A NONDIFFERENTIABLE EXTENSION OF A THEOREM OF PUCCI AND SERRIN AND APPLICATIONS

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Abstract. We study the multiplicity of critical points for functionals which are only differentiable along some directions. We extend to this class of functionals the three critical point theorem of Pucci and Serrin and we apply it to a one-parameter family of functionals J_λ, $\lambda \in I \subset \mathbb{R}$. Under suitable assumptions, we locate an open subinterval of values λ in I for which J_λ possesses at least three critical points. Applications to quasilinear boundary value problems are also given.

1. Introduction

For a C^1-functional J defined in a reflexive real Banach space $(X, \|\cdot\|_X)$ and satisfying the standard Palais-Smale compactness condition, it is proved in [9] (see also [10]) that there exists a third critical point provided that J has two local minima. The main aim of this paper is to extend this result to the case of functionals which are only differentiable along directions in a subspace $Y \subset X$. Specifically, Y denotes a subspace of X, which is itself a normed space endowed with a norm $\|\cdot\|_Y$ such that $(Y, \|\cdot\|_X + \|\cdot\|_Y)$ is a Banach space. We consider functionals $J : X \longrightarrow \mathbb{R}$ such that the restriction of J to Y is continuous with respect to the norm $\|\cdot\|_X + \|\cdot\|_Y$. We also assume that

a) J has a directional derivative $\langle J'(u), v\rangle$ at each $u \in X$ through any direction $v \in Y$.

b) For fixed $u \in X$, the function $\langle J'(u), v\rangle$ is linear in $v \in Y$, and, for fixed $v \in Y$, the function $\langle J'(u), v\rangle$ is continuous in $u \in X$.

This kind of functional has been considered in [2] where a suitable version of the classical Mountain Pass Theorem [1] was proved. Here we apply this, together with an argument based on the Ekeland Principle [7], in order to prove that if J has two local minima in Y with respect to the norm $\|\cdot\|_X$, then it has at least a third critical point in X. Here, by a critical point u for J we mean $u \in X$ such that $\langle J'(u), v\rangle = 0$ for every $v \in Y$.

As an application, this extension of the Pucci-Serrin theorem allows us to deduce a version for nondifferentiable functionals of the three critical point theorem in [11] (see also [6]). To be precise, we take into consideration a one-parameter ($\lambda \in \mathbb{R}$) family of coercive functionals $J_\lambda = \Phi + \lambda\Psi$, where $\Phi : X \longrightarrow \mathbb{R}$ is a weakly lower semicontinuous functional satisfying the conditions a) and b) and such that the

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restriction of Φ to Y is continuous with respect to the norm $\|\cdot\|_X + \|\cdot\|_Y$; while $\Psi : X \to \mathbb{R}$ is weakly lower semicontinuous and continuously Gateaux differentiable. We give sufficient conditions for the existence of a nonempty open interval $\Lambda \subset \mathbb{R}$ such that $\Phi + \lambda\Psi$ has at least

i) three critical points for $\lambda \in \Lambda$,
ii) two critical points for λ in the boundary of Λ, and
iii) one critical point for λ outside the closure of Λ.

Indeed, case i) is related to [11] where continuously Gateaux differentiable functionals are studied. In this particular framework of differentiable functionals, our proof is simpler than the previous one. In addition, our result improves it because, in contrast with [11] where only the existence of a three critical point interval is proved without a detailed description of it, we localize the interval Λ for the existence of three solutions (see Theorem 3.4). In [5, Theorem B] there is given a different localization of the interval; in the applications we show that, at least in some cases, our localization is better. We mention also that the existence of two critical points for λ in the boundary of the interval Λ seems to be new (even for the differentiable framework).

The last part of the paper is devoted to applications of the previous abstract theorems to boundary value problems associated with quasilinear equations. In particular, if $\Omega \subset \mathbb{R}^N$ is an open bounded set with smooth boundary, and $A(x, u)$ is a Carathodory function satisfying

$$\begin{aligned} &0 < \alpha \leq A(x,u) \leq \beta \quad \text{a.e. } x \in \Omega, \quad \forall u \in \mathbb{R}, \\ &|A'(x,u)| \leq \gamma \quad \text{a.e. } x \in \Omega, \quad \forall u \in \mathbb{R}, \\ &A'(x,u)u \geq 0 \quad \text{a.e. } x \in \Omega, \quad \forall |u| >> 0, \end{aligned} \tag{1.1}$$

then, under various hypotheses on the Carathodory nonlinearity $h(x, u)$, we study the existence of solutions of the equation

$$-\text{div}(A(x,u)\nabla u) + \tfrac{1}{2}A'(x,u)|\nabla u|^2 = \lambda h(x,u), \; x \in \Omega,$$

with zero Dirichlet or Neumann boundary conditions.

The paper is organized as follows. The extension of the three critical point theorem of Pucci and Serrin is given in Section 2. In the third section we deal with the existence of three critical points for a one-parameter family of functionals J_λ with λ in a general interval $I \subset \mathbb{R}$. In Section 4, we consider the existence of solutions for quasilinear boundary value problems.

2. A non-differentiable version of the Pucci-Serrin theorem

For the extension of the Pucci-Serrin three critical point theorem we need a suitable version of the Mountain Pass Theorem [1], which may be found in [2].

Theorem 2.1. *Let $(X, \|\cdot\|_X)$ be a real Banach space and $Y \subset X$ a subspace, which is itself a normed space endowed with a norm $\|\cdot\|_Y$, and such that Y equipped with the norm $\|\cdot\|_X + \|\cdot\|_Y$ is a Banach space. Assume that $J : X \longrightarrow \mathbb{R}$ is a functional on X satisfying the conditions a) and b) and such that the restriction of J to Y is continuous with respect to the norm $\|\cdot\|_X + \|\cdot\|_Y$. Assume that the following Palais-Smale condition is satisfied:*

(PS) *Let* $\{u_n\}$ *be a sequence in* Y *satisfying, for every* $n \in \mathbb{N}$,

$$|J(u_n)| \leq C,$$

$$\|u_n\|_Y \leq 2M_n,$$

$$|\langle J'(u_n), v\rangle| \leq \varepsilon_n \left[\frac{\|v\|_Y}{M_n} + \|v\|_X\right], \quad \forall v \in Y,$$

where C *is a positive constant,* $\{M_n\} \subset \mathbb{R}^+ - \{0\}$ *is any sequence and* $\{\varepsilon_n\} \subset \mathbb{R}^+$ *is a sequence converging to zero. Then* $\{u_n\}$ *has a convergent subsequence in* X.

If there exist $e_1, e_2 \in Y$, $e_1 \neq e_2$ *and* $r \in (0, \|e_2 - e_1\|_X)$ *such that*

$$\inf\{J(v) \,/\, \|v - e_1\|_X = r\} > \max\{J(e_1), J(e_2)\},$$

and we denote by Γ *the family of paths* $\gamma : [0,1] \longrightarrow (Y, \|\cdot\|_Y + \|\cdot\|_X)$ *joining* e_1 *and* e_2 $(\gamma(0) = e_1,\ \gamma(1) = e_2)$*, then*

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)) > \max\{J(e_1), J(e_2)\}$$

is a critical value for J.

We can now prove the nondifferentiable version of the Pucci-Serrin theorem [9].

