Quasilinear elliptic problems interacting with its asymptotic spectrum☆

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Abstract

Under suitable assumptions on the coefficients of the matrix $A(x,u)$ and on the nonlinear term $f(x,u)$, we study the quasilinear problem in bounded domains $\Omega \subset \mathbb{R}^N$

$$\begin{aligned} -\mathrm{div}(A(x,u)\nabla u) &= f(x,u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned}$$

We extend the semilinear results of Landesman–Lazer (J. Math. Mech. 19 (1970) 609) and of Ambrosetti–Prodi (in: A Primer on Nonlinear Analysis, Cambridge University Press, Cambridge, 1993) for resonant problems. The existence of positive solution is also considered extending to the quasilinear case the classical result by Ambrosetti–Rabinowitz (J. Funct. Anal. 14 (1973) 349). In this case, the result is obtained as a corollary of the previous multiplicity result in the Ambrosetti–Prodi framework.

Keywords: Quasilinear elliptic equations; Bifurcation theory; Resonance; Jumping nonlinearities

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary $\partial\Omega$. We consider here the boundary value problem

$$\begin{aligned} -\mathrm{div}(A(x,u)\nabla u) &= f(x,u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

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where $A(x,s) := (a_{ij}(x,s))$, $i,j = 1,\dots,N$ is a symmetric matrix with coefficients $a_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ which are Carathéodory functions (i.e., $a_{ij}(x,s)$ is measurable with respect to x for all $s \in \mathbb{R}$ and continuous in s for almost everywhere (a.e.) $x \in \Omega$). We assume that there exist positive constants α and β satisfying for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and a.e. $x \in \Omega$,

$$|A(x,s)| \leqslant \beta, \tag{A$_1$}$$

$$A(x,s)\xi \cdot \xi \geqslant \alpha|\xi|^2, \tag{A$_2$}$$

$$|A(x,s) - A(x,t)| \leqslant \omega(|s-t|), \quad \forall s,t \in \mathbb{R}, \tag{A$_3$}$$

with $\omega : \mathbb{R}^+ \to \mathbb{R}$ being some Osgood function, that is,

$$\omega \text{ is not decreasing}, \quad \omega(0) = 0, \quad \int_{0^+} \frac{\mathrm{d}s}{\omega(s)} = +\infty.$$

Let $f(x,s)$ be a Carathéodory function satisfying for some positive constants c_1, c_2, that

$$|f(x,s)| \leqslant c_1|s| + c_2, \quad \text{a.e. } x \in \Omega, \ \forall s \in \mathbb{R}. \tag{f$_1$}$$

To establish the definition of solution, we consider the usual Sobolev space $H_0^1(\Omega)$ (endowed with the norm $\|u\| = \|\nabla u\|_2$) and the space $E = C_0(\bar{\Omega})$ of the continuous functions in $\bar{\Omega}$ which vanish on $\partial\Omega$ (endowed with the norm $\|u\|_0 = \sup_\Omega |u|$). Thanks to (A$_1$) and (f$_1$), for a solution of this problem we mean a function $u \in H_0^1(\Omega) \cap E$ satisfying

$$\int_\Omega A(x,u)\nabla u \cdot \nabla v = \int_\Omega f(x,u)v, \quad \forall v \in H_0^1(\Omega).$$

We note that hypotheses (A$_{1\text{–}2}$) mean that the nonlinear differential operator $Q : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by

$$Q(u) = -\mathrm{div}(A(x,u)\nabla u), \quad u \in H_0^1(\Omega),$$

is continuous (by (A$_1$)) and coercive (by (A$_2$)). If $h \in H^{-1}(\Omega)$, the classical result by Leray and Lions [23] allows to deduce the existence of a weak solution u of the problem

$$Q(u) = h, \quad u \in H_0^1(\Omega),$$

i.e., a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega A(x,u)\nabla u \cdot \nabla v = \int_\Omega hv, \quad \forall v \in H_0^1(\Omega).$$

In addition, under (A$_3$), by [6, Théorèm 1.3], this solution u is unique.

As a consequence of this, assuming (A$_{1\text{–}3}$), we can consider the inverse $K : H^{-1}(\Omega) \to H_0^1(\Omega)$ of the operator Q. In addition, K is continuous and compact in $L^2(\Omega)$ (see for instance [4, Remark 2.2]). Moreover, if $q > N/2$, by De Giorgi–Stampacchia Theorem (see [15, Theorem II]; [17, Theorem 8.29] or [26, Theorem 7.3]), K maps the standard Lebesgue space $L^q(\Omega)$ (which norm is denoted by $\|u\|_q$) into E. If $u \in E$, by (f_1) we have that $f(x,u) \in L^q(\Omega)$, and consequently $K(f(x,u)) \in E$. This implies that $\Phi : E \to E$ given by

$$\Phi(u) \equiv u - K(f(x,u)), \quad u \in E$$

is well defined. Solutions of (1) are just the zeros of Φ (see for example [4, Lemma 2.4.i]). Therefore, problem (1) can be studied by means of Leray–Schauder degree. It is shown in [7, Theorem I] that the sub-super-solution method also works in this framework. On the contrary, the problem does not have variational structure if the dependence of the matrix $A(x,s)$ respect to s is nontrivial. Hence, we use only the available topological techniques to study (1) according to different hypotheses on the interaction between the asymptotic behavior of $f(x,s)/s$ and the spectrum of a semilinear eigenvalue problem determined by the asymptotic behavior of the matrix $A(x,s)$. Specifically, we assume

$$\lim_{|s| \to \infty} A(x,s) = A_\infty(x) \tag{A$_4$}$$

and that for some $q > N/2$ there exist positive functions $f'_{\pm\infty}(x)$ belonging to $L^q(\Omega)$ such that

$$\lim_{s \to \pm\infty} \frac{f(x,s)}{s} = f'_{\pm\infty}(x), \quad \text{uniformly in } x \in \Omega. \tag{f$_2$}$$

We consider the eigenvalue problem

$$\begin{aligned} -\operatorname{div}(A_\infty(x)\nabla u) &= \mu u, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{2}$$

Let $\{\lambda_k\}$ be denoting the sequence of associated eigenvalues.

We handle three kinds of problems. The first one is the nonresonant case in which either $f'_{\pm\infty}(x) < \lambda_1$ or $\lambda_k < f'_{\pm\infty}(x) < \lambda_{k+1}$ for some $k \in \mathbb{N}$. We obtain an slight improvement of the previous existence result in [12, Theorem 1] where the author considers the Dirichlet b.v.p. (even with boundary nonzero data) imposing the additional hypotheses on the existence of limit of $\nabla_x A(x,s)$ as $|s|$ tends to $+\infty$.

Next, the resonant case $f'_{+\infty}(x) \equiv f'_{-\infty}(x) \equiv \lambda_1$ is also treated. This kind of resonant problem at infinity has been extensively studied in the semilinear case. Among others we can cite here [1,5,8,18,19,21,24]. Extending to the quasilinear case the ideas in [5,24], we study resonance phenomena by the determination of the side of every possible bifurcation from (λ_1, ∞) for a certain one-parameter problem. This allows us to deduce conditions to assure an a priori bound for solutions on the side where bifurcation does not occur. This a priori bound is the only requirement to use compactness arguments and to prove in a standard way the existence of solution (see [18]). In this way, we improve the results in [11, Theorem 10] because, in addition to cancel the assumption about the behavior of $\nabla_x A(x,s)$ as $|s|$ goes to $+\infty$, we also avoid the condition used by the author connecting the infimum κ_1 of the set consisting by the first eigenvalue of every eigenvalue problem (associated to every $v \in L^2(\Omega)$)

$$\left.\begin{aligned} &-\operatorname{div}(A(x,v)\nabla u) = ku, \quad x \in \Omega, \\ &u = 0, \quad x \in \partial\Omega. \end{aligned}\right\} \quad v \in L^2(\Omega).$$

Specifically, in contrast with the cited paper, we do not need to assume that $\kappa_1 = \lambda_1$.

In the third case we suppose, in addition to (f_{1-2}), that there exists a positive constant ε such that for a.e. $x \in \Omega$

$$-\infty < f'_{-\infty}(x) < \lambda_1 - \varepsilon < \lambda_1 < \lambda_1 + \varepsilon < f'_{+\infty}(x) < \lambda_2 - \varepsilon < \lambda_2, \qquad (f_3)$$

obtaining existence and multiplicity results for the one parameter problem

$$\begin{aligned} -\operatorname{div}(A(x,u)\nabla u) &= f(x,u) + t\varphi, \quad x \in \Omega, \\ u &= 0, \qquad\qquad\quad x \in \partial\Omega, \end{aligned} \qquad (P_t)$$

where $\varphi \in L^\infty(\Omega)$ is a positive function and $t \in \mathbb{R}$.

Semilinear problems with nonlinearity having a derivative *jumping* the eigenvalues of the linear part have been extensively studied and this problems are known by Ambrosetti–Prodi type problems. From the paper by Ambrosetti–Prodi [2], we can cite for example [9,19] or more recently [14] (see also the references therein). Chabrowski in [11] and Lefton–Shapiro in [22] handle quasilinear operators, but they again need a condition about jumping the first eigenvalue of all a class of linear operators, namely that $f'_{-\infty} < \kappa_1 \leqslant \lambda_1 < f'_{+\infty}$.

By combining degree theory and sub-supersolution method, we obtain the existence of $t^* \in \mathbb{R}$ such that (P_t) has no solution for $t > t^*$, has at least one solution for $t = t^*$ and at least two solutions for $t \ll t^*$. Moreover, for a particular class of matrices $A(x,s)$, this multiplicity result is improved by showing the existence of a continuum $\mathscr{C}$ of solutions in $\mathbb{R} \times E$ that emanates from $\{t\} \times E$, reaches $\{t^*\} \times E$ and then returns again to a different solution in $\{t\} \times E$. We have to point out that, at least for the knowledge of the authors, the existence of this continuum with $\supset$-shape seems to be unknown even in the semilinear case.

Finally, we extend to quasilinear operators the classical semilinear result of Ambrosetti–Rabinowitz about the existence of positive solution for super-linear ($f'_{+\infty} \equiv +\infty$) problems. The key idea and novelty here is to embed this problem into the Ambrosetti–Prodi framework that we have previously studied. Indeed, adding $t\varphi$ in the right-hand side we embed the above problem into an Ambrosetti–Prodi problem. In this case, we prove that $t^* > 0$ and then there exist two solutions for $t = 0$, one of them has to be nontrivial and nonnegative.

The paper is organized as follows. In Section 2 we are concerned with existence of solutions for both the nonresonant and the resonant problem. In Section 3 we apply all these results jointly with the sub-super-solution method and a priori estimates in order to obtain an existence result for the Ambrosetti–Prodi type problem (P_t). Section 4 is devoted to obtain a multiplicity result for (P_t) as a consequence of the existence of a continuum of solutions. Finally, in Section 5 we give the applications to problems such as Ambrosetti–Rabinowitz type ones via Ambrosetti–Prodi ones.

