TOTAL BOUNDEDNESS IN PROBABILISTIC NORMED SPACES

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In this paper, we study total boundedness in probabilistic normed space and we give criterion for total boundedness and D-boundedness in these spaces. Also we show that in general a totally bounded set is not D-bounded.

Keywords: Total boundedness, Probabilistic normed spaces, Triangle functions.

1. Introduction

In this paper, we shall consider the space of all distance probability distribution functions (briefly, d.f. 's), namely the set of all left--continuous and non--decreasing functions from $\overline{\mathbb{R}}$ into $[0,1]$ such that $F(0)=0$ and $F(+\infty)=1$; here as usual, $\overline{\mathbb{R}}:=\mathbb{R}\cup\{-\infty,+\infty\}$. The spaces of these functions will be denoted by Δ^+, while the subset $D^+\subseteq\Delta^+$ will denote the set of all proper distance d.f. 's, namely those for which $\ell^-F(+\infty)=1$. Here $\ell^- f(x)$ denotes the left limit of the function f at the point x, $\ell^- f(x):=\lim_{t\to x^-} f(t)$. For any $a\geq 0$, ε_a is the d.f. given by $\varepsilon_a=0$ if $x\geq a$ and $\varepsilon_a=1$ if $x<a$. In particular, under the usual point-wise ordering of functions, ε_0 is the maximal element of Δ^+. A triangle function is a binary operation on Δ^+, namely a function $\tau:\Delta^+\times\Delta^+\to\Delta^+$ that is associative, commutative, nondecreasing and which has ε_0 as unit, continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+.

Probabilistic normed spaces were introduced by Sherstnev in 1962 [1] by means of a definition that was closely modeled on the theory of (classical) normed spaces, and used to study the problem of best approximation in statistics. Then a new definition was proposed by Alsina, Schweizer and Sklar [2]. The properties

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of these spaces were studied by several authors; here we shall mention [3-9] (but see also the survey paper [10]).

Definition 1.1 A Probabilistic Normed space (briefly, PN space) is a quadruple (V,ν,τ,τ^*), where V is a real vector space, τ and τ^* are continuous triangle functions with $\tau\le\tau^*$ and ν is a mapping (the probabilistic norm) from V into Δ^+, such that for every choice of p and q in V the following hold: (N1) $\nu_p=\varepsilon_0$ if, and only if, $p=\theta$ (θ is the null vector in V); (N2) $\nu_{-p}=\nu_p$, (N3) $\nu_{p+q}\ge\tau(\nu_p,\nu_q)$; (N4) $\nu_p\le\tau^*(\nu_{\lambda p},\nu_{(1-\lambda)p})$ for every $\lambda\in[0,1]$.

A PN space is called a Šerstnev space if it satisfies (N1), (N3) and the following condition: For every $\alpha\ne 0\in\mathbb{R}$ and $x>0$ one has

$$(NS) \qquad \nu_{\alpha p}(x)=\nu_p(x/|\alpha|),$$

which clearly implies (N2) and also (N4) in the strengthened form $\nu_p=\tau_M(\nu_{\lambda p},\nu_{(1-\lambda)p})$. The triple (V,ν,τ) where V is a real vector space, τ is a continuous triangle functions and ν is a mapping from V into Δ^+, such that (N1), (NS) and (N3) hold is a Šerstnev space.

A PN space in which $\tau=\tau_T$ and $\tau^*=\tau_{T^*}$ for a suitable continuous t--norm T and its conorm T^* is called a *Menger PN space*. In the case of PN spaces, the concepts of boundedness are based on the consideration of the *probabilistic radius* rather than that of the *probabilistic diameter*; the probabilistic radius R_A of a set $A\subset V$ is defined by $R_A(+\infty)=1$ and, for $x>0$, by $R_A(x):=\lim_{y\to x,y<x}\inf\{\nu_p(y):p\in A\}$. In a PN space there is an easy characterization of a D-bounded set A: A is D-bounded if, and only if,there exists a *proper* distance distribution function G, i.e. one for which $\lim_{x\to+\infty}G(x)=1$, such that $\nu_p\ge G$ for every $p\in A$.

Definition 1.2 Let (V,ν,τ,τ^*) be a PN-space. For each p in V and $\lambda>0$, the strong λ–neighborhood of p is the set $N_p(\lambda)=\{q\in V:\nu_{p-q}(\lambda)>1-\lambda\}$, and the strong neighborhood system for V is the union $\cup_{p\in V}\mathrm{N}_p$ where $\mathrm{N}_p=\{N_p(\lambda):\lambda>0\}$.