Theorem 2.2. *Let* $(X, \|\cdot\|_X)$ *be a real Banach space and* $Y \subset X$ *a subspace, which is itself a normed space endowed with a norm* $\|\cdot\|_Y$*, and such that* Y *equipped with the norm* $\|\cdot\|_X + \|\cdot\|_Y$ *is a Banach space. Assume that* $J : X \longrightarrow \mathbb{R}$ *is a functional on* X *satisfying the conditions a) and b) and such that the restriction of* J *to* Y *is continuous with respect to the norm* $\|\cdot\|_X + \|\cdot\|_Y$*. Suppose also that the* (PS) *condition in the form stated above is satisfied.*

If J *has two local minima in* Y *with respect to the norm* $\|\cdot\|_X$*, then it has at least one more critical point in* X.

Proof. Assume that $e_1, e_2 \in Y$ are local minima for the restriction $J_{|(Y,\|\cdot\|_X)}$, i.e. there exists $\varepsilon_0 > 0$ such that

$$J(e_i) \leq J(v), \quad \text{if } v \in Y,\ \|v - e_i\|_X \leq \varepsilon_0, \quad i = 1, 2.$$

Assume without loss of generality that $J(e_2) \leq J(e_1)$.

Notice that if there exists $\varepsilon \in (0, \varepsilon_0)$ such that

$$J(e_1) < \inf\{J(v) \,/\, v \in Y,\ \|v - e_1\|_X = \varepsilon\} \tag{2.1}$$

then the above Mountain Pass Theorem implies that J has a third critical point.

In case (2.1) fails for every ε, we adapt the arguments in [8]. We fix $\varepsilon \in (0, \varepsilon_0)$ and we choose $\{v_n\} \in Y$ such that

$$\|v_n - e_1\|_X = \varepsilon, \quad J(v_n) \leq J(e_1) + \frac{1}{2n}, \quad \forall n \in \mathbb{N},$$

and $\delta > 0$ such that $0 < \varepsilon - \delta < \varepsilon + \delta < \varepsilon_0$. We point out that

$$\inf\{J(v) \,/\, v \in Y,\ \varepsilon - \delta \leq \|v - e_1\|_X \leq \varepsilon + \delta\} = J(e_1).$$

Therefore, if $M_n = 1 + \|v_n\|_Y$, applying the Ekeland variational principle [7] with Y equipped with the complete norm $\|\cdot\|_n \equiv \|\cdot\|_X + \|\cdot\|_Y / M_n$ we obtain a new sequence $z_n \in Y$ such that

$$\varepsilon - \delta \le \|z_n - e_1\|_X \le \varepsilon + \delta,$$

$$J(z_n) \le J(v_n) \le J(e_1) + \frac{1}{2n},$$

$$\|v_n - z_n\|_X + \frac{\|v_n - z_n\|_Y}{M_n} \le \frac{1}{\sqrt{n}} \tag{2.2}$$

and, for every $v \in Y$ such that $\varepsilon - \delta \le \|v - e_1\|_X \le \varepsilon + \delta$,

$$J(z_n) \le J(v) + \frac{1}{\sqrt{n}} \left[\|v - z_n\|_X + \frac{\|v - z_n\|_Y}{M_n} \right]. \tag{2.3}$$

From (2.2), $\|v_n - z_n\|_X \le 1/\sqrt{n}$ and hence, for large n, we have $\varepsilon - \delta < \|z_n - e_1\|_X < \varepsilon + \delta$. Hence, if we consider $w \in Y$ with $\|w\|_X \le 1$, we can assume that $v = z_n + tw$ satisfies $\varepsilon - \delta \le \|v - e_1\|_X \le \varepsilon + \delta$ for $t > 0$ small enough. By taking limits as t tends to zero, we deduce from (2.3) that

$$|\langle J'(z_n), w \rangle| \le \frac{1}{\sqrt{n}} \left[\|w\|_X + \frac{\|w\|_Y}{M_n} \right],$$

for n large enough.

In addition, $\|z_n\|_Y \le \|z_n - v_n\|_Y + \|v_n\|_Y \le M_n(1 + 1/\sqrt{n}) \le 2M_n$. The (PS) condition shows that there exists a subsequence $\{z_{n_k}\}$ which converges to some z, which is necessarily a critical point for J, with $\|z - e_1\|_X = \varepsilon$ (again from (2.2)) and hence is different from e_1 and e_2. □

Remark 2.3. In the applications to quasilinear elliptic partial differential equations, by regularity results, the functional J usually verifies

$$\text{every local minimizer in } X \text{ for } J \text{ belongs to } Y. \tag{2.4}$$

In that case, the assumption of the preceding theorem can be relaxed by imposing only the existence of two local minima in X. (Note also that (2.4) is trivially satisfied in the differentiable case, i.e. if $X = Y$).

3. Three critical point intervals.

In this section, we take a real interval I and for $\lambda \in I$ we consider a one-parameter family of coercive functionals $J_\lambda = \Phi + \lambda\Psi$, i.e. satisfying

$$\lim_{\|u\| \to +\infty} \Phi(u) + \lambda\Psi(u) = +\infty. \tag{3.1}$$

In addition, we suppose that X is reflexive and $\Phi : X \longrightarrow \mathbb{R}$ is a weakly lower semicontinuous functional satisfying the conditions a) and b) and such that the restriction of Φ to Y is continuous with respect to the norm $\|\cdot\|_X + \|\cdot\|_Y$; while $\Psi : X \to \mathbb{R}$ is weakly lower semicontinuous, continuously Gateaux differentiable and non-constant.

Taking into account that Ψ is weakly lower semicontinuous, the set $\Psi^{-1}(-\infty, r])$ is weakly closed, and thus the weakly lower semicontinuous and coercive functional $\Phi + \lambda\Psi$ attains its infimum on this set for every $r \in \Psi(X)$. If, in addition, Ψ is weakly (upper semi) continuous, $\Psi^{-1}([r, +\infty))$ is also weakly closed and, for $r \in \Psi(X)$ we deduce that the restriction of $\Phi + \lambda\Psi$ to $\Psi^{-1}([r, +\infty))$ attains its infimum. In the following two lemmas we give sufficient conditions to assure that these infima are in fact critical values.

Lemma 3.1. *Assume that* $r \in \Psi(X) \setminus \{\inf_{u\in X} \Psi(u)\}$. *Then, the infimum of* $\Phi + \lambda\Psi$ *in* $\Psi^{-1}((-\infty, r])$ *is attained at some point in* $\Psi^{-1}((-\infty, r))$ *provided that* $\lambda \in I$ *satisfies*

$$\lambda > \inf \left\{ \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \quad / \quad u \in \Psi^{-1}((-\infty, r)) \right\}. \tag{3.2}$$

Remark 3.2. Let $\lambda_0 \in I$ and $v_n \in X$ be such that

$$\lim_{n\to+\infty} \Phi(v_n) + \lambda_0\Psi(v_n) = \inf_{\Psi^{-1}(r)} [\Phi + \lambda_0\Psi] \in [-\infty, +\infty).$$

Since $\Phi + \lambda_0\Psi$ is coercive, we can assume that v_n is weakly convergent to some $v \in X$. Using that $\Phi + \lambda_0\Psi$ is weakly lower semicontinuous, we get

$$\inf_{\Psi^{-1}(r)} [\Phi + \lambda_0\Psi] = \lim_{n\to+\infty} \Phi(v_n) + \lambda_0\Psi(v_n) \geq \Phi(v) + \lambda_0\Psi(v) > -\infty.$$

In particular,

$$\inf_{\Psi^{-1}(r)} \Phi = -\lambda_0 r + \inf_{\Psi^{-1}(r)} [\Phi + \lambda_0\Psi] > -\infty.$$

Therefore $(\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u))/(\Psi(u) - r) < +\infty$ for every $u \in \Psi^{-1}((-\infty, r))$ and the infimum in $\Psi^{-1}((-\infty, r))$ appearing in (3.2) is strictly smaller than $+\infty$.