2. Resonance for quasilinear operators. Landesman–Lazer results

Before studying resonance phenomena we begin with an existence result for solutions of (1) in the case in which the nonlinearity $f(x,s)$ does not interact at $\pm\infty$ with the spectrum of (2). The basic fact to do that will be the existence of an a priori bound in the E-norm of the solutions. Indeed, we prove the following.

Theorem 1. *Suppose that* (A_{1-4}) *and* (f_{1-2}) *are satisfied. Assume also that* $\lambda_k < f'_{\pm\infty}(x) < \lambda_{k+1}$ *for some* $k \in \mathbb{N}$ *or* $f'_{\pm\infty}(x) < \lambda_1$, *a.e.* $x \in \Omega$. *Then problem* (1) *has at least one solution.*

Proof. We prove the theorem by constructing a homotopy from problem (1) to a semilinear one and using then the invariance property of the Leray–Schauder Degree.

For each $s \in [0,1]$ we consider the matrix $A_s(x,u) = sA(x,u) + (1-s)A_\infty(x)$, which satisfies hypotheses (A_{1-4}). Let also $f_s(x,u) = sf(x,u) + (1-s)[f'_{+\infty}(x)u^+ + f'_{-\infty}(x)u^-]$ (with $u^- \equiv \min\{u, 0\}$). We study now the problem

$$\begin{aligned} -\mathrm{div}(A_s(x,u)\nabla u) &= f_s(x,u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Since A_s fulfills condition (A_3), there exist the continuous and compact inverse $K_s : L^2(\Omega) \to H_0^1(\Omega)$ of the differential operator defined by $Q_s(u) \equiv -\mathrm{div}(A_s(x,u)\nabla u)$. In this way, the above problem is equivalent to find zeros of $\Phi_s : E \to E$, given by

$$\Phi_s(u) \equiv u - K_s[sf(x,u) + (1-s)(f'_{+\infty}(x)u^+ + f'_{-\infty}(x)u^-)] = 0.$$

Claim. *There exists* $R \in \mathbb{R}^+$ *such that* $\Phi_s(u) \neq 0$, *for every* $u \in E$ *with* $\|u\|_0 = R$ *and* $s \in [0,1]$.

Suppose on the contrary that there exist $s_n \in [0,1]$ and $u_n \in H_0^1(\Omega)$ satisfying $\Phi_{s_n}(u_n) = 0$ with $\|u_n\|_0 \to \infty$. First we note that $z_n = u_n/\|u_n\|$ is bounded and thus there exists $z \in H_0^1(\Omega)$ such that, up to a subsequence, z_n weakly converges to z. Moreover, $z_n \to z$ strongly in $L^p(\Omega)$, $p < 2^*$ (where 2^* is denoting the Sobolev exponent related to $H_0^1(\Omega)$, i.e., $2^* = 2N/(N-2)$ if $N > 2$ and $2^* = \infty$ if $N \leqslant 2$), and $z_n(x) \to z(x)$ a.e. $x \in \Omega$. In addition, z_n satisfies

$$\int_\Omega A_{s_n}(x,u_n)\nabla z_n \cdot \nabla v = (1-s_n)\left[\int_\Omega f'_{+\infty}(x)z_n^+ v + \int_\Omega f'_{-\infty}(x)z_n^- v\right] + s_n \int_\Omega \frac{f(x,u_n)}{\|u_n\|} v, \quad \forall v \in H_0^1(\Omega). \tag{3}$$

Taking $v = z_n - z$ as test function, subtracting $\int_\Omega A_{s_n}(x,u_n)\nabla z \cdot \nabla(z_n - z)$ and using hypothesis (A_2) we have

$$\begin{aligned} \alpha\|z_n - z\|^2 &\leqslant \int_\Omega A_{s_n}(x,u_n)\nabla(z_n - z)\cdot\nabla(z_n - z) \\ &= (1-s_n)\left[\int_\Omega f'_{+\infty}(x)z_n^+(z_n - z) + \int_\Omega f'_{-\infty}(x)z_n^-(z_n - z)\right] \\ &\quad + s_n \int_\Omega \frac{f(x,u_n)}{\|u_n\|}(z_n - z) - \int_\Omega A_{s_n}(x,u_n)\nabla z \cdot \nabla(z_n - z). \end{aligned}$$

To prove the strong convergence of z_n to z in $H_0^1(\Omega)$ it suffices to show that the terms on the right-hand side of the above inequality converge to zero. In order to see this, first we observe that, from L^p-convergence we obtain that

$$(1-s_n)\left[\int_\Omega f'_{+\infty}(x)z_n^+(z_n-z)+\int_\Omega f'_{-\infty}(x)z_n^-(z_n-z)\right]\to 0.$$

Regularity theorems (see [15, Theorem II; 17, Theorem 8.29] or [26, Theorem 7.3]) imply that $\|u_n\|$ is unbounded and from (f_1)

$$s_n\int_\Omega \frac{f(x,u_n)}{\|u_n\|}(z_n-z)\to 0.$$

Now, condition (A_1) and the Lebesgue Theorem imply the strong convergence of $A_{s_n}(x,u_n)\nabla z$ to $A_\infty(x)\nabla z$ in $L^2(\Omega)$. Since $\nabla(z_n-z)$ weakly converges to zero in $L^p(\Omega)$ we have

$$\int_\Omega A_{s_n}(x,u_n)\nabla z\cdot\nabla(z_n-z)\to 0.$$

Thus, we deduce the strong convergence in $H_0^1(\Omega)$ of z_n to z. To get now the equation satisfied by z we take limit in (3). First we note arguing as above that

$$\int_\Omega A_{s_n}(x,u_n)\nabla z_n\cdot\nabla v\to\int_\Omega A_\infty(x)\nabla z\cdot\nabla v,\quad \forall v\in H_0^1(\Omega).$$

On the other hand, since s_n is bounded, using (f_{1-2}) and Lebesgue Theorem we deduce for every $v\in H_0^1(\Omega)$ that

$$(1-s_n)\int_\Omega f'_{+\infty}(x)z_n^+v+(1-s_n)\int_\Omega f'_{-\infty}(x)z_n^-v$$
$$+s_n\int_\Omega\frac{f(x,u_n)}{\|u_n\|}v\to\int_\Omega f'_{+\infty}(x)z^+v+\int_\Omega f'_{-\infty}(x)z^-v.$$

Then, the equation satisfied by z is

$$\int_\Omega A_\infty(x)\nabla z\cdot\nabla v=\int_\Omega f'_{+\infty}(x)z^+v+\int_\Omega f'_{-\infty}(x)z^-v,\quad \forall v\in H_0^1(\Omega).$$

Denoting by χ_B the characteristic function of the set B and taking $m(x)=f'_{-\infty}(x)\chi_{\{z\leqslant 0\}}+f'_{+\infty}(x)\chi_{\{z>0\}}$, the equation above means that $\mu=1$ is an eigenvalue for the weighted eigenvalue problem

$$\begin{aligned} -\mathrm{div}(A_\infty(x)\nabla u) &= \mu m(x)u, & x\in\Omega,\\ u &= 0, & x\in\partial\Omega. \end{aligned}\tag{4}$$

Let us denote by $\{\mu_k(m(x))\}$ the sequence of positive eigenvalues for (4) and suppose that $1=\mu_j(m(x))$ for some j. From the hypothesis on $f'_{\pm\infty}$, either $m(x)<\lambda_1$ (in the case $f'_{\pm\infty}<\lambda_1$), or $\lambda_k<m(x)<\lambda_{k+1}$ (if $\lambda_k<f'_{\pm\infty}<\lambda_{k+1}$ for some $k\geqslant 1$). Thus, it is known (see [13, Proposition 1.12A]) that either

$$\mu_j(m(x))>\mu_j(\lambda_1)=\frac{\lambda_j}{\lambda_1}$$

or

$$\frac{\lambda_j}{\lambda_{k+1}} = \mu_j(\lambda_{k+1}) < \mu_j(m(x)) < \mu_j(\lambda_k) = \frac{\lambda_j}{\lambda_k}.$$

In the first case we have that $\lambda_1 > \lambda_j$ and in the second one that $\lambda_k < \lambda_j < \lambda_{k+1}$. This is in both cases a contradiction, which proves the claim.

By virtue of the claim and by the homotopy invariance property, we deduce that the Leray–Schauder degree

$$\deg(\Phi_s, B_R(0), 0) = \text{constant},$$

where $B_R(0) = \{u \in E/\|u\|_0 < R\}$ is the open ball centered at zero of radius R, $R > 0$. For $s = 0$ we can compute the above degree and show that it is different from zero. Indeed, as we have seen in the proof of the claim, since $f'_{\pm\infty}(x)$ does not interact with the spectrum of (2) then 1 is not an eigenvalue of $K_0(f'_{+\infty}(x)u^+ + f'_{-\infty}(x)u^-)$. In this case, it is proved in [20] that

$$\deg(\Phi_0, B_R(0), 0) = (-1)^v,$$

where v is the sum of the algebraic multiplicities of the eigenvalues μ of the compact operator $K_0(f'_{+\infty}(x)u^+ + f'_{-\infty}(x)u^-)$ with $1 < \mu$. Consequently, for $s = 1$

$$\deg(\Phi_1, B_R(0), 0) \neq 0,$$

which implies that (1) has at least one solution $u \in E$ with $\|u\|_0 < R$. $\square$

Next, we consider the easiest case of interaction of the nonlinearity with the spectrum. Namely, we are concerned with the resonant problem

$$\begin{aligned} &-\operatorname{div}(A(x,u)\nabla u) = \mu_1(m(x))m(x)u + g(x,u), \quad x \in \Omega, \\ &u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{5}$$

where as in the proof of the previous lemma $\mu_1(m(x))$ is the first positive eigenvalue of (4) and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$\lim_{|s|\to\infty} \frac{g(x,s)}{s} = 0, \quad \text{uniformly in } \Omega. \tag{6}$$

With respect to $m(x)$, we suppose that it is an $L^q(\Omega)$-function with Lebesgue measure $|\{x \in \Omega:\ m(x) > 0\}| > 0$ (i.e., $m(x)^+ \not\equiv 0$).

We employ the quasilinear extension in [4] of the approach in [5, Theorem 19] and [24].

This problem can be studied by means of bifurcation theory if we embeds it into the one-parameter problem

$$\begin{aligned} -\operatorname{div}(A(x,u)\nabla u) &= \lambda m(x)u + g(x,u), && x \in \Omega, \\ u &= 0, && x \in \partial\Omega. \end{aligned} \tag{Q$_\lambda$}$$

We have to point out that for $f(x,u) = m(x)u + g(x,u)/\mu_1(m(x))$, $f'_{+\infty}(x) = f'_{-\infty}(x) \equiv m(x)$ and the claim proved in the proof of the previous theorem means that $\mu_k(m(x))$ are the only possible positive bifurcation points from infinity for the problem (Q$_\lambda$).

