On two problems studied by A. Ambrosetti

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Dedicated to Antonio Ambrosetti on the occasion of his sixtieth birthday

Abstract

In this paper, we study the Ambrosetti-Prodi and Ambrosetti-Rabinowitz problems. We prove for the first one the existence of a continuum of solutions with shape of a reflected C ($\supset$-shape). Next, we show that there is a relationship between these two problems.

1 Introduction

In the early seventies, Antonio Ambrosetti wrote in collaboration with Giovanni Prodi [3] and Paul H. Rabinowitz [4] two of the most seminal papers in the theory of Nonlinear Functional Analysis. Both of them have become key-stones in the study of the existence and multiplicity of solutions for nonlinear P.D.E.

The problem considered in [3] is

$$\begin{aligned} -\Delta u = f(u) + h, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where f is, roughly speaking, a nonlinearity whose derivative *crosses* the first eigenvalue associated with the Laplacian operator with zero Dirichlet boundary condition. By using their abstract result, they are able to describe the exact number of solutions in terms of h.

On the other hand, in [4], using the Mountain Pass Theorem, the authors prove the existence of a positive (nontrivial and nonnegative) solution for the problem

$$\begin{aligned} -\Delta u &= f(x,u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{2}$$

where, roughly speaking, f is a superlinear nonlinearity ($\lim_{u\to+\infty} \frac{f(x,u)}{u} = +\infty$) with $f(x,0) = 0$ a.e. $x \in \Omega$ and zero derivative with respect to u at zero.

These problems are known in the literature respectively as "Ambrosetti-Prodi" and "Ambrosetti-Rabinowitz" problems. Given the large amount of papers written on this subject , it is an almost impossible task to give an original look of them. However, at least, our wish is to convince the reader that these "Ambrosetti problems" are not so different as they seem.

The ideas rely on the study of some quasilinear extensions of the results in [5] (see also [6]). Indeed, in [5] we study the case of a non-variational differential operator, while in [6] the p-laplacian operator ($p > 1$) is considered. The intrinsic difficulties of the considered quasilinear equations are the reason for having developed there a new bifurcation approach to the Ambrosetti-Prodi problem. Moreover, we deduce from it the quasilinear Ambrosetti-Rabinowitz result, unifying in this way those two classical results. As a tribute to the pioneering works by Ambrosetti et al. we devote this note to a survey of these results in the simpler semilinear case. We remark explicitly that, in this case, the interest is to give a new perspective in applying the Leray-Schauder topological degree, in conjunction with the remarkable *a priori* bound of B. Gidas and J. Spruck [16], instead of variational methods. The interested reader can see [5] for the more complicated non-variational quasilinear case.

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2 Continua of solutions for Ambrosetti-Prodi type problems

In [3] (see also [2]), A. Ambrosetti and G. Prodi, using some results on the global inversion of differentiable mappings between Banach spaces, proved the following existence result for (1). In order to state it, we denote by $\{\lambda_k\}$ the sequence of eigenvalues for the Laplacian operator with zero Dirichlet boundary condition.

Theorem 2.1 (Ambrosetti-Prodi, 1972) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^{2,\alpha}$-boundary ($0 < \alpha < 1$) and let $f \in C^2(\mathbb{R})$ satisfy:*

(i) $f(0) = 0$*,*

(ii) $f''(s) > 0$ *for every* $s \in \mathbb{R}$*,*

(iii) $\lim_{s\to-\infty} f'(s) = f'(-\infty) < \lambda_1 < f'(+\infty) = \lim_{s\to+\infty} f'(s) < \lambda_2$.

Then, there exists a connected and closed C^1*-manifold,* $\mathcal{M}$*, of codimension 1 in* $C^{0,\alpha}(\overline{\Omega})$ *such that* $C^{0,\alpha}(\overline{\Omega}) \setminus \mathcal{M}$ *has exactly two connected components,* $\mathcal{A}_1$*,* $\mathcal{A}_2$ *with the following properties:*

(a) *if* $h \in \mathcal{A}_1$*, the problem* (1) *has no solution in* $C^{2,\alpha}(\overline{\Omega})$*,*

(b) *if* $h \in \mathcal{A}_2$*, it has exactly two solutions in* $C^{2,\alpha}(\overline{\Omega})$*,*

(c) *if* $h \in \mathcal{M}$*, it admits a unique solution in* $C^{2,\alpha}(\overline{\Omega})$*.*