Proof. Let be $\inf_X \Psi \neq r \in \Psi(X)$. Since Ψ is weakly lower semicontinuous, $\Psi^{-1}((-\infty, r])$ is weakly closed and (3.1) implies that the restriction of $\Phi + \lambda\Psi$ to $\Psi^{-1}((-\infty, r])$ attains its infimum at some $u_\lambda \in \Psi^{-1}((-\infty, r])$.

Observe that if $\Psi(u_\lambda) = r$ then

$$\begin{aligned} \inf_{\Psi^{-1}(r)} \Phi &\leq \Phi(u_\lambda) = \Phi(u_\lambda) + \lambda\left(\Psi(u_\lambda) - r\right) \\ &\leq \Phi(u) + \lambda\left(\Psi(u) - r\right), \quad \forall u \in \Psi^{-1}((-\infty, r]), \end{aligned}$$

which yields

$$\lambda \leq \inf \left\{ \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \quad / \quad u \in \Psi^{-1}((-\infty, r)) \right\}.$$

Therefore, if $\lambda \in I$ verifies (3.2), it has to be satisfied that $\Psi(u_\lambda) < r$ and hence that u_λ is a local minimizer. $\square$

Similarly, the following result can be proved.

Lemma 3.3. *Assume that* Ψ *is also weakly (upper semi) continuous and let be* $r \in \Psi(X) \setminus \{\sup_{u\in X} \Psi(u)\}$. *Then the infimum of* $\Phi + \lambda\Psi$ *in* $\Psi^{-1}([r, +\infty))$ *is attained at some point in* $\Psi^{-1}((r, +\infty))$ *provided that* $\lambda \in I$ *satisfies*

$$\lambda < \sup \left\{ \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \quad / \quad u \in \Psi^{-1}((r, +\infty)) \right\}. \tag{3.3}$$

$\square$

By convenience we will denote in the sequel by φ_1 and φ_2 the functions given by

$$\varphi_1(r) = \inf \left\{ \frac{\inf\limits_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \quad / \quad u \in \Psi^{-1}((-\infty, r)) \right\}, \tag{3.4}$$

$$\varphi_2(r) = \sup \left\{ \frac{\inf\limits_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \quad / \quad u \in \Psi^{-1}((r, +\infty)) \right\}, \tag{3.5}$$

for every $r \in (\inf_{u\in X} \Psi(u), \sup_{u\in X} \Psi(u))$. We can now prove the main theorem of this section.

Theorem 3.4. *Let $(X, \|\cdot\|_X)$ be a reflexive real Banach space and $Y \subset X$ a subspace, which is itself a normed space endowed with a norm $\|\cdot\|_Y$, and such that $(Y, \|\cdot\|_X + \|\cdot\|_Y)$ is a Banach space. Assume that $\Phi : X \longrightarrow \mathbb{R}$ is a weakly lower semicontinuous functional on X satisfying the conditions a) and b) and such that the restriction of Φ to Y is continuous with respect to the norm $\|\cdot\|_X + \|\cdot\|_Y$. Let also $\Psi : X \to \mathbb{R}$ be a continuously Gateaux differentiable functional with compact derivative Ψ'. Assume that* (3.1) *holds and that $\Phi + \lambda\Psi$ satisfies (PS) for every λ in some real interval I. Let us suppose that*

$$\text{there exists } r \in \left(\inf_{u\in X} \Psi(u), \sup_{u\in X} \Psi(u) \right) \text{ such that } \varphi_1(r) < \varphi_2(r). \tag{3.6}$$

Then

i) *The functional $\Phi + \lambda\Psi$ admits at least one critical point for every $\lambda \in I$.*

ii) *Even more, in case $(\varphi_1(r), \varphi_2(r)) \cap I \neq \emptyset$,*
   - a) *If $J \equiv \Phi + \lambda\Psi$ satisfies* (2.4) *then it has at least three critical points for every $\lambda \in (\varphi_1(r), \varphi_2(r)) \cap I$.*
   - b) *If $\varphi_1(r) \in I$ then $\Phi+\lambda\Psi$ has at least two critical points for $\lambda = \varphi_1(r)$.*
   - c) *If $\varphi_2(r) \in I$ then $\Phi+\lambda\Psi$ has at least two critical points for $\lambda = \varphi_2(r)$.*

Remarks 3.5.
i) Assume that $(\varphi_1(r), \varphi_2(r)) \cap I \neq \emptyset$. If the interval I contains a point in $(-\infty, \varphi_1(r)]$, then $\varphi_1(r) \in I$ and this theorem states that for every $\lambda \in I$, there exist, respectively one, two or three critical points provided that, respectively $\lambda < \varphi_1(r)$, $\lambda = \varphi_1(r)$ or $\varphi_1(r) < \lambda < \varphi_2(r)$. A similar remark can be done if the interval I contains points to the right of $\varphi_2(r)$.

ii) For differentiable functionals, the existence of a three critical point interval (without a detailed description of it) is proved in [11]. Here we locate the three critical point interval. A previous location related with Theorem 3.4 was given in [5, Theorem B], where only the case $I = [0, +\infty)$ is considered and the assumptions involve weak closure. We have in some cases a bigger three critical point interval. See Remarks 3.9 below.

iii) We remark explicitly that in the boundary of the three critical point interval we state the existence of at least two critical points.

Proof. Thanks to the previous two lemmas we infer that $\Phi + \lambda\Psi$ admits a local minimum in $\Psi^{-1}(-\infty, r)$ for every $\lambda \in I \cap (\varphi_1(r), +\infty)$ and it also admits a local minimum in $\Psi^{-1}(r, +\infty)$ for every $\lambda \in I \cap (-\infty, \varphi_2(r))$. Thus, since $\varphi_1(r) < \varphi_2(r)$, case i) follows. Moreover, we have just proved the existence of two local minima for every $\lambda \in (\varphi_1(r), \varphi_2(r)) \cap I$ if this intersection is not empty. Now the proof of case ii)-a) follows directly from the Theorem 2.2.