The previous theorem also proves that if $\lambda \neq \mu_k(m(x))$ for every k, then (Q_λ) have at least one solution.

We prove our existence result for (5), which improves that in [11, Theorem 10]. The idea here is to analyze carefully the side of every possible bifurcation from infinity at $\mu_1(m(x))$. Let us denote by ψ the positive eigenfunction associated to $\mu_1(m(x))$ with $\|\psi\| = 1$. By $[A(x,s) - A_\infty(x)] \leqslant 0$ (respect. $[A(x,s) - A_\infty(x)] \geqslant 0$) we mean that the quadratic form induced by the matrix $A(x,s) - A_\infty(x)$ is definite nonpositive (respect. nonnegative).

Theorem 2. *Assume conditions* (6) *and* $(\mathrm{A}_{1\text{–}4})$ *with the matrix* $A_\infty(x)$ *having* $C^1(\bar{\Omega})$-*coefficients. Suppose also that the following condition*

$$a_{ij}(x,s) \in C^{1,\gamma}(\bar{\Omega} \times \mathbb{R}), \quad 0 < \gamma < 1 \tag{7}$$

and

$$f(x,s) \equiv m(x) + g(x,s) \quad \text{is a } C^1\text{-function in } \bar{\Omega} \times \mathbb{R},$$

holds and there exist $q > N$, $\sigma \in (0, 3 - 1/q)$ *and* $C \in L^q(\Omega)$ *such that*

$$\begin{aligned} &|g(x,s)|\,|s|^{\sigma-1} \leqslant C(x), \quad \text{a.e. } x \in \Omega, \ \forall s \in \mathbb{R}, \\ &\lim_{s \to \pm\infty} [g(x,s)]|s|^{\sigma-1} = \rho_{\pm\infty}(x), \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{8}$$

Then resonant problem (5) *has at least one solution, provided that one of the following conditions holds: either*

$$\begin{aligned} &[A(x,s) - A_\infty(x)] \leqslant 0, \quad \text{a.e. in } \Omega, \ \forall s \in \mathbb{R}, \\ &\int_\Omega \rho_{+\infty}(x)\psi^{2-\sigma}(x) > 0 > \int_\Omega \rho_{-\infty}(x)\psi^{2-\sigma}(x), \end{aligned} \tag{9}$$

or

$$\begin{aligned} &[A(x,s) - A_\infty(x)] \geqslant 0, \quad \text{a.e. in } \Omega, \ \forall s \in \mathbb{R}, \\ &\int_\Omega \rho_{+\infty}(x)\psi^{2-\sigma}(x) < 0 < \int_\Omega \rho_{-\infty}(x)\psi^{2-\sigma}(x). \end{aligned} \tag{10}$$

Remarks 3.

(1) We note that in the semilinear case, conditions (9) and (10) for $q = \infty$ and $\sigma = 1$ are in fact the classical Landesman–Lazer conditions (see [21, p. 611]). For the general $\sigma \in (0, 3 - 1/q)$ the integral inequality condition in (9) and (10) appear in [5, Theorems 3,4] (also for semilinear problems).

(2) We have to point out that the regularity assumption on $\lambda m(x) + g(x,s)$ in (7) is a technical condition to obtain that if u_n is an unbounded sequence of solutions of (Q_{λ_n}), up to a subsequence, $u_n/\|u_n\|$ converges to $\pm\psi$ in $C_0^1(\bar{\Omega})$.

Proof. First we note that if (7) is satisfied then, using [17, Theorem 14.17], we have that $K(L^q(\Omega)) \subset C_0^1(\bar{\Omega})$ (the subspace of the functions in $C_0(\bar{\Omega})$ which are C^1 in $\bar{\Omega}$). We claim that if (9) holds (respectively (10)) then every possible bifurcation from

$(\mu_1(m(x)), \infty)$ for the problem (Q_λ) occurs always to the left (respectively to the right). Notice that if the claim is proved, using Theorem 1 we can take u_n solution of (Q_{λ_n}) with $\mu_1(m(x)) < \lambda_n$ and $\lambda_n \to \mu_1(m(x))$ (respectively $\lambda_n < \mu_1(m(x))$). Since every possible bifurcation from infinity is to the left, we have that there exists $M \in \mathbb{R}^+$ such that $\|u_n\|_0 \leqslant M$, and from the compactness of K, there exists $u \in H_0^1(\Omega)$ such that up to a subsequence, u_n strongly converges to u and u is solution of $(Q_{\mu_1(m(x))})$, i.e., u is solution of (5), proving the theorem.

The claim is partially proven in [4, Theorem 3.4]. We include here for convenience of the reader the proof that condition (9) implies that the bifurcation occurs to the left. Let us argue by contradiction and take a sequence (λ_n, u_n) satisfying

$$\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla v = \lambda_n \int_\Omega m(x) u_n v + \int_\Omega g(x,u_n)v, \qquad \forall v \in H_0^1(\Omega),$$

with $\lambda_n \to \mu_1(m(x))$, $\lambda_n \geqslant \mu_1(m(x))$ and $\|u_n\|_0 \to \infty$.

Dividing by $\|u_n\|$, using (6) and (7), we deduce that $z_n = u_n/\|u_n\|$ converges in $C_0^1(\bar{\Omega})$ to ψ or $-\psi$. Thus, up to a subsequence, every solution u_n belongs to the interior of the cone P of nonnegative functions in $C_0^1(\bar{\Omega})$ or it belongs to $-P$. Thus, we can take $v = \psi^2/u_n$ as test function in the equation above to get

$$\int_\Omega A(x,u_n)\nabla u_n \left[2\frac{\psi}{u_n}\nabla\psi - \left(\frac{\psi}{u_n}\right)^2 \nabla u_n \right] = \lambda_n \int_\Omega m(x)\psi^2 + \int_\Omega g(x,u_n)\frac{\psi^2}{u_n}.$$

Hence we yield that

$$\int_\Omega g(x,u_n)\frac{\psi^2}{u_n} = \int_\Omega \left[A(x,u_n) - \frac{\lambda_n}{\mu_1(m(x))} A_\infty(x) \right] \nabla\psi \cdot \nabla\psi$$
$$- \int_\Omega A(x,u_n)\left(\nabla\psi - \frac{\psi}{u_n}\nabla u_n\right) \cdot \left(\nabla\psi - \frac{\psi}{u_n}\nabla u_n\right).$$

By the fact that $\lambda_n \geqslant \mu_1(m(x))$ we get, applying conditions (A_2) and (9), that

$$0 \geqslant \int_\Omega \left[A(x,u_n) - \frac{\lambda_n}{\mu_1(m(x))} A_\infty(x) \right] \nabla\psi \cdot \nabla\psi$$
$$- \int_\Omega A(x,u_n)\left(\nabla\psi - \frac{\psi}{u_n}\nabla u_n\right) \cdot \left(\nabla\psi - \frac{\psi}{u_n}\nabla u_n\right).$$

In particular,

$$0 \geqslant \|u_n\|^\sigma \int_\Omega g(x,u_n)\frac{\psi^2}{u_n}.$$

Using (8) and Fatou Lemma, we obtain that

$$0 \geqslant \liminf_{n\to\infty} \int_\Omega g(x,u_n)u_n^{\sigma-1}\frac{\psi^2}{z_n^\sigma} \geqslant \int_\Omega \rho_{+\infty}(x)\psi^{2-\sigma} > 0,$$

if $z_n \to \psi$, and

$$0 \geqslant \liminf_{n\to\infty} -\int_\Omega g(x,u_n)|u_n|^{\sigma-1}\frac{\psi^2}{|z_n|^\sigma} \geqslant -\int_\Omega \rho_{-\infty}(x)\psi^{2-\sigma} > 0,$$

if $z_n \to -\psi$. In both cases we have a contradiction with (9), proving the claim, and consequently the theorem. $\square$

3. Ambrosetti–Prodi problems

We are interested in this section with the problem of existence and multiplicity of solution of (P_t) in the case that several eigenvalues of (2) are contained in the interval $(f'_{-\infty}(x), f'_{+\infty}(x))$ for almost every $x \in \Omega$. More precisely we study two classes of nonlinearities f. The first kind of nonlinearity f corresponds to an asymptotically linear one satisfying $(\mathrm{f}_{1\text{–}3})$ (f interacts with the first eigenvalue). In this case we improve the results in [11] for this quasilinear operators. We consider also nonlinearities f which interact with all the spectrum. Specifically, we study problem (P_t) with nonlinearities f satisfying (f_2) and

$$-\infty < f'_{-\infty}(x) < c_+ < \alpha\mu < f'_{+\infty}(x) \equiv +\infty, \qquad \text{a.e. } x \in \Omega, \tag{f_3'}$$

where μ denotes the first eigenvalue associated to the Laplacian operator. In addition, in this case we suppose that there exists $h(x) \in L^\infty(\Omega)$ and $1 < p < 2^* - 1$ such that

$$\lim_{s \to +\infty} \frac{f(x,s)}{s^p} = h(x) > c > 0, \qquad \text{uniformly in } x \in \Omega. \tag{f_4}$$

We remark explicitly that for the knowledge of the authors the quasilinear problem (P_t) is unknown for this kind of “super-linear” nonlinearities.

The main tools are the sub-super-solution method and the Leray–Schauder degree. Some a priori bound on t and on the E-norm of the solutions will be useful.