Thus, a precise description of the number of solutions of (1) is given. This result has been the motivation of a very large amount of works on the number of solutions of b.v.p. with nonlinearities $f(x,u)$ whose derivative *jumps* the first eigenvalue λ_1 or higher eigenvalues. It would be impossible to give here a complete list of references, but, at least, we wish to cite some of the most classical ones and we refer the reader to the surveys by D. G. de Figueiredo [11, 12] for a more extensive list. In [8], M. S. Berger and E. Podolak write the function h as $h = t\varphi + \tilde{h}$, where $t \in \mathbb{R}$, φ is a positive function and $\tilde{h}$ is orthogonal to φ in $L^2(\Omega)$. Existence and multiplicity of solutions are described in terms of the values of t. Fixed point theory is applied by H. Amann and P. Hess in [1] and also by S. Fučik, [14] who introduces the term *jumping* nonlinearity. In particular, in [1] $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that there exist positive functions $f'_{\pm\infty}(x) \in L^\infty(\Omega)$ satisfying

$$\lim_{s\to\pm\infty} \frac{f(x,s)}{s} = f'_{\pm\infty}(x), \quad \text{uniformly in } x \in \Omega. \tag{3}$$

The result in [1] states:

Theorem 2.2 *Suppose that* $\varphi \in C^{0,\alpha}(\overline{\Omega})$ *with* $\varphi > 0$ *in* Ω*. Assume also* (3) *and that there exists a positive constant* ε *such that for a.e.* $x \in \Omega$

$$f'_{-\infty}(x) < \lambda_1 - \varepsilon < \lambda_1 < \lambda_1 + \varepsilon < f'_{+\infty}(x) < \lambda_2 - \varepsilon < \lambda_2. \tag{4}$$

Then, there exists $t^* \in \mathbb{R}$ *with the following properties:*

(a) *if* $t > t^*$*, the problem*

$$\begin{aligned} -\Delta u = f(x,u) + t\varphi, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{aligned} \tag{P_t}$$

has no solution.

(b) *if* $t = t^*$*, it has, at least, one solution.*

(c) *if* $t < t^*$*, it admits, at least, two solutions.*

A different approach based on Morse theory was given by H. Berestycki [7]. We must mention also the papers by D.G. de Figueiredo and S. Solimini [13] and K.C. Chang [10] where the case of subcritical superlinear nonlinearities jumping all eigenvalues λ_k, i.e. satisfying for some $\varepsilon > 0$

$$f'_{-\infty}(x) < \lambda_1 - \varepsilon < f'_{+\infty}(x) \equiv +\infty, \quad \text{a.e. } x \in \Omega, \tag{5}$$

is considered. The main tool in [13] is the Mountain Pass Theorem, developed for the Ambrosetti-Rabinowitz problem. Indeed, they prove that one solution of (P_t) can be obtained by the sub- and super-solution method. As clarified by H. Brézis and L. Nirenberg [9], such solution is a local minimum with respect to the $H^1_0(\Omega)$-topology of the associated Euler functional. Then, the second one is obtained by application of the cited Mountain Pass Theorem (provided that, in addition, the nonlinearity f satisfies the technical Ambrosetti-Rabinowitz condition [4] for the standard Palais-Smale condition). We remark also that the case of a critical superlinear nonlinearity is studied in [9] by using this variational technique.

As far as we are concerned the motivation of our results lies in the study of some quasilinear elliptic equations ([5] and [6]). Other works for quasilinear Ambrosetti-Prodi problems are [17, 18]. The particular difficulties of these do not allow to extend the previous ideas and, consequently, we have developed a new approach based on proving the existence of a continuum of solutions with $\supset$-shape (see Remark 2.7). We consider the two classes of cited nonlinearities, i.e. either the case in which the asymptotically linear nonlinearity f satisfies (3) and (4) or the superlinear case of nonlinearities satisfying (3) and (5). In addition, in this case, we suppose that there exists $h(x) \in L^\infty(\Omega)$ and $1 < p < 2^* - 1$ such that

$$\lim_{s \to +\infty} \frac{f(x,s)}{s^p} = h(x)c > 0, \quad \text{uniformly in } x \in \Omega. \tag{6}$$

Here 2^* stands for the Sobolev critical exponent, i.e. $2^* = 2N/(N-2)$, if $N \geq 3$, while $2^* = +\infty$, provided $N = 2$.