The strong neighborhood system for V determines a Hausdorff topology for V which is also first countable.

Definition 1.3 Let (V,ν,τ,τ^*) be a PN space, a sequence $\{p_n\}$ in V is said to be strongly convergent to p in V if for each $\lambda>0$, there exists a positive integer N such that $p_n\in N_p(\lambda)$, for $n\ge N$. Also the sequence $\{p_n\}$ in V is called strongly Cauchy sequence if for every $\lambda>0$, there exists a positive integer N such that $\nu_{p_n-p_m}(\lambda)>1-\lambda$, whenever $m,n>N$. A PN space (V,ν,τ,τ^*) is said to be strongly complete in the strong topology if and only if every strongly Cauchy sequence in V is strongly convergent to a point in V.

Lemma 1.4 ([2]) *If* $|\alpha| \leq |\beta|$ *then* $\nu_{\beta p} \leq \nu_{\alpha p}$, *for every* p *in* V.

Definition 1.5 A subset A of TVS (topological vector space) V is said to be topologically bounded if for every sequence $\{\alpha_n\}$ of real numbers that converges to zero as $n \to +\infty$ and for every $\{p_n\}$ of elements of A, one has $\alpha_n p_n \to \theta$, in the strong topology. The PN space (V, ν, τ, τ^*) is called *characteristic* whenever $\nu(V) \subseteq D^+$.

Example 1.6 The triple (V, ν, τ_π), where $\nu : V \to \Delta^+$ is defined by $\nu_p(x) = \frac{x}{x + \|p\|}$ is a characteristic Šerstnev space (see [11, Theorem 9]).

Theorem 1.7 ([11]) *A Šerstnev space* (V, ν, τ) *is a TVS if and only if it is characteristic.*

Lemma 1.8 ([11]) *In a characteristic Šerstnev space* (V, ν, τ) *a subset* A *of* V *is topologically bounded if and only if it is D-bounded .*

Lemma 1.9 *Let* τ *be a continuous triangle function. Then for every* $F \in D^+$ *and* $F < \varepsilon_0$ *there exists* $G \geq F$ *such that* $\tau(G, G) > F$.

Proof. Let there exists $F \in D^+$ and $F < \varepsilon_0$ such that for every $G \geq F$ we have $\tau(G, G) \leq F$. Consider the sequence of d.f. 's defined by $G_n = \max(\varepsilon_{\frac{1}{n}}, F)$, then $G_n \geq F$ for every $n \in \mathbb{N}$, therefore $\tau(G_n, G_n) \leq F$. Taking $n \to \infty$ in the above inequality then we have $\varepsilon_0 \leq F$ which is a contradiction.

2. The Main Results

Definition 2.1 Let (V, ν, τ, τ^*) be a PN space and $A \subset V$. We say A is a probabilistic strongly totally bounded set if for every $F \in D^+$ and $F < \varepsilon_0$, there exists a finite subset S_F of A such that

$$A \subseteq \bigcup_{p \in S_F} D_p(F). \quad (2.1)$$

Where $D_p(F) = \{q \in V : \nu_{p-q} > F\}$.

Lemma 2.2 *Let* (V, ν, τ, τ^*) *be a PN space and* $A \subset V$. A *is a probabilistic strongly totally bounded set if and only if for every* $F \in D^+$ *with* $F < \varepsilon_0$, *there exists a finite subset* S_F *of* V *such that*

$$A \subseteq \bigcup_{p \in S_F} D_p(F). \quad (2.2)$$

Proof. Let $F \in D^+$, $F < \varepsilon_0$ and condition (2.2) holds. By continuity of τ, there exists $G \geq F$ such that $\tau(G,G) > F$. Now, applying condition (2.2) for G, there exists a subset $S_G = \{p_1,...,p_n\}$ of V such that $A \subset \bigcup_{p_i \in S_G} D_p(G)$. We assume that $D_{p_j}(G) \cap A \neq \phi$, otherwise we omit p_j from S_G and so we have $A \subset \bigcup_{p_i \in S_G \setminus \{p_j\}} D_{p_i}(G)$. For every $i = 1,...,n$ we select q_i in $D_{p_i}(G) \cap A$, and we put $S_F = \{q_1,...,q_n\}$. Now for every q in A, there exists $i \in \{1,...,n\}$ such that $\nu_{q-p_i} > G$. Therefore we have (by using property N3 of a PN space), $\nu_{q-q_i} \geq \tau(\nu_{q-p_i}, \nu_{p_i-q_i}) \geq \tau(G,G) > F$. Which implies that $A \subset \bigcup_{p_i \in S_F} D_{p_i}(F)$. The converse is trivial.