Lemma 4. *Assume that* $(\mathrm{A}_{1,2,4})$ *hold. Suppose also that either*

(i) *conditions* $(\mathrm{f}_{1\text{–}3})$ *hold,*

or

(ii) *A and f satisfies* (A_3), (f_2), (f_3'), (f_4), (7), *and for some positive constants* $c_1, c_2 \geqslant 0$,

$$|f(x,s)| \leqslant c_1|s| + c_2, \qquad \text{a.e. } x \in \Omega, \ \forall s \in \mathbb{R}^-. \tag{f_1'}$$

Then solutions of (P_t) are uniformly bounded in compact sets of t, i.e. for every compact interval $\Gamma \subset \mathbb{R}$, there exists $c \in \mathbb{R}$ such that every solution $u \in E$ of (P_t) with $t \in \Gamma$ satisfies

$$|u(x)| \leqslant c, \qquad \forall x \in \Omega.$$

Proof. Let us begin with the proof in the case that assumptions of item (i) are satisfied. Suppose on the contrary that u_n is a solution of (P_{t_n}) with t_n bounded and $\|u_n\|_0 \to \infty$. Using that $t_n/\|u_n\|$ converges to zero and similar arguments to those in the proof of Theorem 1 we deduce that $z_n = u_n/\|u_n\|$ strongly converges to a solution $z \in H_0^1(\Omega)$

of (4) with $\mu = 1$ $(m(x) = f'_{-\infty}(x)\chi_{\{z\leqslant 0\}} + f'_{+\infty}(x)\chi_{\{z>0\}})$. This means that 1 is an eigenvalue for this weighted eigenvalue problem. However, we claim that this is a contradiction. Indeed, since $m(x) < \lambda_2$, it is known (see [13, Proposition 1.12 A]) that the sequence of positive eigenvalues $\mu_j(m(x))$ satisfies

$$\mu_j(m(x)) > \mu_j(\lambda_2) = \frac{\lambda_j}{\lambda_2}.$$

Thus, $\mu_j(m(x)) > 1$ for every $j \geqslant 2$. Therefore if $\mu=1$ would be an eigenvalue, it would be $\mu_1(m(x))=1$. Using that the nonzero eigenfunctions associated to the first eigenvalue do not change sign we would obtain that either $m(x) \equiv f'_{-\infty}(x)$ or $m(x) \equiv f'_{+\infty}(x)$. In the first case we would have that

$$\mu_1(f'_{-\infty}(x)) = \mu_1(m(x)) > \mu_1(\lambda_1) = \frac{\lambda_1}{\lambda_1} = 1.$$

Similarly in the second one we would have that

$$\mu_1(f'_{+\infty}(x)) = \mu_1(m(x)) < 1.$$

In any case we would reach a contradiction, proving that $\mu = 1$ is not an eigenvalue for (4) and consequently the lemma in the case of item (i).

In the second case, when the conditions of item (ii) are satisfied, we divide the proof into two steps. In the step 1 we prove that every solution of (P_t) is bounded from below. The second step is devoted to prove that they are also bounded from above. Let u be a solution of (P_t) with $t \in \Gamma$. Let us denote during all the proof by c a positive constant, independent of t and u (possibly different between steps).

Step 1. First we prove the existence of a uniform bound in the $H_0^1(\Omega)$-norm of the negative part of solutions of (P_t). Indeed, taking $v=u^-$ as test function in the equation satisfied by u and using (A_1) we have that

$$\alpha\|u^-\|^2 \leqslant \int_\Omega A(x,u)\nabla u \nabla u^- = \int_\Omega f(x,u)u^- + t\int_\Omega \varphi u^-.$$

Observe that from hypotheses (f_1') and (f_3') we get

$$f(x,s) \geqslant \bar{c}_1 s - \bar{c}_2, \qquad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}^-,$$

with $\bar{c}_1 < \alpha\mu$. Thus, we deduce, using Poincaré and Cauchy–Schwartz inequalities, that

$$\alpha\|u^-\|^2 \leqslant \bar{c}_1 \int_\Omega |u^-|^2 + \bar{c}_2 \int_\Omega |u^-| + |t| \int_\Omega \varphi |u^-| \leqslant \frac{\bar{c}_1}{\mu}\|u^-\|^2 + c_3\|u^-\|,$$

where, since $t \in \Gamma$, we can take $c_3 = 1/\sqrt{\mu}(\|\varphi\|_2 \sup_{t\in\Gamma}\{|t|\} + \bar{c}_2|\Omega|^{1/2})$. Hence, we have $\|u^-\| \leqslant c_3\mu/(\alpha\mu - \bar{c}_1)$. Using this estimate we prove the existence of a bound for $\|u^-\|_0$. In order to do that we consider for every $k \in \mathbb{R}^+$ the function G_k given by

$$G_k(s) = \begin{cases} s+k & s \leqslant -k, \\ 0 & -k < s \leqslant k, \\ s-k & s > k. \end{cases}$$

Thus, taking $v = G_k(u^-)$ as test function in the equation satisfied by u and using (A_2) we obtain that

$$\alpha \int_\Omega |\nabla G_k(u^-)|^2 \leqslant \int_\Omega A(x,u^-)\nabla G_k(u^-) \cdot \nabla G_k(u^-)$$

$$= \int_\Omega A(x,u)\nabla u \cdot \nabla G_k(u^-)$$

$$= \int_{\Omega_k} (f(x,u^-) + t\varphi) G_k(u^-),$$

where $\Omega_k \equiv \{x \in \Omega \colon u(x) < -k\}$. Taking into account that there exist a positive constant C such that

$$|f(x,s) + t\varphi| < C|s|, \quad \forall s \leqslant -k,$$

we deduce from above that

$$\alpha \int_\Omega |\nabla G_k(u^-)|^2 \leqslant C \int_{\Omega_k} |u^-||G_k(u^-)|.$$

Using now the Sobolev and Hölder inequalities we get that for some new constant C and $r > 2N/(N+2)$

$$\|G_k(u^-)\|_{2^*}^2 \leqslant C\|u^-\|_r \|G_k(u^-)\|_{2^*} (\operatorname{meas} \Omega_k)^{(1-1/r-1/2^*)}.$$

Notice now that for every $h \geqslant k$, $|G_k(u^-)| \geqslant h - k$ in Ω_h, this implies that

$$(h-k)(\operatorname{meas} \Omega_h)^{1/2^*} \leqslant C\|u^-\|_r (\operatorname{meas} \Omega_k)^{(1-1/r-1/2^*)},$$

or equivalently that

$$\operatorname{meas} \Omega_h \leqslant \frac{C\|u^-\|_r^{2^*} (\operatorname{meas} \Omega_k)^{(2^*-1-2^*/r)}}{(h-k)^{2^*}}.$$

Therefore we can apply Stampacchia Lemma to deduce that

(i) if $r > N/2$ then $u^- \in L^\infty(\Omega)$ and $\|u^-\|_0 \leqslant c\|u^-\|_r$,
(ii) if $r = N/2$ then $u^- \in L^s(\Omega)$ for $s \in [1, \infty)$ and $\|u^-\|_s^s \leqslant c + c'\|u^-\|_r^s$,
(iii) if $r < N/2$ then $u^- \in L^s(\Omega)$ for $s = 2^*r/((2-2^*)r - 2^*) - \delta$ and arbitrary small $\delta > 0$. Moreover, $\|u^-\|_s^s \leqslant c + c'\|u^-\|_r^{2^*r/((2-2^*)r+2^*)}$.

Since $u \in L^{2^*}(\Omega)$ and $2^* > 2N/(N+2)$ we can argue as above for $r_0 = 2^*$. Thus, if $N < 6$ we conclude by item i). In the case $N = 6$ we use item ii) to choose $r_1 > N/2$ and after repeating the argument we lie in the item i) and conclude again. Finally, in the case $N > 6$ we can take

$$r_1 = \frac{2^*r}{(2-2^*)r + 2^*} - \delta_1 > r_0.$$

As before, if $r_1 \geqslant N/2$ we conclude easily. In other cases we take

$$r_2 = \frac{2^*r_1}{(2-2^*)r_1 + 2^*} - \delta_2.$$

Arguing by iteration, we can conclude after a finite number of steps. Indeed, in other cases, we have that r_n is bounded, where r_n is defined in the recurrence form

$$r_0 = 2^*,$$

$$r_{n+1} = \frac{2^* r_n}{(2-2^*)r_n + 2^*} - \delta_{n+1}.$$

where $\lim_{n\to+\infty} \delta_n = 0$. Moreover, r_n is increasing and therefore it is convergent and the limit $r \in (2^*, N/2]$ satisfies

$$r = \frac{2^* r}{(2-2^*)r + 2^*},$$

i.e., $2^* r = (2-2^*)r^2 + 2^* r$, which is a contradiction, proving the claim.

Step 2. Let us suppose (Step 1) that $|u^-| \leqslant c$, denote $v{=}u{+}c \geqslant 0$, $\tilde{f}(x,s){=}f(x,s{-}c)$ and $\tilde{A}(x,s) = A(x, s-c)$. Thus, v satisfies

$$\begin{aligned} -\operatorname{div}(\tilde{A}(x,v)\nabla v) &= \tilde{f}(x,v) + t\varphi, \quad && x \in \Omega, \\ v &= c, && x \in \partial\Omega, \end{aligned}$$

with $\tilde{f}$ satisfying (f_4).

We observe that the result in [4, Theorem 5.11] remains valid for solutions of the equation with bounded Dirichlet data instead of zero Dirichlet data. This result gives the existence of $\tilde{c} \in \mathbb{R}^+$ such that $v(x) \leqslant \tilde{c}$, $\forall x \in \Omega$, i.e., u is bounded from above. $\square$

Let us denote by ϕ the solution of the problem

$$-\operatorname{div}(A_\infty(x)\nabla u) = \varphi, \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial\Omega.$$

Inspired by some ideas for semilinear problems contained in the paper by McKenna–Walter [25] we prove the following lemma.

Lemma 5. *Assume that* ($A_{1\text{–}4}$), ($f_{1\text{–}3}$) *with* $q > N$, *are satisfied and that the coefficients of* A_∞ *are of class* C^1. *For each* $0 < \varepsilon < \mu^{1/2}\lambda_1^2\|\phi\|_2^2/\lambda_2\|\varphi\|_2 + \lambda_1^2\|\phi\|_2$ *there exists* $t_\varepsilon \in \mathbb{R}$ *such that for every* $t < t_\varepsilon$ *and* $\lambda \in [0,1]$, *the problem*

$$-\operatorname{div}((\lambda A(x,u) + (1-\lambda)A_\infty(x))\nabla u) = \lambda f(x,u) + t\varphi, \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial\Omega,$$

has no solution in $\partial B_{|t|\varepsilon}(t\phi) = \{u \in H_0^1(\Omega)\colon \|u - t\phi\| = |t|\varepsilon\}$.

Proof. We argue by contradiction and suppose that there exist sequences $t_n \in \mathbb{R}$ with $t_n \to -\infty$, $\lambda_n \in [0,1]$ (up to a subsequence we can assume that $\lambda_n \to \lambda \in [0,1]$) and $u_n \in H_0^1(\Omega)$ with $\|u_n/t_n - \phi\| = \varepsilon$, satisfying that

$$-\operatorname{div}((\lambda_n A(x,u_n) + (1-\lambda_n)A_\infty(x))\nabla u_n) = \lambda_n f(x,u_n) + t_n\varphi.$$

From the fact that $\|u_n/t_n - \phi\| = \varepsilon$ we can deduce that $z_n = u_n/t_n$ is bounded. Moreover, we know that $\|u_n\|_0 \to \infty$, since in other cases $z_n \to 0$ and so $0 \in B_\varepsilon(\phi)$ which is impossible because from the Poincaré inequality

$$\varepsilon < \sqrt{\mu}\|\phi\|_2 \frac{\lambda_1^2\|\phi\|_2}{\lambda_2\|\varphi\|_2 + \lambda_1^2\|\phi\|_2} \leqslant \|\phi\|_2\sqrt{\mu}$$

$$\text{(Poincaré inequality)} \leqslant \|\phi\|.$$

On the other hand, there exists $z \in H_0^1(\Omega)$ such that (up to a subsequence) $z_n \to z$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and $z_n(x) \to z(x)$ a.e. $x \in \Omega$. Arguing as in Theorem 1 we deduce the strong convergence of z_n to z. Consequently $\|z - \phi\| = \varepsilon$.