In order to prove case ii)-b) (a similar argument works for ii)-c)), let us suppose, in addition to $(\varphi_1(r), \varphi_2(r)) \cap I \neq \emptyset$, that $\varphi_1(r) \in I$. From Lemma 3.1 there exist sequences $\{\lambda_n\} \subset (\varphi_1(r), \varphi_2(r)) \cap I$ and $\{u_n\} \subset X$ such that

$$\lambda_n \searrow \varphi_1(r),$$
$$\Psi(u_n) < r, \quad \Phi(u_n) + \lambda_n \Psi(u_n) = \inf_{u \in \Psi^{-1}(-\infty, r)} \Phi(u) + \lambda_n \Psi(u).$$

Since $\lambda_n \leq \lambda_1$ and $\Psi(u_n) < r$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \Phi(u_n) \quad + & \quad \lambda_1(\Psi(u_n) - r) \leq \limsup_{n\to\infty} \Phi(u_n) + \lambda_n(\Psi(u_n) - r) \\
= & \quad \limsup_{n\to\infty} \left[\inf_{u \in \Psi^{-1}(-\infty, r)} \Phi(u) + \lambda_n(\Psi(u) - r) \right] \\
= & \quad \inf_{u \in \Psi^{-1}(-\infty, r)} \Phi(u) + \varphi_1(r)(\Psi(u) - r) \\
& \quad + \limsup_{n\to\infty} (\lambda_n - \varphi_1(r))(\Psi(u) - r) \\
\leq & \quad \Phi(u) + \varphi_1(r)(\Psi(u) - r),
\end{aligned}$$

for every $u \in \Psi^{-1}(-\infty, r)$. By (3.1), this implies that u_n is bounded. Then, up to a subsequence, u_n is weakly convergent to some $u \in \Psi^{-1}((-\infty, r])$. Taking into account that $\Phi'(u_n) + \lambda_n \Psi'(u_n) = 0$, we get for every $v \in Y$,

$$\begin{aligned}
\langle \Phi'(u_n) + \varphi_1(r)\Psi'(u_n)), v\rangle \quad &= \quad \langle \Phi'(u_n) + \lambda_n \Psi'(u_n), v\rangle + \langle (\varphi_1(r) - \lambda_n)\Psi'(u_n), v\rangle \\
&= \quad \langle (\varphi_1(r) - \lambda_n)\Psi'(u_n), v\rangle.
\end{aligned}$$

Using that Ψ' is compact, we have

$$|\langle \Phi'(u_n) + \varphi_1(r)\Psi'(u_n)), v\rangle| \leq \varepsilon_n \|v\|_X, \quad \forall v \in Y,$$

with $\varepsilon_n \longrightarrow 0$. Furthermore, by recalling that Ψ is weakly continuous (since Ψ' is compact), the sequence $\{\Psi(u_n)\}$ is bounded. Consequently, from the convergence of λ_n to $\varphi_1(r)$, we deduce that

$$\Phi(u_n) + \varphi_1(r)\Psi(u_n) = \Phi(u_n) + \lambda_n \Psi(u_n) + (\varphi_1(r) - \lambda_n)\, \Psi(u_n)$$

is bounded.

Since the functional $\Phi + \varphi_1(r)\Psi$ satisfies the (PS) condition, we see that, up to a subsequence, u_n strongly converges to u and by the continuity assumption b)

$$\langle \Phi'(u) + \varphi_1(r)\Psi'(u), v\rangle = \lim_{n\to\infty} \langle \Phi'(u_n) + \lambda_n \Psi'(u_n), v\rangle = 0, \quad \forall v \in Y,$$

i.e. u is a critical point for $\Phi + \varphi_1(r)\Psi$.

To finish the proof of case ii) we observe that $u \in \Psi^{-1}((-\infty, r])$ is a critical point different from the local minimum in $\Psi^{-1}(r, +\infty)$ given by Lemma 3.3. □

Remark 3.6. Some remarks about the hypothesis (3.6) are in order. We begin by observing that it is equivalent to the following one

$$\text{There exist } r \in \left(\inf_{u\in X} \Psi(u), \sup_{u \in X} \Psi(u) \right) \text{ and } u_1, u_2 \in X \text{ such that}$$
$$\Psi(u_1) < r < \Psi(u_2) \text{ and}$$
$$\inf_{v \in \Psi^{-1}(r)} \Phi(v) > \frac{(\Psi(u_2) - r)\Phi(u_1) - (\Psi(u_1) - r)\Phi(u_2)}{\Psi(u_2) - \Psi(u_1)}.$$

Remark 3.7. For $u_1, u_2 \in X$ such that $\Psi(u_1) < r < \Psi(u_2)$, the quotient

$$\frac{(\Psi(u_2) - r)\Phi(u_1) - (\Psi(u_1) - r)\Phi(u_2)}{\Psi(u_2) - \Psi(u_1)}$$

is a convex combination of $\Phi(u_1)$ and $\Phi(u_2)$ and so (3.6) implies, by the previous remark, that $\inf_{v\in\Psi^{-1}(r)} \Phi(v) > \inf_{u\in X} \Phi(u)$.

The converse is not true in general. Indeed, the condition

$$\inf_{v\in\Psi^{-1}(r)} \Phi(v) > \inf_{u\in X} \Phi(u)$$

leads to one of the following three possibilities:

i) $\inf_{v\in\Psi^{-1}(r)} \Phi(v) = \inf_{u\in\Psi^{-1}([r,+\infty))} \Phi(u) > \inf_{u\in\Psi^{-1}((-\infty,r])} \Phi(u)$, which implies that $\varphi_1(r) < 0$ and $\varphi_2(r) \le 0$.

ii) $\inf_{v\in\Psi^{-1}(r)} \Phi(v) = \inf_{u\in\Psi^{-1}((-\infty,r])} \Phi(u) > \inf_{u\in\Psi^{-1}([r,+\infty))} \Phi(u)$, which implies that $\varphi_1(r) \ge 0$ and $\varphi_2(r) > 0$.

iii) $\inf_{v\in\Psi^{-1}(r)} \Phi(v) > \max\left\{\inf_{u\in\Psi^{-1}((-\infty,r])} \Phi(u), \inf_{u\in\Psi^{-1}([r,+\infty))} \Phi(u)\right\}$, which implies that $\varphi_1(r) < 0 < \varphi_2(r)$.

In particular, we have proved that a sufficient condition for hypothesis (3.6) is that

$$\inf_{v\in\Psi^{-1}(r)} \Phi(v) > \max\left\{\inf_{u\in\Psi^{-1}((-\infty,r])} \Phi(u), \inf_{u\in\Psi^{-1}([r,+\infty))} \Phi(u)\right\}.$$

(Moreover, in this case, $\varphi_1(r) < 0 < \varphi_2(r)$).

The following corollary is a improvement of [12, Theorem 2].

Corollary 3.8. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be defined as in Theorem 3.4. Suppose that there exist $u_1, u_2 \in X$ and $r \in \mathbb{R}$ such that*

i) $\Psi(u_1) < r < \Psi(u_2)$,

ii) $\Phi(u_1), \Phi(u_2) < \inf_{u\in\Psi^{-1}(r)} \Phi(u)$.

Then the assertion of Theorem 3.4 holds and $\varphi_1(r) < 0 < \varphi_2(r)$.

Proof. The proof is a direct consequence of Theorem 3.4, since case iii) of Remark 3.7 applies. □

Remarks 3.9. i) In [12, Theorem 2], the case of continuously Gateaux differentiable functional Φ, i.e. continuously Frchet differentiable, is studied. In contrast with our result above, the author of that reference imposes additionally that $I = \mathbb{R}$ and that $\Phi(u_1) = \Phi(u_2) = \inf_{u\in X} \Phi(u)$.

ii) We would like to stress that our localization of the three critical point interval improves the previous one given in [5]. For instance, under the conditions of the preceding corollary with $I = [0, +\infty)$, we deduce the existence of three critical points for every $\lambda \in [0, \varphi_2(r))$. In contrast, in [5] any possible three critical point interval has the form $]a, b[$ with $a > 0$.

