spaces and PTPN spaces are considered.
As an example of F-norm one has the function $x \to |x|$ on V into $\mathbb{R}$ such that
(i) $|\lambda| \leq 1$ implies $|\lambda x| \leq |x|$ for all $x \in V$.
(ii) $|x+y| \leq |x| + |y|$ for all $x \in V$, $y \in V$.
(iii) $|x| = 0$ equivalent with $x = 0$. The metric $(x, y) \to |x - y|$ generates the topology on V.

Theorem 4.1. *If* (V, ν, τ) *is a PTPN space then there exists a F-norm* $|\cdot|$ *such that* $(V, |\cdot|)$ *is F-normed space.*

Proof. Let (V, ν, τ) be a PTPN space. By Corolary 2.1 (V, ν, τ) is a topological vector space and the set $\{N_8(1/n) : n \in \mathbb{N}\}$ is a countable 0-neighborhood base. Let $\{N_8(1/n_i) : i \in \mathbb{N}\}$ be a base of circled 0-neighborhoods satisfying

$$N_8(1/n_{i+1}) + N_8(1/n_{i+1}) \subset N_8(1/n_i) \qquad (i \in \mathbb{N}) . \tag{4.1}$$

For each non-empty finite subset K of $\mathbb{N}$, define the circled 0–neighborhood V_K by $V_K = \sum_{n \in K} N_8(1/n_k)$ and the real number P_K by $P_K = \sum_{n \in K} 2^{-n}$. It follows from (2.1) by induction on the number of elements of K that these implications hold:

$$p_K < 2^{-n} \Rightarrow n < K \Rightarrow V_K \subset N_8(1/n_k), \tag{4.2}$$

where $n < K$ means that $n < k$ for $k \in K$. We define the real–valued function $x \to |x|$ on V by $|x| = 1$ if x is not contained in any V_K, and by

$$|x| = \inf_K \{P_K : x \in V_K\}$$

otherwise; the range of this function is contained in the real unit interval. The function $|\cdot|$ is the desired F-norm.

□

For any $\epsilon > 0$, let $B_\epsilon = \{x \in V : |x| \leq \epsilon\}$; then we have

$$B_{2^{-k-1}} \subset N_8(1/n_k) \subset B_{2^{-k}} \qquad (n \in \mathbb{N}). \tag{4.3}$$

The inclusion $N_8(1/n_k) \subset B_{-k}$ is obvious since $x \in N_8(1/n_k)$ implies $|x| \leq 2^{-k}$. On the other hand , if $|x| \leq 2^{-n-1}$, then there exist K such that $x \in V_K$ and $p_K < 2^{-k}$; hence (4.2) implies that $x \in N_8(1/n_k)$
The following Theorem and Corollary are geometrical forms of the Hahn- Banach Theorem in the PTPN spaces.

Theorem 4.2. *Let* (V, ν, τ) *be a PTPN space,* A *an open convex set in* V *and* L *a vector subspace such that not meeting* A*, then there exists a continuous function* f *on* V *such that* $f(L) = 0$ *and* $f(A) \neq 0$.

Proof. (V, ν, τ) is a PTPN space. Now by Corolary 2.1 V is a topological vector space with strong topology. A is convex therefore A may be represented by a equation $P(x - x_0) < 1$, where x_0 is a point of V and P is subadditive, and positive- homogeneous function on V. P is continuous since A is open. Since $L \cap A = \phi$, $P(y - x_0) \geq 1$ for $y \in L$. Define the linear function f_0 on $L + \mathbb{R}x_0$ by $f(y - \alpha x_0) = \alpha$ for $y \in L$ and $\alpha \in \mathbb{R}$. $f_0 \leq P$ at all point of $L + \mathbb{R}x_0$. Since P is a positive gauge function on E, the Hahn Banach theorem shows that f_0 may be extended to a linear functional f on V such that $f \leq P$ at points of V.
Moreover $H = f^{-1}(0)$ is a closed hyperplane in V which contains $L = (L + \mathbb{R}x_0) \cap f^{-1}(0)$. $f(x) = 0$ for $x \in H$ and so $0 = f(x) = f(x - x_0) + f(x_0) = f(x - x_0) + f_0(x_0) = f(x - x_0) - 1 \leq P(x - x_0) - 1$, showing that $p(x - x - x_0) \geq 1$ for $x \in H-$ that is , that A dos not meet H. Since $L \subset H$, so $f(L) = 0$. Since H does not meet A and A is convex, $f(A)$ must be a real interval not containing 0. Therefore $f(a) > 0$. □

The next Corollary is immediate from the Theorem 4.2.

Corollary 4.3. *Let (V, ν, τ) be a P p-N space A an open convex set and B a convex set in strong topology of (V, ν, τ) and suppose that $A \cap B = \phi$. Then there exists a continuous linear form $f \neq 0$ on V and a real number α such that*

$$f(A) < \alpha, \qquad f(B) \geq \alpha.$$

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