Dividing by t_n the equation satisfied by u_n and taking limits we deduce from (A_4) and (f_3) that z satisfies the following equation:

$$-\mathrm{div}(A_\infty(x)\nabla z) = \lambda m(x)z + \varphi,$$

where $m(x) = f'_{-\infty}(x)\chi_{\{z\leqslant 0\}} + f'_{+\infty}(x)\chi_{\{z>0\}}$. We claim now that z is nonnegative. Indeed, this is a consequence of taking z^- as test function in the equation satisfied by z to obtain from (f_3) that

$$\lambda_1\|z^-\|_2^2 \leqslant \int_\Omega A_\infty(x)\nabla z \cdot \nabla z^- = \lambda \int_\Omega m(x)(z^-)^2 + \int_\Omega \varphi z^-$$

$$= \lambda \int_\Omega f'_{-\infty}(x)(z^-)^2 + \int_\Omega \varphi z^- < \lambda\lambda_1\|z^-\|_2^2,$$

which implies that $z^- \equiv 0$, proving the claim.

As a direct consequence of the claim we have that $m(x) = f'_{+\infty}(x)$.

We take now ϕ as test function in the equation satisfied by z and z as test function in the equation satisfied by ϕ, obtaining that

$$\int_\Omega A_\infty(x)\nabla z \cdot \nabla\phi = \lambda \int_\Omega f'_{+\infty}(x)z\phi + \int_\Omega \varphi\phi$$

and

$$\int_\Omega A_\infty(x)\nabla\phi \cdot \nabla z = \int_\Omega z\varphi.$$

This implies, since $A_\infty(x)$ is symmetric, that

$$\int_\Omega \varphi z = \lambda \int_\Omega f'_{+\infty}(x)z\phi + \int_\Omega \varphi\phi.$$

Using now the Hölder inequality and (f_3) we have that

$$\|\varphi\|_2\|z - \phi\|_2 \geqslant \int_\Omega \varphi(z - \phi) = \lambda \int_\Omega f'_{+\infty}(x)z\phi \geqslant \lambda\lambda_1 \int_\Omega z\phi.$$

Since $\|z - \phi\| = \varepsilon$, we can write $z = \phi + \varepsilon z_1$ with $\|z_1\| = 1$. Thus, applying the Poincaré inequality to the above one, we get

$$\frac{\varepsilon}{\lambda\lambda_1\sqrt{\mu}}\|\varphi\|_2 \geqslant \int_\Omega z\phi$$

$$= \|\phi\|_2^2 + \varepsilon \int_\Omega z_1\phi$$

$$\geqslant \|\phi\|_2^2 - \varepsilon \int_\Omega |z_1\phi|$$

$$\geqslant \|\phi\|_2^2 - \varepsilon\|z_1\|_2\|\phi\|_2$$

$$(\text{Poincaré inequality}) \geqslant \|\phi\|_2^2 - \varepsilon\frac{\|\phi\|_2}{\sqrt{\mu}}.$$

This implies $\lambda \leqslant \dfrac{\varepsilon\|\varphi\|_2}{\lambda_1\mu^{1/2}\|\phi\|_2^2 - \varepsilon\lambda_1\|\phi\|_2} < \dfrac{\lambda_1}{\lambda_2}$.

On the other hand, z is a positive super-solution for the problem

$$-\text{div}(A_\infty(x)\nabla u) = \lambda f'_{+\infty}(x)u, \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial\Omega.$$

We also find a sub-solution for this problem of the form $w = \delta\lambda\varphi_1$ with $\delta < 1/\lambda_1$. Note that since $z \neq \phi$, $\lambda \neq 0$. Moreover, we can choose δ small enough to conclude that $w \leqslant z$. The method of sub and super-solution allows to deduce the existence of a nonnegative and nontrivial solution for it. As a consequence of this λ is the first eigenvalue, $\mu_1(f'_{+\infty}(x))$ for the above weighted eigenvalue problem.

Since $f'_{+\infty}(x) < \lambda_2$, it is known that

$$\mu_1(f'_{+\infty}) = \lambda > \mu_1(\lambda_2) = \frac{\lambda_1}{\lambda_2},$$

which is a contradiction with the fact that $\lambda < \frac{\lambda_1}{\lambda_2}$. □

Theorem 6. *Assume* $(A_{1–4})$, $(f_{1–3})$. *Suppose also that the coefficient functions of* $A_\infty(x)$ *are of class* C^1, *and that for a.e.* $x \in \Omega$, $f(x,\cdot)$ *is increasing. Then there exists* $t^* \in \mathbb{R}$ *such that*

(1) (P_t) *has at least two solutions for* $t \ll t^*$,
(2) (P_t) *has at least one solution for* $t \leqslant t^*$,
(3) (P_t) *has no solution for every* $t > t^*$.

Proof. Let us denote $S \equiv \{t \in \mathbb{R}\colon (P_t) \text{ admits a solution}\}$. First we prove that S is not the empty set. In order to do that we use Leray–Schauder degree. Let $t < t_\varepsilon$ (where t_ε is given by Lemma 5) and $\Phi_t(u) = u - K(t\varphi + f(x,u))$. By the homotopy invariance of the Leray–Schauder degree, Lemma 5 implies that

$$\deg(\Phi_t, B_{|t|\varepsilon}(t\phi), 0) = \deg(I - K_1(t\phi), B_{|t|\varepsilon}(t\phi), 0) = 1,$$

where K_1 is the inverse operator of $-\text{div}(A_\infty(x)\nabla u)$. Then there exists a solution of (P_t) in $B_{|t|\varepsilon}(t\phi)$. This means that the interval $(-\infty, t_\varepsilon)$ is a subset of S.

The following step is to observe that S is an interval (unbounded from below). Indeed, if $t \in S$ then there exist a solution $u \in E$ of (P_t). Since u is a super-solution for $(P_{t'})$ with $t' < t$, the sub-super-solution method allows to conclude that $(-\infty, t] \subset S$ provided that we are able to find a sub-solution of $(P_{t'})$ less than u for every $t' \ll t$. In order to prove the existence of such a sub-solution we observe first that, since $(f_{1–3})$

are satisfied, for every $\delta_1 < \lambda_1$ we deduce the existence of $C \in \mathbb{R}^+$ such that

$$f(x,s) \geqslant \delta_1 s - C, \qquad \text{a.e. } x \in \Omega, \ \forall s \in \mathbb{R}. \tag{11}$$

Let us denote

$$l(t,x,s) = t\varphi + \min\{f(x,u), \delta_1 s - C\}.$$

By Theorem 1 with $f(x,s) = l(t',x,s)$ and $f'_{+\infty}(x) = 0$, $f'_{-\infty}(x) = \delta_1$, we can take a solution $\underline{u}$ of

$$\begin{aligned} -\mathrm{div}(A(x,u)\nabla u) &= l(t',x,u), \quad x \in \Omega, \\ u &= 0, \qquad\qquad x \in \partial\Omega. \end{aligned}$$

Thanks to (11) and the comparison theorem in [7] , $\underline{u}$ is sub-solution of (P'_t). Finally, since $l(t',x,s) \leqslant t'\varphi + f(x,u)$ we have that

$$-\mathrm{div}(A(x,\underline{u})\nabla\underline{u}) = l(t',x,\underline{u}) \leqslant t'\varphi + f(x,u) \leqslant -\mathrm{div}(A(x,u)\nabla u)$$

by the comparison principle we yield $\underline{u} \leqslant u$, proving the existence of a sub-solution and thus that S is an interval.

Now we prove that S is bounded from above. Suppose on the contrary that u_n would be a solution of (P_{t_n}), with $t_n \to +\infty$. Since $f'_{-\infty}(x) < c_+ < \lambda_1 < c_- < f'_{+\infty}(x)$, let us consider $\delta_1 < \lambda_1 < \delta_2$ satisfying (11) for every $s \in \mathbb{R}$ and

$$f(x,s) \geqslant \delta_2 s - C, \qquad \text{a.e. } x \in \Omega, \ \forall s \in \mathbb{R},$$

for some $C > 0$. Taking $v = \psi_1$ (the normalized eigenfunction associated to λ_1) as a test function in the equation satisfied by u_n, adding and subtracting $\lambda_1 \int_\Omega u_n\psi_1$, we deduce from above that

$$t_n \int_\Omega \varphi\psi_1 \leqslant \int_\Omega [A(x,u_n) - A_\infty(x)]\, \nabla u_n \cdot \nabla\psi_1 - (\delta_2 - \lambda_1) \int_\Omega u_n\psi_1 + \int_\Omega C\psi_1.$$

Similarly, using (11) we obtain that

$$t_n \int_\Omega \varphi\psi_1 \leqslant \int_\Omega [A(x,u_n) - A_\infty(x)]\, \nabla u_n \cdot \nabla\psi_1 - (\delta_1 - \lambda_1) \int_\Omega u_n\psi_1 + \int_\Omega C\psi_1.$$

Since $(\delta_1 - \lambda_1)(\delta_2 - \lambda_1) < 0$, we have that either the term $(\delta_1 - \lambda_1) \int_\Omega u_n\psi_1 < 0$ or $(\delta_2 - \lambda_1) \int_\Omega u_n\psi_1 < 0$. Hence we yield to

$$\begin{aligned} t_n \int_\Omega \varphi\psi_1 &\leqslant \int_\Omega [A(x,u_n) - A_\infty(x)]\nabla u_n \cdot \nabla\psi_1 + \int_\Omega C\psi_1 \\ &\leqslant 2\beta\|u_n\|\,\|\psi_1\| + C\|\psi_1\|_1. \end{aligned}$$

Therefore, since t_n is unbounded, we have that u_n is unbounded. Using the same arguments in the proof of Theorem 1 we have that $z_n = u_n/\|u_n\|$ strongly converges to a function $z \in H_0^1(\Omega)$. Thus, from (A_4) we deduce that

$$\frac{t_n}{\|u_n\|} \int_\Omega \varphi\psi_1 \leqslant \int_\Omega [A(x,u_n) - A_\infty(x)]\, \nabla z_n \cdot \nabla\psi_1 + \frac{1}{\|u_n\|} \int_\Omega C\psi_1 \to 0,$$

i.e., $t_n/\|u_n\| \to 0$. In this case the equation satisfied by z is (4) with $\mu = 1$ ($m(x) = f'_{-\infty}(x)\chi_{\{z\leqslant 0\}} + f'_{+\infty}(x)\chi_{\{z>0\}}$). In the proof of Lemma 4 is proved that this is a contradiction, proving that S is bounded from above.