Lemmas 3.1 and 3.3 can also be applied to obtain a perturbation result that slightly improves on Theorem 3.4.

Theorem 3.10. *Let X be a reflexive real Banach space, and let Φ, Ψ defined as in Theorem 3.4. Let $\Psi_1 : X \to \mathbb{R}$ a continuously Gateaux differentiable functional with compact derivative Ψ_1'. Assume that for every λ in some real interval I and every $\mu \in [-\eta, \eta]$,*

$$\lim_{\|u\| \to +\infty} \Phi(u) + \lambda(\Psi(u) + \mu\Psi_1(u)) = +\infty, \tag{3.7}$$

and that $\Phi + \lambda(\Psi + \mu\Psi_1)$ satisfies (PS) and (2.4). Let us suppose (3.6) where $\varphi_1(r)$ and $\varphi_2(r)$ are given respectively by (3.4) and (3.5). If $(\varphi_1(r), \varphi_2(r)) \cap I \neq \emptyset$, then for each interval $[a, b] \subset (\varphi_1(r), \varphi_2(r)) \cap I$ there exists $\delta \in (0, \eta)$ such that if $|\mu| < \delta$, the functional $\Phi + \lambda(\Psi + \mu\Psi_1)$ admit at least three critical points for every $\lambda \in [a, b]$.

Remark 3.11. This theorem is applied in Subsection 4.2 to improve some previous results in [11] on the existence of solutions of Dirichlet boundary value problems.

Proof. Thanks to Lemma 3.1 and Lemma 3.3 the functional $\Phi + \lambda\Psi$ has a local minimum in $\Psi^{-1}((-\infty, r))$ and $\Psi^{-1}((r, +\infty))$.

We are going to prove that for some $\delta > 0$ the functional $\Phi + \lambda(\Psi + \mu\Psi_1)$ still has a local minimum in the interior of each of the sets provided that $\lambda \in [a, b]$ and $|\mu| \leq \delta$. This implies the existence of two critical points of $\Phi + \lambda(\Psi + \mu\Psi_1)$. The third one follows arguing as in Theorem 3.4. Let us deal with the local minimum in the interior of $\Psi^{-1}([r, +\infty))$, similar ideas allows to conclude for $\Psi^{-1}((-\infty, r])$.

First we denote by Θ the functional given by:

$$\Theta(u) = \Phi(u) + \min_{\kappa \in \{0,1\}} \{a\Psi(u) \pm a\eta\Psi_1(u) + \kappa(b-a)r, b\Psi(u) \pm b\eta\Psi_1(u) + \kappa(a-b)r\}$$

for every $u \in X$. By (3.7) (with $\lambda = a, b$ and $\mu = -\eta, \eta$), we deduce that Θ is coercive. Thus, for any arbitrary fixed $v \in \Psi^{-1}([r, +\infty))$, there exists $R > 0$ such that if $u \in X$ satisfies $\|u\| > R$ then

$$\begin{aligned} \Theta(u) &> 1 + \Phi(v) + (|b| + |a|)|\Psi(v)| + (|b| + |a|)|\eta||\Psi_1(v)| \\ &\geq 1 + \Phi(v) + \lambda\Psi(v) + \lambda\mu\Psi_1(v), \end{aligned} \tag{3.8}$$

for every $\lambda \in [a, b]$ and $\mu \in [-\eta, \eta]$. If, in addition $u \in \Psi^{-1}(r)$, we have

$$\Theta(u) = \min\{\Phi(u) + \lambda\Psi(u) + \overline{\lambda}\mu\Psi_1(u) \ / \ \lambda, \overline{\lambda} \in \{a, b\},\ \mu \in \{-\eta, \eta\}\},$$

and, using that $a \leq \lambda \leq b$ and $-\eta \leq \mu \leq \eta$, it follows from (3.8) that

$$\Phi(u) + \lambda\Psi(u) + \lambda\mu\Psi_1(u) \geq \Theta(u) > 1 + \Phi(v) + \lambda\Psi(v) + \lambda\mu\Psi_1(v),$$

for every $u \in \Psi^{-1}(r)$ such that $\|u\| > R$. Therefore,

$$\begin{aligned} \inf_{\Psi(u) = r, \|u\| > R} \Phi(u) + \lambda\Psi(u) &+ \lambda\mu\Psi_1(u) > \Phi(v) + \lambda\Psi(v) + \lambda\mu\Psi_1(v) \\ &\geq \inf_{u \in \Psi^{-1}([r, +\infty))} \Phi(u) + \lambda\Psi(u) + \lambda\mu\Psi_1(u). \end{aligned}$$

If we denote by $u_{\lambda,\mu} \in \Psi^{-1}([r, +\infty))$ the infimum of the functional $\Phi + \lambda\Psi + \lambda\mu\Psi_1$ in $\Psi^{-1}([r, +\infty))$, i.e.

$$\Phi(u_{\lambda,\mu}) + \lambda\Psi(u_{\lambda,\mu}) + \lambda\mu\Psi_1(u_{\lambda,\mu}) = \inf_{u \in \Psi^{-1}([r, +\infty))} \Phi(u) + \lambda(\Psi(u) + \mu\Psi_1(u)),$$

then only one of the following possibilities may occur:

i) $u_{\lambda,\mu} \in \Psi^{-1}(r, +\infty)$,
ii) $u_{\lambda,\mu} \in \Psi^{-1}(r) \cap \overline{B}(0, R)$.

We will choose $\delta > 0$ such that for every $\lambda \in [a,b]$ and $\mu \in [-\delta,\delta]$ only the case i) is possible. To do that, let us recall that for $\varepsilon = (\varphi_2(r)-b)/2 > 0$, there exists $u_\varepsilon \in \Psi^{-1}(r,+\infty)$ such that

$$\lambda + \varepsilon \le \varphi_2(r) - \varepsilon < \frac{\inf_{\Psi^{-1}(r)} \Phi - \Phi(u_\varepsilon)}{\Psi(u_\varepsilon) - r}, \quad \forall \lambda \in [a,b].$$

Hence

$$\begin{aligned}
\inf_{\Psi^{-1}(r)} \Phi &> \Phi(u_\varepsilon) + \lambda(\Psi(u_\varepsilon) - r) + \varepsilon(\Psi(u_\varepsilon) - r) \\
&= \Phi(u_\varepsilon) + \lambda(\Psi(u_\varepsilon) - r) + \lambda\mu\Psi_1(u_\varepsilon) + \varepsilon(\Psi(u_\varepsilon) - r) - \lambda\mu\Psi_1(u_\varepsilon) \\
&\ge \Phi(u_{\lambda,\mu}) + \lambda(\Psi(u_{\lambda,\mu}) - r) + \lambda\mu\Psi_1(u_{\lambda,\mu}) + \varepsilon(\Psi(u_\varepsilon) - r) - \lambda\mu\Psi_1(u_\varepsilon).
\end{aligned}$$