Lemma 2.3 *Let (V,ν,τ,τ^*) be a PN space and $A \subset V$. If A is a probabilistic strongly totally bounded set then so is its closure $\overline{A}$.*

Proof. Let $F \in D^+$, $F < \varepsilon_0$, then there exists a finite subset $S_G = \{q_1,...,q_n\}$ of V with $G \geq F$ and $\tau(G,G) > F$, such that $A \subseteq \bigcup_{q_i \in S_G} D_{q_i}(G)$. Since for every r in $\overline{A}$, $N_r(\frac{1}{n}) \bigcap A$ is non-empty for every $n \in \mathbb{N}$ (see Definition 1.2 and first countability property) therefore we can find $p \in A$ such that $\nu_{p-r} \geq G$ and there exists $1 \leq i \leq n$ such that $\nu_{p-q_i} \geq G$, therefore $\nu_{r-q_i} \geq \tau(\nu_{r-p}, \nu_{p-q_i}) \geq \tau(G,G) > F$.

Hence $\overline{A} \subset \bigcup_{q_i \in S_F} D_{q_i}(F)$, i.e. $\overline{A}$ is probabilistic strongly totally bounded set.

Theorem 2.4 *Let (V,ν,τ,τ^*) be a PN space and $A \subset V$. A is a probabilistic strongly totally bounded set if and only if every sequence in A has a strongly Cauchy subsequence.*

Proof. Let A be a probabilistic strongly totally bounded set. Let $\{p_n\}$ be a sequence in A. For every $k \in \mathbb{N}$, there exists a finite subset S_{F_k} of V such that $A \subseteq \bigcup_{q \in S_{F_k}} D_q(F_k)$, here $F_k = \varepsilon_{\frac{1}{k}}$. Hence, for $k = 1$, there exists $q_1 \in S_{F_1}$ and a subsequence $\{p_{1,n}\}$ of $\{p_n\}$ such that $p_{1,n} \in D_{q_1}(F_1)$, for every $n \in \mathbb{N}$. Similarly, there exists $q_2 \in S_{F_2}$ and a subsequence $\{p_{2,n}\}$ of $\{p_{1,n}\}$ such that $p_{2,n} \in D_{q_2}(F_2)$, for every $n \in \mathbb{N}$. Continuing this process, we get $q_k \in S_{F_k}$ and subsequences $\{p_{k,n}\}$ of $\{p_{k-1,n}\}$ such that $p_{k,n} \in D_{q_k}(F_k)$, for every $n \in \mathbb{N}$. Now we consider the subsequence $\{p_{n,n}\}$ of $\{p_n\}$. For every $F \in D^+$ and $F < \varepsilon_0$, by continuity of τ, there exists an $n_0 \in \mathbb{N}$ such that $\tau(F_{n_0}, F_{n_0}) > F$ and $F_{n_0} \geq F$. Therefore for every $k, m \geq n_0$, we have

$$\nu_{p_{k,k}-p_{m,m}} \geq \tau(\nu_{p_{k,k}-q_{n_0}}, \nu_{q_{n_0}-p_{m,m}}) \geq \tau(F_{n_0}, F_{n_0}) > F.$$

Hence $\{p_{n,n}\}$ is a strongly Cauchy sequence. Conversely, suppose that A is not a probabilistic strongly totally bounded set. Then there exists $F \in D^+$ such that for every finite subset S_F of V, A is not a subset of $\bigcup_{q \in S_F} D_q(F)$. Fix $p_1 \in A$. Since A is not a subset of $\bigcup_{q \in \{p_1\}} D_q(F)$, there exists $p_2 \in A$ such that $\nu_{p_1-p_2} \leq F$. Since A is not a subset of $\bigcup_{q \in \{p_1, p_2\}} D_q(F)$, there exists a $p_3 \in A$ such that $\nu_{p_1-p_3} \leq F$ and $\nu_{p_2-p_3} \leq F$. Continuing this process, we construct a sequence $\{p_n\}$ of distinct points in A such that $\nu_{p_i-p_j} \leq F$, for every $i \neq j$. Therefore $\{p_n\}$ has not strongly Cauchy subsequence.

Every probabilistic strongly totally bounded set is not D-bounded set , in general, as can see from the next example.