To conclude the items (2) and (3) it suffices to show that S is closed. To this end let $\{t_n\}$ be a sequence in S converging to $t \in \mathbb{R}$. For every t_n, let u_n be a solution of (P_{t_n}), i.e. $u_n = K(f(x, u_n) + t_n\varphi)$. By Lemma 4, $\|u_n\|$ is bounded and from the compactness of K we deduce that—up to a subsequence—u_n strongly converges to a solution of (P_t).

We have just proved items (2) and (3). With respect to the item (1), since $S = (-\infty, t^*]$, only remains to find another solution of (P_t), for every $t \ll t^*$. Take $t < t_\varepsilon \leqslant t^* < t_1$. We can prove using ($\mathrm{A}_2$) that there exists $R \in \mathbb{R}^+$ such that $B_{|t|\varepsilon}(t\phi) \subset B_R(0)$ and (Lemma 4) for every solution u of ($\mathrm{P}_{t'}$), with $t' \in [t, t_1]$, we have that $\|u\| < R$. In addition, since $t_1 > t^*$, the equation $\Phi_{t_1}(u) = 0$ has no solution and thus

$$\deg(\Phi_{t_1}, B_R(0), 0) = 0.$$

Using again the homotopy invariance property of the Leray–Schauder degree we have

$$\deg(\Phi_{t'}, B_R(0), 0) = 0, \qquad \forall t' \in [t, t_1].$$

In particular,

$$\deg(\Phi_t, B_R(0), 0) = 0.$$

Then from the excision property we have that

$$\begin{aligned}\deg(I - \Phi_t, B_R(0) \setminus B_{|t|\varepsilon}(t\phi), 0) &= \deg(I - \Phi_t, B_R(0), 0)\\ &\quad - \deg(I - \Phi_t, B_{|t|\varepsilon}(t\phi), 0) = -1,\end{aligned}$$

i.e. there exists a second solution of $\Phi_t(u) = 0$ in $B_R(0) \setminus B_{|t|\varepsilon}(t\phi)$, which together with the first one found in $B_{|t|\varepsilon}(t\phi)$ implies item (1). □

Next we study the case where the nonlinearity f crosses all eigenvalues of (2). This is the case when we impose f to satisfy (f'_3). We have the following.

Theorem 7. *Assume that* ($\mathrm{A}_{1\text{–}4}$) *and* (f_2), (f'_3), (f_4) *hold. Assume also* (7) *and that* $f(x,\cdot)$ *is increasing for a.e.* $x \in \Omega$. *Then, there exists* $t^* \in \mathbb{R}$ *such that*

(1) (P_t) *has at least one solution for* $t \leqslant t^*$,
(2) (P_t) *has no solution for every* $t > t^*$.

Proof. The proof follows the outline of the previous one. We use the same notation: $S \equiv \{t \in \mathbb{R}\colon (\mathrm{P}_t) \text{ admits a solution}\}$. In order to prove that S is nontrivial, it suffices to find $t_0 \in \mathbb{R}$ such that (P_{t_0}) admits a super-solution. Indeed, arguing as in Theorem 6 we can find a well-ordered sub-solution and the method of sub-super-solution (see [7, Theorem I]) allow to deduce that in this case $t_0 \in S$. In order to find t_0, we follow closely the ideas in [11]. For each $j > 0$, define $M_j = \sup\{|f(x,s)|\colon x \in \Omega,\ s \in [0, j]\}$. By [17, Theorem 8.16] there exists a positive constant c such that $\|w\|_0 \leqslant c\|b\|_{N+1}$

for every solution w of

$$\begin{aligned} \operatorname{div}(A(x,w)\nabla w) &= b, \quad x\in\Omega, \\ u &= 0, \quad x\in\partial\Omega, \end{aligned}$$

with $b\in L^{N+1}(\Omega)$. Let $\delta=(j/cM_j)^{N+1}>0$ and choose open sets Ω_1, Ω_2 in the following way:

$$\Omega_1\subset\bar\Omega_1\subset\Omega_2\subset\bar\Omega_2\subset\Omega,$$

with Lebesgue measure $|\Omega-\Omega_1|\leqslant\delta$. We consider a continuous function b in $\bar\Omega$ such that

$$\begin{aligned} &b(x)=0, && \forall x\in\Omega_1, \\ &0\leqslant b(x)\leqslant M_j, && \forall x\in\bar\Omega_2\setminus\Omega_1, \\ &b(x)=M_j, && \forall x\in\Omega\setminus\Omega_2. \end{aligned}$$

For such b, let $\bar u$ be a solution of the above problem. Using $\bar u^-$ as test function we easily deduce that $\bar u^-\equiv 0$ and we have that $\bar u\geqslant 0$. Then,

$$0\leqslant\bar u(x)\leqslant c\|b\|_{N+1}\leqslant cM_j(|\Omega-\Omega_1|^{\frac{1}{N+1}})\leqslant cM_j\delta^{\frac{1}{N+1}}\leqslant j,\qquad \text{a.e. } x\in\Omega$$

and by definition of M_j, $f(x,\bar u(x))\leqslant M_j$ a.e. in Ω. Choosing now $t_0<0$ such that $M_j+t_0\varphi\leqslant 0$ in Ω_2, then

$$\begin{aligned} -\operatorname{div}(A(x,\bar u)\nabla\bar u) &= b = b\chi_{\{\Omega\setminus\Omega_2\}}+b\chi_{\{\Omega_2\}} \\ &\geqslant M_j\chi_{\{\Omega\setminus\Omega_2\}}+(M_j+t_0\varphi)\chi_{\{\Omega_2\}} \\ &\geqslant M_j+t_0\varphi \\ &\geqslant f(x,\bar u)+t_0\varphi, \end{aligned}$$

i.e. $\bar u$ is a super-solution for (P_{t_0}).

The proof that S is a closed interval unbounded from below is the same as Theorem 6. The main difference is to prove that S is bounded from above. In order to do that we argue by contradiction and suppose that there exists a solution u_n of (P_{t_n}) with $t_n\to+\infty$. Let denote by $\lambda(u_n)$ the first eigenvalue for the problem

$$\begin{aligned} -\operatorname{div}(A(x,u_n)\nabla v) &= \mu v, \quad x\in\Omega, \\ v &= 0, \quad x\in\partial\Omega, \end{aligned}$$

and ψ_n one positive eigenfunction associated to $\lambda(u_n)$. Recall that (A_{1-2}) imply that $\alpha\mu\leqslant\lambda(u_n)\leqslant\beta\mu$.

Now we observe that since (f_2), (f_3') (thus, (f_1') is deduced from the regularity of f and (f_4) are satisfied, we can use (11) for $\delta\geqslant\alpha\mu$. Taking $w=\psi_n$ as test function in the equation satisfied by u_n, for $\delta=\beta\mu$, we have that

$$\begin{aligned} \lambda(u_n)\int_\Omega\phi_n u_n &= \int_\Omega A(x,u_n)\nabla u_n\cdot\nabla\phi_n = \int_\Omega f(x,u_n)\phi_n + t_n\int_\Omega\varphi\phi_n \\ &> \beta\mu\int_\Omega\phi_n u_n - C\int_\Omega\phi_n + t_n\int_\Omega\varphi\phi_n. \end{aligned}$$

Analogously, for $\delta = \alpha\mu$

$$\lambda(u_n)\int_\Omega \phi_n u_n > \alpha\mu\int_\Omega \phi_n u_n - C\int_\Omega \phi_n + t_n\int_\Omega \varphi\phi_n.$$

Since $(\lambda(u_n)-\alpha\mu)(\lambda(u_n)-\beta\mu)<0$, we obtain from the above inequalities that $-C\int_\Omega \psi_n + t_n\int_\Omega \varphi\psi_n < 0$. Thus, $t_n < C\int_\Omega \psi_n/\int_\Omega \varphi\psi_n$. Moreover, by the Poincaré inequality we have that $t_n < c_1/\int_\Omega \varphi\psi_n/\|\psi_n\|$. We claim now that $\int_\Omega \varphi\psi_n/\|\psi_n\|$ is away from zero. Otherwise, passing to a subsequence if necessary, it converges to zero. On the other hand, $\psi_n/\|\psi_n\|$ is bounded in $H_0^1(\Omega)$ and consequently there exists $z\in H_0^1(\Omega)$ such that—up to a subsequence—$\psi_n/\|\psi_n\| \to z \geqslant 0$ strongly in $L^2(\Omega)$. By $(\mathrm{A}_{1\text{–}2})$,

$$\frac{\alpha}{\beta\mu} \leqslant \frac{\alpha}{\lambda(u_n)} \leqslant \left\|\frac{\psi_n}{\|\psi_n\|}\right\|_2^2 \leqslant \frac{\beta}{\lambda(u_n)} \leqslant \frac{\beta}{\alpha\mu}.$$

Therefore $z\neq 0$, and the Lebesgue Theorem implies that

$$\int_\Omega \varphi\frac{\psi_n}{\|\psi_n\|} \to \int_\Omega \varphi z.$$

Thus $\int_\Omega \varphi z = 0$, which is a contradiction, proving the claim.

The claim allows to deduce that there exist some positive constant c_2 such that $\int_\Omega \varphi\psi_n/\|\psi_n\| > c_2$. Therefore, $t_n < c_1/\int_\Omega \varphi\psi_n/\|\psi_n\| < c_1/c_2$, which contradicts that $t_n \to +\infty$. □

4. Continua of solutions for Ambrosetti–Prodi problems

We assume in this section that the nonlinearity f is a C^1-function satisfying $(\mathrm{f}_{2\text{–}3})$ and the matrix A satisfies the following condition:

$$A(x,s) = a(s)A(x), \qquad \forall s\in\mathbb{R}, \tag{A_5}$$

with $A(x) := (a_{ij}(x))$ a positive definite matrix with continuously differentiable coefficients $a_{ij}(x)$ in Ω and $a:\mathbb{R}\to\mathbb{R}$ a C^1-function satisfying

$$0 < k_1 \equiv \inf_{s\in\mathbb{R}} a(s) \leqslant k_2 \equiv \sup_{s\in\mathbb{R}} a(s) < +\infty.$$

Note that in this case, conditions $(\mathrm{A}_{1\text{–}3})$ hold. We improve the results of the previous section by proving the existence of continua in the set $\Sigma = \{(t,u)\in\mathbb{R}\times E\colon u$ solution of $(\mathrm{P}_t)\}$. In order to do that we begin with an abstract result

Lemma 8. *Let E be a Banach space and $T:\mathbb{R}\times E\to E$ a compact operator. Let us denote by Σ the set of pairs $(t,u)\in\mathbb{R}\times E$ such that u is a solution of*

$$u - T(t,u) = 0. \tag{Π_t}$$

Let U be a bounded subset in E, such that (Π_t) has no solution on ∂U for every $t\in[a,b]$. Assume also that (Π_b) has no solution in $\bar U$. Let $U_1\subset U$ such that (Π_a) has no solution on ∂U_1 and $\deg(I - T(a,\cdot), U_1, 0)\neq 0$. Then there exists a continuum C in Σ such that

$$C\cap(U_1\times\{a\})\neq\emptyset, \quad C\cap((U\backslash U_1)\times\{a\})\neq\emptyset.$$

Proof. Let us use the following notation:

$$\mathscr{S} = \{(t,u)\in[a,b]\times\bar{U}\colon u \text{ solution of } (\Pi_t)\},$$

$$\mathscr{A} = (\{a\}\times\bar{U}_1)\cap\mathscr{S},$$

$$\mathscr{B} = (\{a\}\times\overline{(U\setminus U_1)})\cap\mathscr{S}.$$

Since (Π_b) has no solution in $\bar{U}$, and $\mathscr{S}$ is compact, we can consider that $\mathscr{S}\subset[a,s]\times\bar{U}$ for some $s\in(a,b)$.