Therefore if $u_{\lambda,\mu} \in \Psi^{-1}(r) \cap \overline{B}(0,R)$ we have

$$\Phi(u_{\lambda,\mu}) + \lambda(\Psi(u_{\lambda,\mu}) - r) = \Phi(u_{\lambda,\mu}) \ge \inf_{\Psi^{-1}(r)} \Phi$$

and thus

$$\begin{aligned}
\inf_{\Psi^{-1}(r)} \Phi &> \inf_{\Psi^{-1}(r)} \Phi + \lambda\mu(\Psi_1(u_{\lambda,\mu}) - \Psi_1(u_\varepsilon)) + \varepsilon(\Psi(u_\varepsilon) - r) \\
&\ge \inf_{\Psi^{-1}(r)} \Phi - \max\{|a|,|b|\}|\mu| \max_{u \in \Psi^{-1}(r) \cap \overline{B}(0,R)} |(\Psi_1(u) - \Psi_1(u_\varepsilon))| \\
&\quad + \varepsilon(\Psi(u_\varepsilon) - r),
\end{aligned}$$

proving that

$$\begin{aligned}
|\mu| &> \frac{\varepsilon(\Psi(u_\varepsilon) - r)}{\max\{|a|,|b|\} \max_{u \in \Psi^{-1}(r) \cap \overline{B}(0,R)} |(\Psi_1(u) - \Psi_1(u_\varepsilon))|} \\
&\ge \min\left\{\eta, \frac{\varepsilon(\Psi(u_\varepsilon) - r)}{\max\{|a|,|b|\} \max_{u \in \Psi^{-1}(r) \cap \overline{B}(0,R)} |(\Psi_1(u) - \Psi_1(u_\varepsilon))|}\right\} \equiv \delta.
\end{aligned}$$

This means that if $|\mu| \le \delta$, then the case i) holds, concluding the proof. $\square$

4. Applications to nonlinear boundary value problems

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with smooth boundary and $A(x,u)$ a Carathodory function satisfying (1.1). Let also $h(x,u)$ be a subcritical Carathodory nonlinearity. We study the existence of solutions of the boundary value problem

$$-\mathrm{div}(A(x,u)\nabla u) + \frac{1}{2}A'(x,u)|\nabla u|^2 = \lambda h(x,u), \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial\Omega.$$

Indeed, this will be addressed in Subsections 4.1 and 4.2, while in the last application of the section we will consider the corresponding Neumann boundary value problem.

4.1. **Application 1.** Here we consider the case $h(u) = u + f(u)$, where

$$|f(s)| \le C(1 + |s|^{q-1}), \quad \forall s \in \mathbb{R}, \tag{4.1}$$

for some positive constant C and $1 \le q < 2$, that is, we are considering the boundary value problem

$$\begin{gathered} -\mathrm{div}(A(x,u)\nabla u) + \frac{1}{2}A'(x,u)|\nabla u|^2 = \lambda(u + f(u)), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{4.2}$$

We set $X = H_0^1(\Omega)$ with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2$ and define the functionals $\Phi, \Psi : H_0^1(\Omega) \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\int_\Omega A(x,u)|\nabla u|^2, \quad \Psi(u) = -\frac{1}{2}\int_\Omega u^2 - \int_\Omega F(u), \quad \forall u \in H_0^1(\Omega),$$

where $F(s) = \int_0^s f$ for every $s \in \mathbb{R}$. Observe that Ψ is continuously Gateaux differentiable in X. Moreover, by [2], Φ is weakly lower semicontinuous satisfying the conditions a) and b) with $Y = H_0^1(\Omega) \cap L^\infty(\Omega)$ ($\|\cdot\|_Y = \|\cdot\|_\infty$). Thanks to (4.1) we see also that Ψ' is compact. Furthermore, this hypothesis also implies that, for some $C_1 > 0$,

$$|F(s)| \le C_1(1 + |s|^q), \quad \forall s \in \mathbb{R}.$$

This, together with the inequality $A(x,u) \ge \alpha$, assures that if λ_1 denotes the first eigenvalue of the Laplacian operator with zero Dirichlet boundary condition and associated positive eigenfunction ϕ_1 with $\|\phi_1\|_2 = 1$, then for every $u \in H_0^1(\Omega)$,

$$\begin{aligned} \Phi(u) + \lambda\Psi(u) &\ge \frac{\alpha}{2}\|u\|^2 - \frac{\lambda}{2\lambda_1}\|u\|^2 - |\lambda| C_1 \left(|\Omega| + \|u\|_q^q\right) \\ &\ge \left(\frac{\alpha}{2} - \frac{\lambda}{2\lambda_1}\right)\|u\|^2 - |\lambda| C_1 \left(|\Omega| + \|u\|_2^{\frac{q}{2}} |\Omega|^{1-\frac{q}{2}}\right) \\ &\ge \left(\frac{\alpha}{2} - \frac{\lambda}{2\lambda_1}\right)\|u\|^2 - |\lambda| C_1 \left(|\Omega| + \lambda_1^{\frac{q}{2}} \|u\|^q |\Omega|^{1-\frac{q}{2}}\right). \end{aligned}$$

Hence $\Phi + \lambda\Psi$ satisfies (3.1) for every $\lambda \in (-\infty, \alpha\lambda_1)$. In addition, by [2], $\Phi + \lambda\Psi$ satisfies (PS) and (2.4).

Theorem 4.1. *Let* (1.1) *and* (4.1) *be satisfied and assume that for some* $\gamma > 2$,

$$\limsup_{s \to 0} \frac{|F(s)|}{|s|^\gamma} < +\infty. \tag{4.3}$$

If $\int_\Omega F(\phi_1) > (\beta/\alpha - 1)/2$, *then there exists* $\underline{\lambda} < \alpha\lambda_1$ *such that problem* (4.2) *admits at least two non trivial solutions for every* $\lambda \in (\underline{\lambda}, \alpha\lambda_1)$.

Remarks 4.2.

i) Notice that (4.3) implies that $f(0) = 0$ and thus $u = 0$ is a trivial solution of (4.2).
ii) It will be observed in the proof of the theorem that

$$\underline{\lambda} = \varphi_1(0) = \inf_{u \in \Psi^{-1}((-\infty,0))} -\frac{\Phi(u)}{\Psi(u)},$$

where the function φ_1 is given by (3.4).