Example 2.5 The quadruple $(\mathbb{R}, \nu, \tau_\pi, \tau_\pi^*)$ where $\nu : \mathbb{R} \to \Delta^+$ is defined by $\nu_p(x)=0$ if $x = 0$, $\nu_p(x) = \exp(-\sqrt{|p|})$, if $0 < x < +\infty$ and $\nu_p(x) = 1$ if $x = \infty$. And $\nu_0 = \varepsilon_0$ is a PN space (see, [12]). In this space, since the set $\{\frac{1}{n} : n \in \mathbb{N}\}$ has strongly Cauchy subsequence then it is probabilistic strongly totally bounded but it is not D-bounded set (note that $\nu_p(x) = \exp(-\sqrt{|p|}) < 1$, for all $p \neq 0$). Note that in this space only $\{0\}$ is a D-bounded set.

Lemma 2.6 *In a characteristic Šerstnev space* (V, ν, τ) *every strongly Cauchy sequence is topologically bounded set.*

Proof. Let $\{p_m\}$ be a strongly Cauchy sequence. Then there exists a n_0 such that for every $m, n \geq n_0$, $\nu_{p_m-p_n} \geq \varepsilon_{\frac{1}{m+n}}$. Now let $\alpha_m \to 0$ and $0 < \alpha_m < 1$, then we have (by using a property of Šerstnev space in which $\nu_{\alpha_m p}(x) = \nu_p(x/|\alpha_m|) > \nu_p(x)$)

$$\nu_{\alpha_m p_m} \geq \tau(\nu_{\alpha_m(p_m-p_{n_0})}, \nu_{\alpha_m p_{n_0}}) \geq \tau(\nu_{p_m-p_{n_0}}, \nu_{\alpha_m p_{n_0}})$$

$$\geq \tau(\varepsilon_{\frac{1}{m+n_0}}, \nu_{\alpha_m p_{n_0}}) \to \tau(\varepsilon_0, \varepsilon_0) = \varepsilon_0,$$

as m tends to infinity.

Lemma 2.7 *In a characteristic Šerstnev space* (V,ν,τ) *every probabilistic strongly totally bounded set is D-bounded.*

Proof. We show that if A is a probabilistic strongly totally bounded set then it is topologically bounded, and so by Lemma 1.8, it is D-bounded. If A is not topologically bounded, there exists a sequence $\{p_m\}\subseteq A$ and a real sequence $\alpha_m \to 0$ such that $\alpha_m p_m$ doesn't tend to the null vector in V. There is an infinite set $J\subseteq N$ such that the sequence $\{\alpha_m p_m\}_{m\in J}$ stays off a neighborhood of the origin. Since $\{p_m\}$ is probabilistic strongly totally bounded, then has a Cauchy subsequence say $\{p_{m_l}\}$ which by Lemma 2.6 is topologically bounded and since $\alpha_{m_l}\to 0$ then $\nu_{\alpha_{m_l}p_{m_l}} \to \varepsilon_0$ and hence $\{\alpha_{m_l}p_{m_l}\}$ is a strongly Cauchy subsequence of $\{\alpha_m p_m\}$. Then $\{\alpha_m p_m\}$ is probabilistic strongly totally bounded and so is $\{\alpha_m p_m\}_{m\in J}$, therefore there is a strong Cauchy subsequence of $\{\alpha_m p_m\}_{m\in J}$, say $\alpha_{m_k}p_{m_k}$ which stays off a neighborhood of the origin, hence it doesn't tend to the null vector in V, on the other hand, since $\{\alpha_{m_k}p_{m_k}\}$ is a strongly Cauchy sequence then there is a $k_0\in \mathrm{N}$ such that for every $k,t\ge k_0$ we have $\nu_{p_{m_k}-p_{m_t}} \ge \varepsilon_{\frac{1}{k+t}}$. Thus

$$\nu_{\alpha_{m_k}p_{m_k}} \ge \tau(\nu_{\alpha_{m_k}(p_{m_k}-p_{m_{k_0}})},\nu_{\alpha_{m_k}p_{m_{k_0}}}) \ge \tau(\nu_{p_{m_k}-p_{m_{k_0}}},\nu_{\alpha_{m_k}p_{m_k}}) \ge \tau(\varepsilon_{\frac{1}{k+k_0}},\nu_{\alpha_{m_k}p_{m_k}})$$

$$\to \tau(\varepsilon_0,\varepsilon_0)=\varepsilon_0,$$

as k tends to infinity. Which is a contradiction.

Every D-bounded set is not probabilistic strongly totally bounded set, in general, as can see from the next example.