We argue by contradiction and suppose that the lemma is false. By a topological lemma of Whyburn [27] (see also [16, Lemma 29.1]) there exist $K_{\mathscr{A}}, K_{\mathscr{B}}$ disjoint compact sets containing, respectively to $\mathscr{A}$ and $\mathscr{B}$, such that $\mathscr{S}=K_{\mathscr{A}}\cup K_{\mathscr{B}}$. We can take N_δ a δ-neighborhood of $K_{\mathscr{A}}$ such that $\operatorname{dist}(N_\delta, K_{\mathscr{B}})>0$. Therefore the Leray–Schauder degree is well defined in $(N_\delta)_t=\{u\in\bar{U}/(t,u)\in N_\delta\}$ for all $t\in[a,b]$. Moreover, by homotopy invariance we have that

$$\deg(I-T(t,\cdot),(N_\delta)_t,0)=\text{constant}$$

and consequently,

$$\deg(I-T(t,\cdot),(N_\delta)_a,0)=\deg(I-T(t,\cdot),(N_\delta)_b,0).$$

On the other hand, since $N_\delta\cap K_{\mathscr{B}}=\emptyset$, we have that there is no solution of (Π_a) in $(N_\delta)_a\setminus\bar{U}_1$ and thus

$$\deg(I-T(t,\cdot),(N_\delta)_a,0)=\deg(I-T(t,\cdot),U_1,0)\neq 0.$$

Since $(N_\delta)_b=\emptyset$, we also have that $\deg(I-T(t,\cdot),(N_\delta)_b,0)=0$. Hence we have reached a contradiction, proving the lemma. □

Let $\hat{a}:\mathbb{R}\to\mathbb{R}$ be defined by

$$\hat{a}(s)=\int_0^s a(t)\,\mathrm{d}t.$$

From the chain rule, u is a solution of (P_t), i.e.,

$$\int_\Omega a(u)A(x)\nabla u\cdot\nabla v=\int_\Omega f(x,u)v+t\int_\Omega \varphi v,\qquad \forall v\in H_0^1(\Omega),$$

if and only if $w=\hat{a}(u)$ satisfies

$$\int_\Omega A(x)\nabla w\cdot\nabla v=\int_\Omega f(x,u)v+t\int_\Omega \varphi v,\qquad \forall v\in H_0^1(\Omega),$$

i.e., $w=\hat{a}(u)$ verifies

$$\begin{aligned} -\operatorname{div}(A(x)\nabla w) &= f(x,u)+t\varphi, && x\in\Omega,\\ w &= 0, && x\in\partial\Omega. \end{aligned}$$

We give here a result, proved in [4, Lemma 5.15] with slight less generality.

Lemma 9. *Suppose that $f\in C^1(\bar{\Omega}\times\mathbb{R})$. Let $\bar{u}\in C_0^1(\bar{\Omega})$ be a super-solution of (P_t) and $u\in C_0^1(\bar{\Omega})$ be a solution of (P_t) such that $\bar{u}\geqslant u$ and $\bar{u}\not\equiv u$. Then, $\bar{u}(x)>u(x)$ in Ω and $a(\bar{u})\partial\bar{u}/\partial n<a(0)\partial u/\partial n$ on $\partial\Omega$.*

Remark 10. Similarly if $\underline{u} \in C_0^1(\bar{\Omega})$ is a sub-solution of (P_t) and $u \in C_0^1(\bar{\Omega})$ is a solution of (P_t) such that $\underline{u} \leqslant u$ and $\underline{u} \not\equiv u$, then $\underline{u} < u$ in Ω and $a(\underline{u})\partial\underline{u}/\partial n > a(u)\partial u/\partial n$ on $\partial\Omega$.

Proof. Assume that $\bar{u}$ is a super-solution and u is a solution of (P_t) with $\bar{u} \geqslant u$, $\bar{u} \not\equiv u$. Let us define

$$k = \max\left\{\left|\frac{\partial f}{\partial s}(x,s)/a(s)\right| : x \in \bar{\Omega},\ u(x) \leqslant s \leqslant \bar{u}(x)\right\}.$$

Observe that for fixed $x \in \bar{\Omega}$, the function $f(x,s) + t\varphi + k\hat{a}(s)$ is nondecreasing in $s \in [u(x), \bar{u}(x)]$. Thus,

$$\begin{aligned} -\mathrm{div}(A(x)\nabla\hat{a}(\bar{u})) + k\hat{a}(\bar{u}) &\geqslant f(x,\bar{u}) + t\varphi + k\hat{a}(\bar{u}) \\ &\geqslant f(x,u) + t\varphi + k\hat{a}(u) = -\mathrm{div}(A(x)\nabla\hat{a}(u)) + k\hat{a}(u). \end{aligned}$$

Denoting $w = \hat{a}(\bar{u}) - \hat{a}(u)$ we obtain that

$$-\mathrm{div}(A(x)\nabla w) + kw \geqslant 0.$$

Using that $\bar{u} \not\equiv u$, the strong maximum principle implies that

$$0 < w = \hat{a}(\bar{u}) - \hat{a}(u) \quad \text{in } \Omega$$

and

$$\frac{\partial(\hat{a}(\bar{u}) - \hat{a}(u))}{\partial n} = a(\bar{u})\frac{\partial \bar{u}}{\partial n} - a(0)\frac{\partial u}{\partial n} < 0 \quad \text{in } \partial\Omega.$$

Moreover, since $\hat{a}$ is strictly increasing we obtain $\bar{u} > u$ in Ω. $\square$

In the theorem below we use Lemmas 8 and 9 to assure that, for matrices satisfying (A_5), (P_t) has at least two solutions for every $t < t^*$, t^* given by Theorem 6.

Theorem 11. *Let f be a C^1-function satisfying* $(\mathrm{f}_{2\text{–}3})$. *Assume that A satisfies* (A_5) *with* $\lim_{s\to+\infty} a(s) = \lim_{s\to-\infty} a(s)$. *Then for every $t_0 < t^* \equiv \sup\{t \in \mathbb{R}$: (P_t) admits a solution$\}$ there exists a continuum $\mathscr{C}$ in Σ satisfying that*

(1) $[t_0, t^*] \subset Proj_{\mathbb{R}}\, \mathscr{C}$.
(2) *For every $t \in [t_0, t^*)$, $Proj_E\, \mathscr{C}$ contains two distinct solutions of* (P_t).

Proof. First we note that from Lemma 4, for every fixed $t_1 > t^* > t_0$, there exists $R \in \mathbb{R}^+$ such that $\|u\| < R$ for every solution u of (P_t) with $t \in [t_0, t_1]$.

For every $h \in H^{-1}(\Omega)$, let us denote $\tilde{K}(h)$ the unique solution of the problem

$$\begin{aligned} -\operatorname{div}(A(x)\nabla \hat{a}(u)) &= h, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Thus the operator $\tilde{K}$ maps $H^{-1}(\Omega)$ into $H_0^1(\Omega)$. In addition, by using Theorem 9.15 in [17] and the Morrey Theorem, we have that for $q > N$, $\tilde{K}(L^q(\Omega)) \subset C_0^1(\bar{\Omega})$.

Hence, denoting by $\tilde{\Phi}_t$ the map $\tilde{\Phi}_t(u) \equiv u - \tilde{K}(f(x,u) + t\varphi)$, and using the homotopy invariance of degree,

$$\deg(\tilde{\Phi}_t, B_r(0), 0) = \text{constant}, \quad \forall t \in [t_0, t_1], \ \forall r \geqslant R,$$

where $B_r(0)$ denote the open ball in $C^1(\bar{\Omega})$. Since problem (P_{t_1}) has no solution, we deduce that the above degree is zero provided $t = t_1$. Thus

$$\deg(\tilde{\Phi}_t, B_r(0), 0) = 0, \quad \forall t \in [t_0, t_1], \ \forall r \geqslant R.$$

Let u^* be the solution of (P_{t^*}) given by item 2 of Theorem 6. Recall that u^* is a super-solution of (P_t) for every $t \in [t_0, t^*)$ and it is not a solution. Moreover, arguing as in Theorem 6, there exists a sub-solution $u_{t_0} < u^*$ of (P_{t_0}) which is not a solution. Clearly, u_{t_0} is also a sub-solution and no solution for (P_t) if $t \in [t_0, t^*)$. Consider the set

$$\mathcal{O} = \left\{ u \in C_0^1(\bar{\Omega}) \colon u_{t_0} < u < u^* \text{ in } \Omega; \ \frac{\partial \bar{u}^*}{\partial n} < \frac{\partial u}{\partial n} < \frac{\partial u_{t_0}}{\partial n} \text{ on } \partial\Omega \right\}.$$

Lemma 9 implies the nonexistence of solutions of (P_t) on $\partial\mathcal{O}$ (boundary taken in $C_0^1(\bar{\Omega})$). In particular, there is no solution in the boundary of $\mathcal{O} \cap B_r(0)$ ($r \geqslant R$) which means that the degree of $\tilde{\Phi}_t$ is well defined in this set.