Proof. Consider the functions φ_1, φ_2 given respectively by (3.4) and (3.5). We observe that the hypothesis on $\int_\Omega F(\phi_1)$ guaranties that ϕ_1 belongs to $\Psi^{-1}(-\infty, 0)$ and

$$\varphi_1(0) = \inf\left\{-\frac{\Phi(u)}{\Psi(u)} \quad / \quad u \in \Psi^{-1}((-\infty,0))\right\} \le \frac{-\frac{1}{2}\int_\Omega A(x,\phi_1)|\nabla\phi_1|^2}{-\frac{1}{2}\int_\Omega |\phi_1|^2 - \int_\Omega F(\phi_1)}.$$

By (1.1) we obtain

$$\varphi_1(0) \le \frac{\beta\lambda_1}{1 + 2\int_\Omega F(\phi_1)} < \alpha\lambda_1. \tag{4.4}$$

The proof (with $\underline{\lambda} = \varphi_1(0)$) will be concluded applying the Theorem 3.4 if we show that for every compact interval $\Lambda \in (\varphi_1(0), \alpha\lambda_1)$ there exists $r < 0$ such that $\varphi_1(r) < \varphi_2(r)$ and $\Lambda \subset (\varphi_1(r), \varphi_2(r))$. To prove it, note that for every $u \in \Psi^{-1}(-\infty, 0)$ we have

$$\varphi_1(r) \le \frac{\inf\limits_{\Psi(v)=r} \Phi(v) - \Phi(u)}{\Psi(u) - r} \le -\frac{\Phi(u)}{\Psi(u) - r}, \quad \forall r \in (\Psi(u), 0).$$

This implies that

$$\limsup_{r\to 0^-} \varphi_1(r) \le -\frac{\Phi(u)}{\Psi(u)}, \quad \forall u \in \Psi^{-1}(-\infty, 0),$$

or equivalently

$$\limsup_{r\to 0^-} \varphi_1(r) \le \varphi_1(0). \tag{4.5}$$

On the other hand, it is deduced from (4.1) and (4.3) that

$$|F(s)| \le c|s|^\gamma, \quad \forall s \in \mathbb{R},$$

with $c > 0$ and (without loss of generality) $0 < \gamma < 2^*$, where 2^* denotes the Sobolev exponent, i.e. $2^* = 2N/(N-2)$ if $N \ge 3$, while $2^* = +\infty$ if $N = 2$.

For every $u \in H_0^1(\Omega)$ we infer from the Sobolev embedding that

$$|\Psi(u)| \le \frac{1}{2}\|u\|_2^2 + c\|u\|_\gamma^\gamma \le \frac{1}{2\lambda_1}\|u\|^2 + c_1\|u\|^\gamma,$$

where, in the sequel, we denote by $c_1, c_2, ...$ positive constants. Thus, given $r < 0$ and $u \in \Psi^{-1}(r)$, we obtain from (1.1)

$$\alpha(-r) = \alpha(-\Psi(u)) \le \frac{\alpha}{2\lambda_1}\|u\|^2 + \alpha c_1\|u\|^\gamma \le \frac{1}{\lambda_1}\Phi(u) + c_2\Phi(u)^{\frac{\gamma}{2}}.$$

In particular, if we choose $u_0 \in H_0^1(\Omega)$ such that $\Phi(u_0) = \inf_{u\in\Psi^{-1}(r)} \Phi(u)$ and use that $\varphi_2(r) \geq -(1/r)\inf_{u\in\Psi^{-1}(r)} \Phi(u)$, we get

$$\begin{aligned}\alpha &\leq \frac{1}{\lambda_1}\frac{\Phi(u_0)}{-r} + c_2\frac{\Phi(u_0)^{\frac{\gamma}{2}}}{-r} \\ &= \frac{1}{\lambda_1}\frac{\Phi(u_0)}{-r} + c_3(-r)^{\frac{\gamma}{2}-1}\left(\frac{\Phi(u_0)}{-r}\right)^{\frac{\gamma}{2}} \\ &\leq \frac{1}{\lambda_1}\varphi_2(r) + c_3(-r)^{\frac{\gamma}{2}-1}\varphi_2(r)^{\frac{\gamma}{2}},\end{aligned}$$

which assures, since $\gamma > 2$, that $\liminf_{r\to 0^-} \varphi_2(r) \geq \alpha\lambda_1$. This inequality, together with (4.4) and (4.5), implies that, for any given compact interval $\Lambda \subset (\varphi_1(0), \alpha\lambda_1)$, we can choose $r < 0$ such that

$$\varphi_1(r) < \inf\Lambda \leq \sup\Lambda < \varphi_2(r).$$

□

Remark 4.3. For the sake of simplicity, we will assume in this remark that $\alpha = \beta = 1$. In this case, observe that the equation in the problem (4.2) is a semilinear one. The previous theorem asserts that there exist two nontrivial solutions provided that λ belongs to an interval to the left of $\lambda = \lambda_1$. We remark explicitly that, even in this simple case, this existence result does not seem easily obtained by applying Bifurcation Theory. Indeed, it is a consequence of the assumption (4.1) that the problem is asymptotically linear at infinity and that λ_1 is a bifurcation point from infinity. Also, if it is additionally assumed that $\lim_{s\to 0} f(s)/s = 0$, we would deduce that $\lambda = \lambda_1$ is a bifurcation point from zero. However, as it is well-known [3], extra hypotheses on the behavior of $f(s)$ for s either near to infinity or near to zero must be imposed in order to be able to decide, for instance, if both bifurcations are to the left of $\lambda = \lambda_1$.

Remark 4.4. We remark explicitly that in the preceding proof we have shown that if $\inf_{\Psi^{-1}(0)} \Phi = 0$ and Ψ takes negative values, then (4.5) holds.

4.2. **Application 2.** Now we deal with the case $h(s) = f(s) + \mu g(s)$, where f, g satisfy (4.1) and $\mu \in \mathbb{R}$, i.e. with the boundary value problem

$$\begin{aligned}&-\mathrm{div}(A(x,u)\nabla u) + \frac{A'(x,u)}{2}|\nabla u|^2 = \lambda(f(u) + \mu g(u)),\ x \in \Omega, \\ &u = 0,\ x \in \partial\Omega.\end{aligned} \tag{4.6}$$

This has been considered in [11] for the case $A(x,u) \equiv 1$. Here, in addition to generalize the results of that work, we give more information about the location of the three solution interval.

We set again $X = H_0^1(\Omega)$ with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2$, and define the functionals $\Phi, \Psi, \Psi_1 : H_0^1(\Omega) \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\int_\Omega A(x,u)|\nabla u|^2, \quad \Psi(u) = -\int_\Omega F(u), \quad \Psi_1(u) = -\int_\Omega G(u), \qquad \forall u \in H_0^1(\Omega),$$

where $F(s) = \int_0^s f$ and $G(s) = \int_0^s g$, for every $s \in \mathbb{R}$. Note that Ψ, Ψ_1 are continuously Gateaux differentiable in X and, thanks to (4.1), Ψ' and Ψ_1' are compact. We observe that we are considering the same functional Φ of the previous application and as before $\Phi + \lambda\left(\Psi + \mu\Psi_1\right)$ satisfies (3.7), (2.4) and (PS) for every $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$.

To apply Theorem 3.10, we just have to look for $r \in \Psi(H_0^1(\Omega))$ such that $\varphi_1(r) < \varphi_2(r)$. In the following lemma we give conditions to assure that Ψ takes negative or positive values.