Example 2.8 Let $\nu: l^\infty \to \Delta^+$ via $\nu_p := \varepsilon_{\|p\|}$ for every $p\in l^\infty$. Let τ,τ^* be continuous triangle functions such that $\tau\le\tau^*$ and $\tau(\varepsilon_a,\varepsilon_b)=\varepsilon_{a+b}$, for all $a,b>0$. For instance,it suffices to take $\tau=\tau_T$ and $\tau^*=\tau_{T^*}$, where T is a continuous t--norm and T^* is its t--conorm. Then $(l^\infty,\nu,\tau,\tau^*)$ is a PN space (see [6, Example 1.1]). Suppose $A=\{p:\|p\|=1, p\in l^\infty\}$, A is D-bounded set but not probabilistic strongly totally bounded set. In fact

$$R_A(x)=\lim_{y\to x, y<x}\inf\{\varepsilon_{\|p\|}(y): p\in A\} = \to 1, (x\to +\infty).$$

therefore A is D-bounded. Let $\{p_n\}_1^\infty$ is a sequence of A, where

$p_1=(1,0,0,...,0,...)$, $p_2=(0,1,0,...,0,...)$,..., $p_n=(0,0,0,...,1,0,...)$,...

In view of Definition 1.3., It is obvious that $\{p_n\}_1^\infty$ is not strongly Cauchy sequence. By Theorem 2.4., we have that A is not probabilistic strongly totally bounded set.

Theorem 2.9 *Let* (V,ν,τ,τ^*) *be a PN space. If* A *and* B *are two probabilistic strongly totally bounded subsets of* V. *Then*

(i) $A\cup B$ is probabilistic strongly totally bounded;

(ii) $A+B$ is probabilistic strongly totally bounded, where the set A+B given by $A+B:=\{p+q: p\in A, q\in B\}$.

Proof. (i). By Definition 2.1., for every $F\in D^+$ and $F<\varepsilon_0$, there exist finite subset S_F of A and S'_F of B such that $A\subseteq \bigcup_{p\in S_F} D_p(F)$ *and* $B\subseteq \bigcup_{p\in S'_F} D_p(F)$, where $D_p(F)=\{q\in V:\nu_{p-q}>F\}$.

So we have that $A\cup B\subset \bigcup_{p\in S_F} D_p(F)\cup(\bigcup_{p\in S'_F} D_p(F))=\bigcup_{p\in S_F\cup S'_F} D_p(F)$. Thus $A\cup B$ is probabilistic strongly totally bounded.

(ii). Let $\{c_n\}$ is a sequence of $A+B$. Suppose $c_n=p_n+q_n$, where $\{p_n\}\in A$ and $\{q_n\}\in B$. Because A and B are probabilistic strongly totally bounded subsets, by Theorem 2.4., there exist subsequence $\{p_{k,n}\}$ of $\{p_n\}$ and $\{q_{k,n}\}$ of $\{q_n\}$, where $\{p_{k,n}\}$ and $\{q_{k,n}\}$ are both strongly Cauchy subsequences, i.e., $\nu_{p_{k,n}-p_{k,m}}\to\varepsilon_0,\ m,n\to\infty,\qquad \nu_{q_{k,n}-q_{k,m}}\to\varepsilon_0,\ m,n\to\infty$.So

$$\nu_{c_{k,n}-c_{k,m}}=\nu_{(p_{k,n}+q_{k,n})-(p_{k,m}+q_{k,m})}=\nu_{(p_{k,n}-p_{k,m})+(q_{k,n}-q_{k,m})}$$

$$\geq\tau(\nu_{(p_{k,n}-p_{k,m})},\nu_{(q_{k,n}-q_{k,m})})\to\tau(\varepsilon_0,\varepsilon_0)=\varepsilon_0,$$

as m,n tends to infinity, i.e., the subsequence $\{c_{k,n}\}$ of $\{c_n\}$ is a strongly Cauchy subsequence. By Theorem 2.4. we have that $A+B$ is probabilistic strongly totally bounded.

Corollary 2.10. *Let* (V,ν,τ,τ^*)*) be a PN space. Let* A_i *be probabilistic strongly totally bounded, where i=1,2,3,...,n.Then we have that* $\cup_{i=1}^n A_i$ *and* $\sum_{i=1}^n A_i$ *are all probabilistic strongly totally bounded, where* $\sum_{i=1}^n A_i := A_1+A_2+...+A_n$.

3. Conclusions

In this paper, we studied the concept of total boundedness in PN space and its relation to D-boundedness . We proved that A is a probabilistic strongly totally bounded set if and only if every sequence in A has a strongly Cauchy subsequence .Next we showed that every probabilistic strongly totally bounded set is not D-bounded set , in general.

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R E F E R E N C E S

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