We claim now that there exists $r \geqslant R$, such that

$$\deg(\tilde{\Phi}_t, \mathcal{O} \cap B_r(0), 0) = 1.$$

Indeed, fixed $t \in [t_0, t^*]$ we take

$$k = \max \left\{ \left| \frac{\partial f}{\partial s}(x,s)/a(s) \right| : x \in \bar{\Omega}, u_{t_0}(x) < s < u^*(x) \right\}$$

and define the truncated function $\tilde{f}$ in the following way:

$$\tilde{f}(x,s) = \begin{cases} f(x, u_{t_0}(x)) + k\hat{a}(u_{t_0}(x)) & \text{if } s \leqslant u_{t_0}(x), \\ f(x,s) + k\hat{a}(s) & \text{if } u_{t_0}(x) < s < u^*(x), \\ f(x, u^*(x)) + k\hat{a}(u^*(x)) & \text{if } s \geqslant u^*(x). \end{cases}$$

Observe that $\tilde{f}(x,s) + t\varphi$ is bounded and nondecreasing in the s variable. Consider now for every $h \in H^{-1}(\Omega)$ the unique solution $u = Th$ of the problem

$$\begin{aligned} -\mathrm{div}(A(x)\nabla\hat{a}(u)) + k\hat{a}(u) &= h, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned}$$

This defines an operator $T : H^{-1}(\Omega) \to H_0^1(\Omega)$. Since the function $\tilde{f}(x,s) + t\varphi$ is bounded, by [17, Theorem 9.15] and the Morrey Theorem, we know that $T(\tilde{f}(x,\cdot)+t\varphi)$ is bounded in $C^1(\bar{\Omega})$. Let

$$r_0 = \sup\{\|T(\tilde{f}(x,v) + t\varphi)\|_{C^1(\bar{\Omega})} \colon v \in C^1(\bar{\Omega})\}$$

and choose $r > \max\{R, r_0\}$. From the definition of $\tilde{f}$ we have that $\tilde{f}(x,u_{t_0}) + t_0\varphi \leqslant \tilde{f}(x,v) + t\varphi \leqslant \tilde{f}(x,u^*) + t^*\varphi$ for every $v \in C_0^1(\bar{\Omega})$ and consequently, Lemma 9 leads to

$$\{T(\tilde{f}(x,v) + t\varphi) \colon v \in C_0^1(\bar{\Omega})\} \subset \mathcal{O} \cap B_r(0).$$

Now, pick $\psi \in \mathcal{O} \cap B_r(0)$ and consider the compact homotopy, $H(s,u) = sT((\tilde{f}(x,u) + t\varphi)) + (1-s)\psi$, $0 \leqslant s \leqslant 1$. Since $\mathcal{O} \cap B_r(0)$ is a convex set and the equation $u = H(1,u)$ has no solution on $\partial\mathcal{O} \cap B_r(0)$ we have that $u \neq H(s,u)$ for all $u \in \partial\mathcal{O} \cap B_r(0)$ and $s \in [0,1]$. Therefore,

$$\deg(I - T(\tilde{f}(x,\cdot) + t\varphi), \mathcal{O} \cap B_r(0), 0) = \deg(I - \psi, \mathcal{O} \cap B_r(0), 0) = 1.$$

Noting now that $\tilde{f}(x,v(x)) = f(x,v(x)) + k\hat{a}(v(x))$ for every $v \in \overline{\mathcal{O} \cap B_r(0)}$, we yield

$$\deg(\tilde{\Phi}_t, \mathcal{O} \cap B_r(0), 0) = \deg(I - T(\tilde{f}(x,\cdot) + t\varphi), \mathcal{O} \cap B_r(0), 0) = 1,$$

proving the claim.

Choose now $R_1 > R$ in such a way that $\mathcal{O} \subset B_{R_1}(0)$. In order to use Lemma 8 we take $[a,b] = [t_0, t_1]$, $U = B_{R_1}(0)$ and $U_1 = \mathcal{O} \cap B_r(0)$. Notice that (P_t) has no solution in $\partial B_{R_1}(0)$ for every $t \in [t_0, t_1]$ and (P_{t_1}) has no solution in $\overline{B_{R_1}(0)}$. Note also that we have just proved that $\deg(I - T(\tilde{f}(x,\cdot) + t\varphi), \mathcal{O} \cap B_r(0), 0) = 1$. In consequence we can apply Lemma 8 to deduce that there exist a continuum $\mathscr{C}$ in Σ such that

$$\mathscr{C} \cap (\mathcal{O} \cap B_r(0) \times \{t_0\}) \neq \emptyset$$

and

$$\mathscr{C} \cap ([B_R(0) \setminus \overline{\mathcal{O} \cap B_r(0)}] \times \{t_0\}) \neq \emptyset.$$

Moreover, using Lemma 9 it is possible to prove that (P_t) has no solution on $\partial\mathcal{O} \cap B_r(0)$ for every $t \in [t_0, t^*)$. Thus, due to the connection of $\mathscr{C}$, we deduce from above that $\mathscr{C}$ intersects $\partial\mathcal{O} \cap B_r(0) \times \{t\}$ with $t \in [t_0, t^*]$ if and only if $t = t^*$. This proves that the projection on E of this continuum contains at least two solutions of (P_t) for every $t \in [t_0, t^*)$, proving the theorem. □

With the same proof, but using item (ii) of Lemma 4 instead of item (i) of this lemma, we can handle the case of "super-linear" nonlinearities:

Theorem 12. *Let f be a C^1-function satisfying* (f_2), (f_3') *and* (f_4). *Assume that A satisfies* (A_5) *with* $\lim_{s\to+\infty} a(s) = \lim_{s\to-\infty} a(s)$. *Then, for every* $t_0 < t^* \equiv \sup\{t \in \mathbb{R}\colon (\mathrm{P}_t)$ *admits a solution*$\}$ *there exists a continuum* $\mathscr{C}$ in Σ satisfying that

(1) $[t_0, t^*] \subset Proj_{\mathbb{R}}\, \mathscr{C}$.
(2) *$Proj_E\, \mathscr{C}$ contains two distinct solutions of* (P_t) *for every* $t \in [t_0, t^*)$.

5. Ambrosetti–Rabinowitz problems via Ambrosetti–Prodi problems

In this section we see how to apply the results obtained in Sections 3 and 4 in order to obtain the quasilinear extension of the Ambrosetti–Rabinowitz result (see [3]) of existence of positive solutions. More precisely, let us consider a Carathéodory function $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x,0) = 0$ a.e. $x \in \Omega$, satisfying (f_4) and

$$\lim_{s\to 0^+} \frac{f(x,s)}{s} = \gamma < \alpha\mu, \qquad \text{uniformly } x \in \Omega. \tag{f_5}$$

We are interested with the existence of nontrivial and nonnegative (positive) solutions of the problem (1) for this kind of nonlinearities. Let $\tilde f : \Omega \times \mathbb{R} \to \mathbb{R}$ the extension of f given by

$$\tilde f(x,s) \equiv f(x, s^+), \qquad x \in \Omega.$$

Notice that by the maximum principle, every nontrivial solution of

$$\begin{aligned} -\mathrm{div}(A(x,u)\nabla u) &= \tilde f(x,u), && x \in \Omega,\\ u &= 0, && x \in \partial\Omega, \end{aligned} \tag{12}$$

is a positive solution of (1). In this way, the search of positive solutions of (1) is reduced to look for nontrivial solutions of (12). Since the extension $\tilde f$ satisfies clearly conditions $(\mathrm{f}_{1,3}')$ and $(\mathrm{f}_{2,4,5})$, problem (12) lies into the "super-linear" Ambrosetti–Prodi framework (problem (P_t) with $t = 0$ and any φ). Applying Theorem 12 we prove

Theorem 13. *Assume that A satisfies* (A_5) *for some positive function a with* $\lim_{s\to+\infty} a(s) = \lim_{s\to-\infty} a(s)$. *Suppose also that f is* $C^1(\bar\Omega \times \mathbb{R}^+)$, *satisfies* $(\mathrm{f}_{4,5})$, *and $f(x,\cdot)$ is increasing for a.e. $x \in \Omega$. Then problem* (1) *has at least a positive solution.*

Proof. The outline of the proof is the following. First (step 1) we embed the problem into a one parameter problem (P_t) for a suitable chosen function φ. Next (step 2) we use the results in the previous sections proving that t^* defined in Theorem 6 is strictly positive.

Step 1. Choice of φ.

In [10] it is proved that for every $r > 0$ there exists a positive solution $u_r \in P$ of the problem

$$\begin{aligned} -\mathrm{div}(A(x,u_r)\nabla u_r) &= \mu_r u_r, && x \in \Omega,\\ u_r &= 0, && x \in \partial\Omega, \end{aligned}$$

with $\alpha\mu \leqslant \mu_r \leqslant \beta\mu$ and $\|u_r\|_0 \leqslant cr$, for some positive constant c.

We claim that there exists $r \in \mathbb{R}^+$ such that $\mu_r > f(x,u_r)/u_r$ a.e. $x \in \Omega$. Indeed, from (f_5) there exist $\delta \in \mathbb{R}^+$ and $\gamma_1 \in (\gamma, \alpha\mu)$ such that $f(x,s)/s \leqslant \gamma_1$ provided $s < \delta$. Choosing $r < \delta/c$ we have that $\|u_r\|_0 \leqslant \delta$, thus

$$\frac{f(x,u_r)}{u_r} \leqslant \gamma_1 < \alpha\mu \leqslant \mu_r, \quad \text{a.e. } x \in \Omega$$

which proves the claim.

Taking now $t \in (0, \mu_r - \gamma_1)$ we get

$$-\mathrm{div}(A(x,u_r)\nabla u_r) = \mu_r u_r \geqslant f(x,u_r) + tu_r,$$

i.e., u_r is a super-solution of

$$\begin{aligned} -\mathrm{div}(A(x,u)\nabla u) &= \tilde{f}(x,u) + tu, \quad && x \in \Omega, \\ u &= 0, && x \in \partial\Omega. \end{aligned}$$

Let us choose $\varphi = u_r$, and recall that $\varphi \in C_0^1(\bar{\Omega})$. To set the framework of the previous sections we embed (1) into the one parameter problem

$$\begin{aligned} -\mathrm{div}(A(x,u)\nabla u) &= \tilde{f}(x,u) + t\varphi, \quad && x \in \Omega, \\ u &= 0, && x \in \partial\Omega. \end{aligned} \tag{$\tilde{\mathrm{P}}_t$}$$

Step 2. Let us denote $\tilde{S} \equiv \{t \in \mathbb{R}$: $(\tilde{\mathrm{P}}_t)$ admits a solution$\}$. From step 1 we know that φ is a super-solution of $(\tilde{\mathrm{P}}_{t_0})$, for some $t_0 > 0$. Arguing as in Theorem 6 there exists $\underline{u}$ sub-solution of $(\tilde{\mathrm{P}}_{t_0})$ with $\underline{u} \leqslant \varphi$. The sub-super-solution method allows to deduce that $(\tilde{\mathrm{P}}_{t_0})$ has at least one solution. Therefore, $0 < t_0 \in \tilde{S}$ and consequently $0 < \tilde{t}^*$, where $\tilde{t}^* = \sup \tilde{S}$ is given by Theorem 12 applied to the problem $(\tilde{\mathrm{P}}_t)$. Moreover, item (1) of that theorem shows the existence of at least two solutions of $(\tilde{\mathrm{P}}_0)$, or equivalently the existence of two nonnegative solutions of (1). Taking into account that (f_5) implies that zero is solution of this problem, we deduce that there exists at least one positive solution of (1). □

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