The proof of the result applies the sub-super-solution method and the Leray-Schauder degree in conjunction with the following *a priori* bound on the $C^0(\overline{\Omega})$-norm of the solutions.

Lemma 2.3 *Let $\varphi \in L^\infty(\Omega)$ be a positive function. Suppose that either*

(i) *conditions* (3) *and* (4) *hold,*

or

(ii) *f satisfies* (3), (5) *and* (6).

Then the solutions of (P_t) are uniformly bounded in compact sets of t, i.e., for every compact interval $\Gamma \subset \mathbb{R}$, there exists $c \in \mathbb{R}$ such that every solution u of (P_t) with $t \in \Gamma$ satisfies

$$|u(x)| \leq c, \ \forall x \in \Omega.$$

Remark 2.4 We remark that, in the case of superlinear nonlinearities satisfying Assumptions (ii), the above a priori bound is an easy extension of the result by B. Gidas and J. Spruck in [16].

Proof. In the case of assumptions (i), the proof is standard [2]. We prove the lemma if (ii) holds. First we show that, given a compact real interval Γ, there exists a positive constant c such that

$$u(x) > -c, \; x \in \Omega, \tag{7}$$

for every solution u of (P_t) with $t \in \Gamma$. In order to prove this a priori bound, we observe that taking $u^- \equiv \min\{u, 0\}$ as test function in the equation satisfied by u, and by using hypothesis (5), we get an uniform bound in the $H_0^1(\Omega)$-norm of u^-. Consider now, for each $k \in \mathbb{R}$, the function G_k given by

$$G_k(s) = \begin{cases} s+k, & \text{if } s \leq -k, \\ 0, & \text{if } -k < s \leq k, \\ s-k, & \text{if } k < s. \end{cases}$$

Thus, taking $v = G_k(u^-)$ as test function in the equation satisfied by u we obtain that

$$\int_\Omega |\nabla G_k(u^-)|^2 = \int_{\Omega_k} (f(x,u^-) + t\varphi) G_k(u^-),$$

where $\Omega_k \equiv \{x \in \Omega : \; u(x) < -k\}$. From (5), there exists a positive constant C such that

$$f(x,s) + t\varphi \geq Cs, \quad \forall s \leq -k, \quad \forall t \in \Gamma.$$

We deduce from above that

$$\int_\Omega |\nabla G_k(u^-)|^2 \leq C \int_{\Omega_k} |u^-||G_k(u^-)|.$$

Using now the Sobolev and Hölder inequalities, we get for some new constant C and $r > 2N/(N+2)$,

$$\|G_k(u^-)\|_{2^*}^2 \leq C\|u^-\|_r \|G_k(u^-)\|_{2^*} (\text{meas } \Omega_k)^{(1-1/r-1/2^*)}.$$

(For $r \geq 1$, we are denoting $\|\cdot\|_r$ the usual norm of the Lebesgue space $L^r(\Omega)$). Notice now that for every $h \geq k$, $|G_k(u^-)| \geq h-k$ in Ω_h, which implies that

$$(h-k)(\text{meas } \Omega_h)^{1/2^*} \leq C\|u^-\|_r (\text{meas } \Omega_k)^{(1-1/r-1/2^*)},$$

or equivalently that

$$\text{meas } \Omega_h \leq \frac{C\|u^-\|_r^{2^*} (\text{meas } \Omega_k)^{(2^*-1-2^*/r)}}{(h-k)^{2^*}}.$$

Using ideas of Stampacchia [19], we deduce the existence of a positive constant c such that $\|u^-\|_\infty \leq c$, for every solution u of (P_t) with $t \in \Gamma$. Therefore, (7) has been proved.

On the other hand, denote $v = u + c \geq 0$ and $\tilde{f}(x,s) = f(x, s-c)$. Thus, v satisfies

$$-\Delta v = \tilde{f}(x,v) + t\varphi, \quad x \in \Omega,$$
$$v = c, \quad x \in \partial\Omega.$$

We observe that the result in [16] remains true for solutions of the equation with bounded Dirichlet data instead of zero Dirichlet data. This result gives the existence of $\tilde{c} \in \mathbb{R}^+$ such that $v(x) \leq \tilde{c}$, for every $x \in \Omega$, i.e. u is bounded from above. □

We need also the following abstract theorem about the existence of a $\supset$-shaped continuum. The proof can be found in [5].