Lemma 4.5. *$F^+ \not\equiv 0$ if and only if $\Psi(H_0^1(\Omega)) \cap \mathbb{R}^- \neq \emptyset$. Similarly, $F^- \not\equiv 0$ if and only if $\Psi(H_0^1(\Omega)) \cap \mathbb{R}^+ \neq \emptyset$.*

Proof. Clearly if $F^+ \equiv 0$ then $\Psi(H_0^1(\Omega)) \subset \mathbb{R}^+ \cup \{0\}$. On the other hand, if $F^+ \not\equiv 0$ there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$. Given $\varepsilon > 0$ we can choose an open bounded subset $\Omega_\varepsilon \subset\subset \Omega$ with $|\Omega \setminus \overline{\Omega}_\varepsilon| < \varepsilon$ and $w_\varepsilon \in C^\infty(\Omega)$ with compact support in Ω such that $w_\varepsilon \equiv s_0$ in Ω_ε and $\|w_\varepsilon\|_\infty = |s_0|$. Therefore we conclude that

$$\begin{aligned} \Psi(w_\varepsilon) &= -\int_\Omega F(w_\varepsilon) \\ &= -\int_{\Omega_\varepsilon} F(w_\varepsilon) - \int_{\Omega\setminus\Omega_\varepsilon} F(w_\varepsilon) \\ &\leq -F(s_0)|\Omega_\varepsilon| + \varepsilon \max_{|s|\leq s_0} |F(s)| \end{aligned}$$

and hence taking limits for $\varepsilon \to 0$ that

$$\limsup_{\varepsilon\to 0^+} \Psi(w_\varepsilon) \leq -F(s_0)|\Omega| < 0.$$

$\square$

Theorem 4.6. *Let* (1.1), (4.1) *and* (4.3) *be satisfied and $F^+ \not\equiv 0$. Then for each compact and non degenerate interval $[a,b] \subset (\varphi_1(0), \infty)$ there exists $\delta > 0$ such that if $|\mu| < \delta$, then problem* (4.6) *admits at least three solutions for every $\lambda \in [a,b]$.*

Proof. By Lemma 4.5, $\Psi^{-1}(-\infty, 0) \neq \emptyset$ and then (4.5) holds (see Remark 4.4).

On the other hand, we recall that (4.3) and (4.1) assure that, for some positive constant c,

$$|F(s)| \leq c|s|^\gamma, \quad \forall s \in \mathbb{R}.$$

Thus, for every $u \in H_0^1(\Omega)$ we have

$$|\Psi(u)| \leq c\|u\|_\gamma^\gamma \leq c_1\|u\|^\gamma,$$

where, in the sequel, we denote by $c_1, c_2, ...$ positive constants. Therefore, given $r < 0$ and $u \in \Psi^{-1}(r)$, we obtain

$$-r = -\Psi(u) \leq c_2\|u\|^\gamma.$$

This implies that ($A(x,u) \geq \alpha$)

$$\varphi_2(r) \geq \frac{\alpha}{2} \frac{\inf\limits_{u\in\Psi^{-1}(r)} \|u\|^2}{-r} \geq c_3 \frac{(-r)^{\frac{2}{\gamma}}}{-r}$$

and hence that

$$\lim_{r\to 0^-} \varphi_2(r) = +\infty. \tag{4.7}$$

For any compact interval $\Lambda \subset (\varphi_1(0), +\infty)$, conditions (4.5) and (4.7) allow to choose $r < 0$ such that

$$\varphi_1(r) < \inf\Lambda \leq \sup\Lambda < \varphi_2(r).$$

The proof is concluded by applying Theorem 3.10. □

Remark 4.7. If, in addition to the hypotheses of the preceding theorem, we assume $\mu = 0$, then there exist at least two nontrivial solutions of (4.6) for every $\lambda \in (\varphi_1(0), \infty)$. Indeed, the proof of this claim is similar to the previous one by applying Theorem 3.4 instead of Theorem 3.10.

4.3. **Application 3.** Given $c(x) \in L^\infty(\Omega)$ with $c(x) > 0$, $m \in (1,2)$ and a Carathodory function $f : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfying (4.1), we study as a third application the Neumann boundary value problem associated with the equation

(4.8)
$$-\mathrm{div}(A(x,u)\nabla u) + \frac{A'(x,u)}{2}|\nabla u|^2 = c(x)\left(|u|^{m-2}u - u\right) + \lambda f(x,u),\ x \in \Omega,$$

i.e., we are looking for $u \in H^1(\Omega)$ such that

$$\int_\Omega A(x,u)\nabla u\nabla v + \int_\Omega \frac{A'(x,u)}{2}|\nabla u|^2 v = \int_\Omega c(x)\left(|u|^{m-2}u - u\right)v - \lambda\int_\Omega f(x,u)v,$$

for every $v \in H^1(\Omega)$. This problem was studied in [12] for the semilinear case ($\alpha = \beta = 1$).

In order to set this problem in our abstract setting we take $X = H^1(\Omega)$ with the norm

$$\|u\|_c = \left(\int_\Omega |\nabla u|^2 + \int_\Omega c(x)u^2\right)^{1/2}.$$

Observe that, since $c(x) > 0$ this norm is equivalent to the usual one. We also set $\Phi, \Psi : H^1(\Omega) \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\int_\Omega A(x,u)|\nabla u|^2 + \frac{1}{2}\int_\Omega c(x)u^2 - \frac{1}{m}\int_\Omega c(x)|u|^m,$$

$$\Psi(u) = -\int_\Omega F(x,u), \quad \forall u \in H^1(\Omega),$$

where $F(x,s) = \int_0^s f(x,t)dt$ for every $x \in \Omega$ and $s \in \mathbb{R}$. Note that Ψ is continuously Gateaux differentiable in X. Moreover, by [2, 4], Φ is weakly lower semicontinuous satisfying the conditions a) and b) with $Y = H^1(\Omega) \cap L^\infty(\Omega)$ ($\|\cdot\|_Y = \|\cdot\|_\infty$). Thanks to (4.1) we also see that Ψ' is compact. Furthermore, this hypothesis also implies that, for some $C_1 > 0$,

$$|F(x,s)| \le C_1(1 + |s|^q), \quad \forall s \in \mathbb{R}.$$

This, together with the inequality $A(x,u) \ge \alpha$, assures that for every $u \in H^1(\Omega)$ and $\lambda \in \mathbb{R}$ we have

$$\Phi(u) + \lambda\Psi(u) \quad \ge \quad \frac{1}{2}\min\{\alpha, 1\}\|u\|_c^2 - \frac{\|c\|_\infty}{m}\|u\|_m^m - |\lambda|C_1\left(|\Omega| + \|u\|_q^q\right),$$

from which, as before, it is deduced that $\Phi + \lambda\Psi$ is coercive. In addition, from [2, 4], $\Phi + \lambda\Psi$ satisfies the (PS) condition.

Theorem 4.8. *Let* (4.1) *be satisfied. Assume that*

$$\int_\Omega F(x,1) \neq \int_\Omega F(x,-1). \tag{4.9}$$

Then there exist $\tau_1 < 0$ *and* $\tau_2 > 0$ *such that for every* $\lambda \in (\tau_1, \tau_2)$ *the Neumann boundary value problem associated with the equation* (4.8) *has at least three solutions.*

Proof. It is easy to check that $w = 1$ and $w = -1$ are the only global minima of Φ. From (4.9) we deduce that either $\Psi(-1) < \Psi(1)$ or $\Psi(1) < \Psi(-1)$ and hence that we can choose $r \in \mathbb{R}$ such that either $\Psi(-1) < r < \Psi(1)$ or $\Psi(1) < r < \Psi(-1)$. Since $1, -1 \notin \Psi^{-1}(r)$ we have

$$\Phi(1), \Phi(-1) < \inf_{u \in \Psi^{-1}(r)} \Phi(u)$$

and we can apply Corollary 3.8 to conclude the proof. □

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