Lemma 2.5 *Let E be a Banach space and $T : \mathbb{R} \times E \to E$ a compact operator. Let us denote by Σ the set of pairs $(t,u) \in \mathbb{R} \times E$ such that u is a solution of*

$$u - T(t,u) = 0. \tag{8}$$

Let U be a bounded subset in E, such that (8) has no solution on ∂U while $t \in [a,b]$. Assume also that, for $t = b$, (8) has no solution in $\overline{U}$. Let $U_1 \subset U$ such that, for $t = a$, (8) has no solution on ∂U_1 and $\deg(I - T(a,\cdot), U_1, 0) \neq 0$. Then there exists a continuum C in Σ such that

$$C \cap (\{a\} \times U_1) \neq \emptyset, \quad C \cap (\{a\} \times (U \setminus \overline{U_1})) \neq \emptyset.$$

Now, we prove the main result of this section.

Theorem 2.6 *Let $\varphi \in L^\infty(\Omega)$ be a positive function and let $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a C^1-function satisfying either*

i) *conditions* (3) *and* (4),

or

ii) *conditions* (3), (5) *and* (6).

Then $t^* \equiv \sup\{t \in \mathbb{R} : (P_t) \text{ admits a solution}\}$ is finite and for every $t_0 < t^*$ there exists a continuum $\mathcal{C}$ in $\Sigma \equiv \{(t,u) \in \mathbb{R} \times C_0^1(\overline{\Omega}) : u \text{ solution of } (P_t)\}$ satisfying that

1. *$[t_0, t^*] \subset Proj_{\mathbb{R}}\, \mathcal{C}$.*
2. *For every $t \in [t_0, t^*)$, $Proj_{C_0^1(\overline{\Omega})}\, \left[\mathcal{C} \cap (\{t\} \times C_0^1(\overline{\Omega}))\right]$ contains two distinct solutions of (P_t) .*

Remark 2.7 We observe that, roughly speaking, the continuum $\mathcal{C}$ of solutions in $\mathbb{R} \times C_0^1(\overline{\Omega})$ emanates from $\{t_0\} \times C_0^1(\overline{\Omega})$, reaches $\{t^*\} \times C_0^1(\overline{\Omega})$ and then, it turns left to meet a different solution in $\{t_0\} \times C_0^1(\overline{\Omega})$ ($\supset$-shaped continuum). As a consequence,

1. (P_t) has, at least, two solutions for $t < t^*$,

2. (P_t) has, at least, one solution for $t \leq t^*$,

3. (P_t) has no solution for every $t > t^*$.

Proof. Let us denote $S \equiv \{t \in \mathbb{R} : (P_t) \text{ admits a solution}\}$. First we show that S is not the empty set. This relies upon two facts [12]:

- (P_t) has a supersolution for some $t \in \mathbb{R}$,
- given a supersolution, $\overline{u}$, of (P_t) for some $t \in \mathbb{R}$, there exists a subsolution $\underline{u}$ of it, such that $\underline{u} < \overline{u}$ in Ω.

Indeed, by means of the sub and super solution method, this implies that S is a closed interval unbounded from below. Moreover, the usual trick of multiplying by one positive eigenfunction associated to λ_1 leads to the nonexistence of solution for $t >> 0$ large enough and thus S is bounded from above. This means that the supremum of the closed interval S is attained. Denote

$$t^* \equiv \sup S = \max S.$$

We now prove the existence of the continuum of solutions. First we observe that, from Lemma 2.3 jointly with some regularity results, if $t_0 < t^* < t_1$, there exists $R > 0$ such that $\|u\|_{C^1} < R$ for each solution u of (P_t) with $t \in [t_0, t_1]$. Denote by Φ_t the map $\Phi_t(u) \equiv u - (-\Delta)^{-1}(f(x,u) + t\varphi)$. Using the homotopy invariance of Leray-Schauder degree and that problem (P_{t_1}) has no solution, we get

$$\deg(\Phi_t, B_R(0), 0) = \deg(\Phi_{t_1}, B_R(0), 0) = 0, \quad \forall t \in [t_0, t_1],$$

where $B_R(0)$ denotes the open ball in $C_0^1(\overline{\Omega})$ of radius R centered at zero.

Let u^* be a solution of (P_{t^*}). Observe that u^* is a super-solution of (P_t) for every $t \in [t_0, t^*)$ and it is not a solution. Moreover, as it has been mentioned above, there exists a sub-solution $u_{t_0} < u^*$ of (P_{t_0}) which is not a solution. Clearly u_{t_0} is also a sub-solution and no solution for (P_t) if $t \in [t_0, t^*)$. Consider the set

$$U_1 = \{u \in C_0^1(\overline{\Omega}) : u_{t_0} < u < u^* \text{ in } \Omega, \ \frac{\partial u^*}{\partial n} < \frac{\partial u}{\partial n} < \frac{\partial u_{t_0}}{\partial n} \text{ on } \partial\Omega\} \cap B_R(0).$$

The strong maximum principle implies the nonexistence of solutions of (P_t) on ∂U_1 if $t < t^*$ (see [15]). Hence, the degree of Φ_t is well defined in this set U_1. In addition, by using the results in [11],

$$\deg(\Phi_t, U_1, 0) = 1, \ \forall t \in [t_0, t^*).$$

Applying Lemma 2.5 with $E = C_0^1(\overline{\Omega})$, $[a,b] = [t_0, t_1]$ and $U = B_R(0)$, we deduce the existence of a continuum $\mathcal{C}$ in Σ such that

$$\mathcal{C} \cap (\{t_0\} \times U_1) \neq \emptyset,$$

and

$$\mathcal{C} \cap (\{t_0\} \times [B_R(0) \setminus \overline{U_1}]) \neq \emptyset.$$

In particular, the continuum $\mathcal{C}$ crosses $\{t\} \times \partial U_1$, for some $t \in (t_0, t^*]$. It has been observed that this is possible if and only if $t = t^*$. This concludes the proof.

3 Ambrosetti-Rabinowitz via Ambrosetti-Prodi

Let us consider a C^1-function $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x, 0) = 0$, ($x \in \Omega$), satisfying (6) and

$$\limsup_{s \to 0^+} \frac{f(x, s)}{s} \leq \gamma < \lambda_1, \quad \text{uniformly } x \in \Omega. \tag{9}$$

We are interested in the existence of positive solutions for the b.v.p.

$$\begin{aligned} -\Delta u = f(x, u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{aligned} \tag{10}$$

As usual, we can assume that f is extended to all of $\Omega \times \mathbb{R}$ by setting $f(x, s) = 0$ for $s < 0$. Notice that by the maximum principle, every nontrivial solution of (10) with the extended nonlinearity is positive.

Since f satisfies conditions (3), (5) and (6), problem (10) fits into the "superlinear" Ambrosetti-Prodi framework (problem (P_t) with $t = 0$ and any φ). As an easy application of Theorem 2.6, we deduce

Theorem 3.1 *Let $f \in C^1(\overline{\Omega} \times \mathbb{R}^+)$ satisfy* (6) *and* (9)*. Then problem* (10) *has, at least, one positive solution.*

Proof. To apply the framework of the previous sections we embed (10) into the one parameter problem

$$\begin{aligned} -\Delta u = f(x, u) + t\varphi_1, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{aligned} \tag{Q_t}$$

Observe that this problem is (P_t) with $\varphi = \varphi_1$. From Theorem 2.6, let t^* denote the supremum of all $t \in \mathbb{R}$ such that (Q_t) has solution. The proof is concluded if we prove that $t^* > 0$. Indeed, if this is done, that theorem (see also item 1. (with $t = 0$) of Remark 2.7) shows the existence of, at least, two solutions of (Q_0), or equivalently, the existence of two nonnegative solutions of (10). Taking into account that zero is solution of this problem, we deduce that there exists, at least, one positive solution of (10).

In order to prove that $t^* > 0$, we use the sub-super-solution method. First, we note that $u = 0$ is a subsolution of (Q_t) for every $t > 0$. Thus, we only have to find a positive supersolution of (Q_t), for some $t > 0$. This is easily deduced as follows. From (9), we take δ small enough such that $\dfrac{f(x, \delta\varphi_1)}{\delta\varphi_1} \leq \lambda_1 - \delta$, for every $x \in \Omega$. Then

$$-\Delta(\delta\varphi_1) = \lambda_1 \delta\varphi_1 \geq f(x, \delta\varphi_1) + \delta^2 \varphi_1,$$

This means that $\delta\varphi_1$ is a super-solution of (Q_t) with $t = \delta^2 > 0$ concluding the proof.

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